LIFTING FUNCTORS TO EILENBERG—MOORE CATEGORY OF MONAD GENERATED BY FUNCTOR C_pC_p

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The second iteration of the contravariant functor of spaces of continuous functions in the pointwise convergence topology is a functorial part of a monad (triple) on the category of Tikhonov spaces. The problem of lifting functors to the Eilenberg–Moore category of this monad is investigated.

1. The triple generated by the functor C_pC_p. A triple $\mathbf{T} = (T, \eta, \mu)$ on the category C consists of an endofunctor $T: C \to C$ and natural transformations $\eta: 1_C \to T$ and $\mu: TT \to T$ satisfying the following conditions: $\mu \circ \eta T = \mu \circ T\eta = 1_T$, $\mu \circ T\mu = \mu \circ \mu T$ [1].

We denote by Tikh the category of Tikhonov topological spaces and continuous maps. The contravariant functor C_p: Tikh $\to$ Tikh is defined in the following way. The space C_pX of all continuous real-valued functions on X is equipped with the pointwise convergence topology [2]; given the map $f: X \to Y$ (X and Y both Tikhonov spaces), we define the map $C_pf: C_pY \to C_pX$ by means of the following formula: $C_pf(\varphi) = \varphi \circ f$, $\varphi \in C_pY$. For every $x \in X$, we let $ev_x: C_pX \to \mathbb{R}$ denote the map defined by the formula $ev_x(\varphi) = \varphi(x)$, $\varphi \in C_pX$. It is well known that the map $\eta X: X \to C_pC_pX$, $\eta X(x) = ev_x$, $x \in X$, is continuous [2]. It is easily seen that $\eta = (\eta X)$ is a natural transformation from 1_{Tikh} into C_pC_p. The natural transformation $\mu: C_pC_pC_pC_p \to C_pC_p$ is constructed by means of the equality $\mu X(\Phi)(\varphi) = \Phi(ev_\varphi)$, $\Phi \in C_pC_pC_pC_pX$, $\varphi \in C_pX$. The map $\mu X(\Phi): C_pX \to \mathbb{R}$ is continuous, as it is the composition of ev: $C_pX \to C_pC_pC_pX$ and $\Phi: C_pC_pC_pX \to \mathbb{R}$, hence μX is well-defined. To show that μX is continuous, it is necesary to verify the simple inclusion $\mu X(\Phi, ev_\varphi, \varepsilon) \subseteq (\mu X(\Phi), \varphi, \varepsilon)$, and to apply the fact that the sets $(\Gamma, \psi, \varepsilon) = \{E \in C_pC_pX \mid |E(\psi) - \Gamma(\psi)| < \varepsilon\}$ form a subbasis in C_pC_pX (for $\Gamma \in C_pC_pX$, $\psi \in C_pX$, $\varepsilon \in \mathbb{R}$).

Proposition 1. $\mathbb{C}_p^2 = (C_pC_p, \eta, \mu)$ is a triple on the Tikh category.

Proof. Since

$$\mu X \circ C_pC_pX(\varphi)(\psi) = C_pC_p\eta X(\varphi)(ev_\psi) = \varphi(C_p(\eta X(ev_\psi))) = \varphi(\psi)$$

and $\mu X \circ \eta C_pC_pX(\varphi)(\psi) = \mu X(ev_\varphi)(\psi) = ev_\varphi(e\psi_\psi) = \varphi(\psi)$ for arbitrary $\varphi \in C_pC_pX$, $\psi \in C_pX$, we deduce $\mu \circ C_pC_p\eta = \mu \circ \eta C_pC_p = 1$. For every $\psi \in C_pC_pC_pC_pX$, $\varphi \in C_pX$, we find that $C_p\mu X(ev_\varphi)(\psi) = ev_\varphi \circ \mu X(\psi) = \mu X(\psi) \times (\varphi) = \psi(ev_\varphi)$ and, therefore, $C_p\mu X(ev_\varphi) = ev$. Combining these equations, we deduce

$$\mu X \circ C_pC_p X(\Phi)(\varphi) = C_pC_p\mu X(\Phi)(ev_\varphi) = \Phi \circ C_p\mu X(ev_\varphi) = \Phi(ev_\varphi) =$$
$$= \mu C_pC_pX(\Phi)(ev_\varphi) = (\mu X \circ \mu C_pC_pX(\Phi))(\varphi) \text{ (i. e. } \mu \circ C_pC_p\mu = \mu \circ \mu C_pC_p).$$

The proposition is proved.

