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The maximum number of K_3-free and K_4-free edge 4-colorings

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Abstract

Let $F(n, r, k)$ denote the maximum number of edge r-colorings without a monochromatic copy of K_k that a graph with n vertices can have. Addressing two questions left open by Alon, Balogh, Keevash and Sudakov [*J. London Math. Soc.* 70 (2004) 273–288], we determine $F(n, 4, 3)$ and $F(n, 4, 4)$ and describe the extremal graphs for all large n.

1. *Introduction*

Given a graph G and integers $k \geqslant 3$ and $r \geqslant 2$, let $F(G, r, k)$ denote the number of distinct edge r-colorings of G that are K_k-*free*, that is, do not contain a monochromatic copy of K_k, the complete graph on k vertices. Note that we do not require that these edge colorings are *proper* (that is, we do not require that adjacent edges get different colors). We consider the following extremal function:

$$F(n, r, k) = \max\{F(G, r, k) : G \text{ is a graph on } n \text{ vertices}\},$$

the maximum value of $F(G, r, k)$ over all graphs of order n.

One obvious choice for G is to take a maximum K_k-free graph of order n. The celebrated theorem of Turán [15] states that $\mathrm{ex}(n, K_k)$, the maximum size of a K_k-free graph of order n, is attained by a unique (up to isomorphism) graph, namely, the *Turán graph* $T_{k-1}(n)$ which is the complete $(k-1)$-partite graph on n vertices with parts of size $\lfloor n/(k-1)\rfloor$ or $\lceil n/(k-1)\rceil$. Thus

$$\mathrm{ex}(n, K_k) = t_{k-1}(n), \quad \text{for all } n, k \geqslant 2, \tag{1.1}$$

where $t_{k-1}(n)$ denotes the number of edges in $T_{k-1}(n)$. This gives the following trivial lower bound on our function:

$$F(n, r, k) \geqslant F(T_{k-1}(n), r, k) = r^{t_{k-1}(n)}. \tag{1.2}$$

Erdős and Rothschild (see [5, 6]) conjectured that this is best possible when $r = 2$ and $k = 3$. Yuster [16] proved that, indeed, $F(n, 2, 3) = 2^{t_2(n)} = 2^{\lfloor n^2/4\rfloor}$ for large enough n. Both sets of authors further conjectured that this holds for all k when we have $r = 2$ colors. Alon, Balogh, Keevash and Sudakov [1] not only settled this conjecture for large n, but also showed that it holds for 3-colorings as well, that is, we have equality in (1.2) when $r = 2, 3$, $k \geqslant 3$ and $n > n_0(k)$.

The generalization of the problem where one has to avoid a monochromatic copy of a general graph F was also studied in [1]. The papers [7, 9–11] studied H-free edge colorings for general hypergraphs H. In particular, Lefmann, Person and Schacht [11] proved that, for every k-uniform hypergraph F and $r \in \{2, 3\}$, the maximum number of F-free edge r-colorings over n-vertex hypergraphs is $r^{\mathrm{ex}(n,F)+o(n^k)}$. Interestingly, this result holds for every F even though

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the value of the Turán function $\mathrm{ex}(n, F)$ is known for very few hypergraphs F. Also, Balogh [3] studied a version of the problem where a specific coloring of a graph F is forbidden. Alon and Yuster [2] considered this problem for directed graphs (where one counts admissible orientations instead of edge colorings).