Note that the space C_pC_pX possesses a self-evident algebraic structure with respect to pointwise addition and multiplication of functions and scalar multiplication. Let L_pX be a linear subspace of C_pX generated by the image of X under the map ηX and let A_pX be the smallest subalgebra of C_pX that includes the set $\eta X(X)$. It is clear that both L_p and A_p are subfunctors of C_p.

The condition $\mu X \circ \eta C_pC_pX = 1_X$ implies that $\mu X(L_pL_pX) \subseteq L_pX$ and $\mu X(A_pA_pX) \subseteq A_pX$. Therefore, two new triples are obtained: $\mathbf{L}_p = (L_p, \eta, \mu \mid L_pL_p)$ and $\mathbf{A}_p = (A_p, \eta, \mu \mid A_pA_p)$ on the Tikh category.

2. Normal functors on the Tikh category. The functor F: Tikh $\to$ Tikh is said to be normal if it is continuous, preserves weight, monomorphisms, intersections, inverse images, empty space, and one-point space, and transforms k-

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covering maps into surjective maps (cf. [3]). Note that a map f: $X \to Y$ is k-covering if and only if for an arbitrary compact subspace $K \subseteq Y$ there exists a compact subspace $L \subseteq X$ such that $F(L) = K$.

A normal functor F: Tikh $\to$ Tikh is said to be of degree $\leq n$ (abbreviated: deg (F) $\leq n$) if for every $a \in FX$ there exist $b \in Fn$ and map f: $n \to X$ such that $a = Ff(b)$. A normal functor is said to be finite if it preserves the class of finite spaces, and is said to be multiplicative if it preserves products.

Proposition 2. A normal multiplicative functor F is isomorphic to the power functor $(-)^i$ for some $i < \infty$ if either one of the following assertions is true:

(a) deg (F) = n (and then i = n);

(b) F is finite.

For the proof see [4, 5].

THEOREM 1. Let F be a normal multiplicative functor F: Tikh $\to$ Tikh such that F(Comp) $\subseteq$ Comp. Then F is a subfunctor of $(-)^\omega$.

Proof. Without loss of generality we may assume that deg (F) = ∞. By [6] there exists a functor isomorphism h: F |Comp $\to (-)^\omega$ |Comp. Let us first prove that for every Tikhonov space X and its compactification i_x: $X \to bX$ the following inclusions hold: $hbX \circ F\ |i_x(FX) \subseteq X^\omega \subseteq (bX)^\omega$. We assume the contrary. Then, without loss of generality, X may be considered a discrete countable space and bX to be αX, the one-point compactification of X. Let $a \in h\alpha X \circ Fi_x(FX) \llcorner X^\omega$. Then there exists a sequence $(b_i)_{i=1,2,\ldots}$ in $F\alpha X$ that converges to a such that the supports of b_i are finite and lie in X.

Suppose that $a = (x_i)_{i=1,2,\ldots} \in (\alpha X)^\omega$. Without loss of generality, it may be assumed that $x_0 \in \alpha X \backslash X$. Suppose that $h\alpha X \circ Fi_x(b_i) = (y_{ij})_{j=1,2,\ldots}$. There exists a map f: $X \to Y$ such that the sequence $(f(y_{i0})_{i=1,2,\ldots})$ is not converging in αX. Therefore, the sequence $(Fi_x \circ Ff(b_i))_{i=1,2,\ldots}$ is not converging in $F(\alpha X)$. We have obtained a contradiction.

Now let us define the natural transformation f: $F \to (-)^\omega$ by means of the following formula: $jX = h\beta X \circ Fi_x$, where i_x: $X \to \beta X$ is the canonical embedding of X in the Stone–Čech compactification βX of the space X. The theorem is proved.

3. Lifting a normal functor to the Eilenberg–Moore category. A pair (X, ξ), where ξ: $TX \to X$ is a C-morphism, is called a T-algebra if and only if $\xi \circ \mu X = 1_x$ and $\xi \circ \mu X\xi \circ T\xi$. A morphism f: $X \to Y$ is said to be a morphism of a T-algebra (X, ξ) into a **T**-algebra (Y, ζ) if $f \circ \xi = \zeta \circ Tf$. **T**-algebras and their morphisms form a category usually denoted $C^{\mathbf{T}}$ (Eilenberg–Moore category). The forgetful functor $U^{\mathbf{T}}$: $C^{\mathbf{T}} \to C$ may be defined by the formula $U^{\mathbf{T}}(X, \xi) = X$, $U^{\mathbf{T}}(f) = f$ (cf. [1] for a detailed discussion).

A lifting of the functor F: $C \to C$ on the category $C^{\mathbf{T}}$ is a functor G: $C^{\mathbf{T}} \to C^{\mathbf{T}}$ such that $U^{\mathbf{T}} \circ G = F \circ U^{\mathbf{T}}$. The next proposition supplies an existence criterion for a lifting that is dual to a result found by Vinárek [7] (cf. [8]).

Proposition 3 (cf. [9]). There exists a bijective correspondence between liftings of a functor to $C^{\mathbf{T}}$ and natural transformations δ: $TF \to FT$ such that $\delta \circ \eta F = F\eta$ and $\delta \circ \mu F = F\mu \circ \delta T \circ T\delta$.

Let **T** denote any one of the triples $\mathbb{C}_p^2$, $\mathbf{A}_p$, or $\mathbf{L}_p$. Below we apply the method of [5] for describing a lifting to the category of compact groups.

THEOREM 2. If a normal functor F may be lifted to the category $\mathrm{Tikh}^{\mathbf{T}}$, F is multiplicative.

Proof. Let T be any one of the functors C_pC_p, L_p, or A_p. We wish to consider the free T-algebra (TQ, μQ), denoting TQ by X (Q is the Hilbert cube). Let us assume that F admits a lifting to $\mathrm{Tikh}^{\mathbf{T}}$. Then the map $f = (Fpr_1, Fpr_2)$: $F(X \times X) \to FX \times FX$ turns out to be bijective. In fact, from the condition that F preserves inverse images and intersections, we find that ker (f) = 0, using the fact that f is a linear map of topological linear spaces. In addition, since the set ker $(Fpr_1) = F(\ker(pr_1))$ is homeomorphically mapped onto FX by means of the map Fpr_2, we find that f is surjective (cf. [5]).

Since Q may be topologically embedded in X, F turns out to be multiplicative (cf. [4]).

COROLLARY. If F is a normal functor that admits a lifting to the $\mathrm{Tikh}^{\mathbf{T}}$ category and F is either finite or deg (F) $< \infty$, F is isomorphic to a power functor.

REFERENCES

1. M. Barr and C. Wells, Toposes, Triples, and Theories, Springer, New York (1985), 345 pp.
2. A. V. Arkhangel'skii, Topological Spaces of Functions [in Russian], Izd-vo Mosk. Un-ta, Moscow (1989), 224 pp.
3. A. Ch. Chigogidze, "On the extension of normal functors," Vestn. Mosk. Un-ta. Ser. Matem., No. 6, 23-26 (1984).
4. E. V. Shchepin, "Functors and odd powers of compacta," Usp. Mat. Nauk, **36,** No. 3, 3-62 (1981).