Let us return to the original question. Surprisingly, Alon *et al.* [1] showed that one can do significantly better than (1.2) for larger values of r. In two particular cases, they were also able to obtain the best possible constant in the exponent; namely, they proved that

$$F(n, 4, 3) = 18^{n^2/8+o(n^2)}, \tag{1.3}$$

$$F(n, 4, 4) = 3^{4n^2/9+o(n^2)}. \tag{1.4}$$

Let us briefly show the lower bounds in (1.3) and (1.4), which are given by $F(T_4(n), 4, 3)$ and $F(T_9(n), 4, 4)$, respectively. Let $W_1, \ldots, W_k$ denote the parts of $T_k(n)$. Consider $T_4(n)$ first. Fix a function π that assigns to each pair $\{i, j\}$ of $\{1, \ldots, 4\}$ a list $\pi(\{i, j\})$ of two or three colors so that each color appears in exactly four lists with the corresponding four pairs forming a 4-cycle. Up to a symmetry, such an assignment is unique and we have two lists of size 2 and four lists of size 3. Generate an edge coloring of $T_4(n)$ by choosing for each edge $\{u, v\}$ with $u \in W_i$ and $v \in W_j$ an arbitrary color from $\pi(\{i, j\})$. Every obtained coloring is K_3-free and if we assume that, for example, $n = 4m$, there are $3^{4m^2} \cdot 2^{2m^2} = 18^{n^2/8}$ such colorings. We proceed similarly for $T_9(n)$ except that we fix the (unique up to a symmetry) assignment where each pair from $\{1, \ldots, 9\}$ gets a list of three colors while every color forms a copy of $T_3(9)$.

The goal of this paper is to determine $F(n, 4, 3)$ and $F(n, 4, 4)$ exactly and describe all extremal graphs for large n. Specifically, we will show the following results.

THEOREM 1.1. *There is N such that, for all $n \geqslant N$, $F(n, 4, 3) = F(T_4(n), 4, 3)$ and $T_4(n)$ is the unique graph achieving the maximum.*

THEOREM 1.2. *There is N such that, for all $n \geqslant N$, $F(n, 4, 4) = F(T_9(n), 4, 4)$ and $T_9(n)$ is the unique graph achieving the maximum.*

Thus, a new phenomenon occurs for $r \geqslant 4$: extremal graphs may have many copies of the forbidden monochromatic graph K_k. This makes the problem more interesting and difficult.

Similarly to [1], our general approach is to establish the stability property first: namely, that all graphs with the number of colorings close to the optimum have essentially the same structure. However, additionally to the approximate graph structure, we also have to describe how typical colorings look like. This task is harder and we do it in stages, getting more and more precise description of typical colorings (namely, the properties called *satisfactory*, *good* and *perfect* in our proofs). We then proceed to show that the Turán graphs are, indeed, the unique graphs that attain the optimum. It is not surprising that our proofs are longer and more complicated than those in [1]. The case of $r \geqslant 4$ colors seems to be much harder than the case $r \leqslant 3$. It is not even clear if there is a simple closed formula for $F(T_4(n), 4, 3)$ and $F(T_9(n), 4, 4)$. Our proofs imply that

$$F(T_4(n), 4, 3) = (C + o(1)) \cdot 18^{t_4(n)/3}, \tag{1.5}$$

$$F(T_9(n), 4, 4) = (20160 + o(1)) \cdot 3^{t_9(n)}, \tag{1.6}$$

where $C = (2^{14} \cdot 3)^{1/3}$ if $n \equiv 2 \pmod 4$ and $C = 36$ otherwise.

Unfortunately, we could not determine $F(n, r, k)$ for other pairs r, k, which seems to be an interesting and challenging problem. Hopefully, our methods may be helpful in obtaining further exact results. It is possible that, for all large n, $n \geqslant n_0(k, r)$, all extremal graphs are complete partite (not necessarily balanced), but we could not prove nor disprove this.

This paper is organized as follows. In Section 2, we state a version of Szemerédi's Regularity Lemma and some auxiliary definitions and results that we use in our arguments. Theorem 1.1 is proved in Section 3 and Theorem 1.2 is proved in Section 4.

2. *Notation and tools*

For a set X and a non-negative integer k, let $\binom{X}{k}$ or $\binom{X}{\leqslant k}$ be the set of all subsets of X with exactly or at most k elements, respectively. Also, we denote $\binom{n}{\leqslant k} = \sum_{i=0}^{k} \binom{n}{i}$ and $[k] = \{1, 2, \ldots, k\}$. We often omit punctuation signs when writing unordered sets, abbreviating, for example, $\{i, j\}$ to ij.

As it is standard in graph theory, we use $V(G)$ and $E(G)$ to refer to the vertex and edge set, respectively, of a graph G. Also, $v(G) = |V(G)|$ and $e(G) = |E(G)|$ denote, respectively, the *order* and *size* of G. In addition, for disjoint $A, B \subseteq V(G)$, we use $G[A]$ to refer to the subgraph induced by A and $G[A, B]$ for the induced bipartite subgraph with parts A and B. Let

$$N_G(x) = \{y \in V(G) : xy \in E(G)\}$$

be the *neighborhood* of a vertex x in G. Let $K(V_1, \ldots, V_l)$ denote the complete l-partite graph with parts $V_1, \ldots, V_l$.

It will be often convenient to identify graphs with their edge sets. Thus, for example, $|G| = e(G)$ denotes the number of edges while $G \triangle H$ is the graph on $V(G) \cup V(H)$ whose edge set is the symmetric difference of $E(G)$ and $E(H)$.

As we make use of a multicolor version of Szemerédi's Regularity Lemma [**14**], we remind the reader of the following definitions. Let G be a graph and A, B be two disjoint non-empty subsets of $V(G)$. The *edge density* between A and B is

$$d(A, B) = \frac{e(G[A, B])}{|A|\,|B|}.$$

For $\epsilon > 0$, the pair (A, B) is called *ϵ-regular* if, for every $X \subseteq A$ and $Y \subseteq B$ satisfying $|X| > \epsilon|A|$ and $|Y| > \epsilon|B|$, respectively, we have

$$|d(X, Y) - d(A, B)| < \epsilon.$$

An *equitable partition* of a set V is a partition of V into pairwise disjoint parts $V_1, \ldots, V_m$ of almost equal size, that is, $|\,|V_i| - |V_j|\,| \leqslant 1$ for all $i, j \in [m]$. An equitable partition of the set of vertices of G into parts $V_1, \ldots, V_m$ is called *ϵ-regular* if $|V_i| \leqslant \epsilon|V|$ for every $i \in [m]$ and all, but at most $\epsilon\binom{m}{2}$ of the pairs (V_i, V_j), $1 \leqslant i < j \leqslant m$, are ϵ-regular.

The following more general result can be deduced from the original Regularity Lemma of Szemerédi [**14**] (cf. [**8**, Theorems 1.8 and 1.18]).

Lemma 2.1 (Multicolor Regularity Lemma). *For every $\epsilon > 0$ and integer $r \geqslant 1$, there is $M = M(\epsilon, r)$ such that, for any graph G on $n > M$ vertices and any (not necessarily proper) edge r-coloring $\chi : E(G) \to [r]$, there is an equitable partition $V(G) = V_1 \cup \ldots \cup V_m$ with $1/\epsilon \leqslant m \leqslant M$, which is ϵ-regular simultaneously with respect to all graphs $(V(G), \chi^{-1}(i))$, $i \in [r]$.*

Also, we need the following special case of the Embedding Lemma (see, for example, [**8**, Theorem 2.1]).

Lemma 2.2 (Embedding Lemma). *For every $\eta > 0$ and integer $k \geqslant 2$ there exists $\epsilon > 0$ such that the following holds for all large n. Suppose that G is a graph of order n with an equitable* *partition $V(G) = V_1 \cup \ldots \cup V_k$ such that every pair (V_i, V_j) for $1 \leqslant i < j \leqslant k$ is ϵ-regular of density at least η. Then G contains K_k.*

While we have $t_k(n) = (1 - 1/k + o(1))\binom{n}{2}$ for $n \to \infty$, the following easy bound holds for all $k, n \geqslant 1$:

$$\max\{e(G) : v(G) = n,\ G \text{ is } k\text{-partite}\} = t_k(n) \leqslant \left(1 - \frac{1}{k}\right)\frac{n^2}{2}. \tag{2.1}$$

We will also use the following stability result for the Turán function (1.1).