5. M. M. Zarichnyy, "Multiplicative normal power functor," Mat. Zametki, **41**, No. 1, 93-100 (1987).
6. M. M. Zarichnyy, "Profinitary multiplicative functors and characterization of projective monads in the category of compacta," Ukr. Mat. Zh., **42**, No. 9, 1271-1275 (1990).
7. J. Vinárek, "Projective monad and extensions of functors," Math. Centr. Afd., No. 195, 1-12 (1983).
8. M. Arbib and E. Manes, "Fuzzy machines in a category," Bull. Austral. Math. Soc., **13**, No. 1, 169-220 (1975).
9. M. M. Zarichnyy, "On covariant topological functors," Q. and A. in General Topology, **8**, No. 2, 317-369 (1990).

STRUCTURE OF INTEGRABLE SUPERSYMMETRIC NONLINEAR DYNAMICAL SYSTEMS ON REDUCED INVARIANT SUBMANIFOLDS

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Based on an analysis of a supersymmetric extension of the algebra of pseudodifferential operators on $\mathbb{R}^1$ *an infinite hierarchy of supersymmetric Lax-integrable nonlinear dynamical systems is constructed by means of the Yang–Baxter* $\mathcal{R}$*-equation method. The structure of these systems on reduced invariant submanifolds specified by a natural invariant Lax-type spectral problem is investigated.*

1. Suppose we are given a Lie superalgebra $\mathfrak{G} = \mathfrak{G}_+ \oplus \mathfrak{G}_-$ over a commutative superalgebra $\mathbb{R}^{1|1}$ of pseudodifferential operators of the form

$$\mathfrak{G}_+ = \bigcup_{\{u\}} \Big\{ \sum_{0\leq j<\infty} u_j \xi^j : u_j \in \mathcal{G}^{(\infty)} (\mathbb{R}^{1|1}; \mathbb{R}^{1|1})\Big\},$$
$$\mathfrak{G}_- = \bigcup_{\{a\}} \Big\{ \sum_{j\in\mathbb{Z}_+} a_j \xi^{-(j+1)} : a_j \in \mathcal{G}^{(\infty)} (\mathbb{R}^{1|1}; \mathbb{R}^{1|1})\Big\}, \tag{1}$$

where $\mathfrak{G}_\pm \subset \mathfrak{G}$ are superalgebras. The Lie operation $[\cdot, \cdot]$ in $\mathfrak{G}$ is denoted thus [1–5]:

$$[a, b] = a \circ b - (-1)^{\tilde{a}\tilde{b}} b \circ a, \quad (\xi c)(x, \theta) = (\partial/\partial\theta + \partial/\partial x)\, c(x, \theta)$$

for all homogeneous elements a, b $\in \mathfrak{G}$ and c(x, θ) $\in \mathbb{R}^{1|1}$, moreover " ~ " denotes the operation of determining the parity of an element and " ∘ " denotes the ordinary composition of operators. The superalgebra $\mathfrak{G}$ of (1) may be transformed into a metrized algebra by means of the following analog of the bilinear Killing symmetric form on $\mathfrak{G}$: for all a, b $\in \mathfrak{G}$,

$$(a, b) = \mathrm{Tr}\,(a \circ b),$$

where $\mathrm{Tr}\,(a) = \int_{R^{1|1}} dx d\theta \operatorname*{res}_{\xi=0} a(\xi)$. On the superalgebra $\mathfrak{G}$ we may introduce [6] still another structure of a Lie superalgebra through the introduction of an $\mathcal{R}$-structure as follows:

$$[a, b]_{\mathcal{R}} = [a, \mathcal{R}b] + [\mathcal{R}a, b],$$

where $\mathcal{R} : \overline{\mathfrak{G}} \to \overline{\mathfrak{G}}$ is a homomorphism of the module $\overline{\mathfrak{G}}$ into itself satisfying the condition that for a, b $\in \mathfrak{G}$,

$$\mathcal{R}\,[a, b]_{\mathcal{R}} - [\mathcal{R}a, \mathcal{R}b] = -[a, b]. \tag{2}$$

By virtue of the decomposition (1), the following operator is a solution of (2):

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