LEMMA 2.3 (Erdős [**4**] and Simonovits [**12**]). *For every $\alpha > 0$ and integer $k \geqslant 1$, there exist $\beta > 0$ and n_0 such that, for all $n > n_0$, any K_{k+1}-free graph G on n vertices with at least $(1 - 1/k)n^2/2 - \beta n^2$ edges admits an equitable partition $V(G) = V_1 \cup \ldots \cup V_k$ with $|G \triangle K(V_1, \ldots, V_k)| < \alpha n^2$.*

3. *Proof of Theorem* 1.1

In this section, we prove Theorem 1.1. Here we have to overcome many new difficulties that are not present for two or three colors. So, unfortunately, the proof is long and complicated. In order to improve its readability, we split it into a sequence of lemmas. Since we use the Regularity Lemma, the obtained value for N in Theorem 1.1 is very large and is of little practical value. Therefore, we make no attempt to determine or optimize it.

First, let us state some important definitions that are extensively used in the whole proof. Fix positive constants

$$c_0 \gg c_1 \gg \ldots \gg c_{10},$$

each being sufficiently small depending on the previous ones. Let $M = 1/c_9$ and $n_0 = 1/c_{10}$.

Typically, the order of a graph under consideration is denoted by n and will satisfy $n \geqslant n_0$. We view n as tending to infinity with $c_0, \ldots, c_9$ being fixed and use the asymptotic terminology (such as, for example, the expression $O(1)$ or the phrase 'almost every') accordingly.

Let $\mathcal{G}_n$ consist of graphs of order n that have many K_3-free edge 4-colorings. Specifically,

$$\mathcal{G}_n = \{G : v(G) = n,\ F(G, 4, 3) \geqslant 18^{n^2/8} \cdot 2^{-c_8 n^2}\}.$$

Let $\mathcal{G} = \bigcup_{n \geqslant n_0} \mathcal{G}_n$. The lower bound in (1.3) (whose proof we sketched in Section 1) shows that $\mathcal{G}_n$ is non-empty for each $n \geqslant n_0$.

Next, for an arbitrary graph G with $n \geqslant n_0$ vertices and a K_3-free 4-coloring χ of the edges of G, we define the following objects and parameters. As the constants c_8 and M satisfy Lemma 2.1 (namely, we can assume that M is at least the function $M(c_8, 4)$ returned by Lemma 2.1), we can find a partition $V(G) = V_1 \cup \ldots \cup V_m$ with $1/c_8 \leqslant m \leqslant M$ that is c_8-regular with respect to each color. Next, we define the *cluster graphs* H_1, H_2, H_3 and H_4 on vertex set $[m]$, where H_ℓ consists of those pairs $ij \in \binom{[m]}{2}$ such that the pair (V_i, V_j) is c_8-regular and has edge density at least c_7 with respect to the ℓ-color subgraph $\chi^{-1}(\ell)$ of G. For $1 \leqslant s \leqslant 4$, let R_s be the graph on vertex set $[m]$ where $ab \in E(R_s)$ if and only if $ab \in E(H_\ell)$ for exactly s values of $\ell \in [4]$. Let $R = \bigcup_{s=1}^{4} R_s$ be the union of the graphs R_s. Let $r_s = 2e(R_s)/m^2$.

We view m, V_i, H_i, R_i, R, r_i as functions of the pair (G, χ). Although we may have some freedom when choosing the c_8-regular partition $V_1, \ldots, V_m$, we fix just one choice for each input (G, χ). We do not require any 'continuity' property from these functions: for example, it may be possible that χ_1 and χ_2 are two colorings of the same graph G that differ on one edge only, but $r_i(G, \chi_1)$ and $r_i(G, \chi_2)$ are quite far apart.

By Lemma 2.2, each cluster graph H_i is triangle-free and, by Turán's theorem (1.1), has at most $t_2(m)$ edges. By (2.1),

$$r_1 + 2r_2 + 3r_3 + 4r_4 = \frac{e(H_1) + e(H_2) + e(H_3) + e(H_4)}{m^2/2} \leqslant 2. \tag{3.1}$$

In addition, note that $R_3 \cup R_4$ is triangle-free because a triangle in $R_3 \cup R_4$ gives a triangle in some H_i. Therefore, by (1.1) and (2.1),

$$r_3 + r_4 \leqslant 1/2. \tag{3.2}$$

We also need the following 'converse' procedure for generating all K_3-free edge 4-colorings of G. Our upper bounds on $F(G, 4, 3)$ and some structural information about typical colorings are obtained by estimating the possible number of outputs. Since the parameters $r_1, \ldots, r_4$ play a crucial role in these estimates, some guesses of the functions m, V_i and H_i (and thus of R_i, R and r_i) are also generated. The procedure is rather wasteful in the sense that it can generate a lot of 'garbage'. But the obtained inequalities (3.1) and (3.2) imply the crucial property that every K_3-free edge 4-coloring of G with the correct guess of m, V_i and H_i is generated at least once provided $v(G) \geqslant n_0$.

The coloring procedure

(1) Choose an arbitrary integer m' between $1/c_8$ and M.
(2) Choose an arbitrary equitable partition $V(G) = V_1' \cup \ldots \cup V_{m'}'$.
(3) Choose arbitrary graphs $H_1', \ldots, H_4'$ with vertex set $[m']$ such that we have

$$r_1' + 2r_2' + 3r_3' + 4r_4' \leqslant 2, \tag{3.3}$$
$$r_3' + r_4' \leqslant 1/2, \tag{3.4}$$

where R_i' and r_i' are defined by the direct analogy with R_i and r_i. (For example, for $i \in [4]$, R_i' is the graph on $[m']$ whose edges are those pairs of $\binom{[m']}{2}$ that are edges in exactly i graphs $H_1', \ldots, H_4'$.)

(4) Assign arbitrary colors to all edges of G that lie inside some part V_i'.

(5) Select at most $4c_8\binom{m'}{2}$ elements of $\binom{[m']}{2}$ and, for each selected pair ij, assign colors to $G[V_i', V_j']$ arbitrarily.

(6) For every color $l \in [4]$ and every $ij \in \binom{[m']}{2}$ color an arbitrary subset of edges of $G[V_i', V_j']$ of size at most $c_7|V_i'|\,|V_j'|$ by color l.

(7) For every edge xy of G that is not colored yet, let us say $x \in V_i'$ and $y \in V_j'$, pick an arbitrary color from the set $C_{ij} = \{s \in [4] : ij \in H_s'\}$. If $C_{ij} = \emptyset$, then we color xy with color 1.

Lemma 3.1. *For every graph G of order $n \geqslant n_0$, the number of choices in Steps (1)–(6) of the Coloring Procedure is at most $2^{c_6 n^2}$.*

Proof. Clearly, the number of choices in Steps (1)–(3) is at most

$$M \cdot n^M \cdot (2^{\binom{M}{2}})^4 = 2^{O(\log n)}. \tag{3.5}$$

Fix these choices. Since $m' \geqslant 1/c_8$, the number of edges that lie inside some part V_i' is at most $m'\binom{\lceil n/m' \rceil}{2} \leqslant c_6 n^2/8$; so the number of choices in Step (4) is at most $4^{c_6 n^2/8}$. In Step (5), we have at most $2^{\binom{m'}{2}} \cdot 4^{4c_8\binom{m'}{2}\lceil n/m' \rceil^2} < 2^{c_6 n^2/4}$ options. The number of choices in Step (6) is at most

$$\binom{\lceil n/m' \rceil^2}{\leqslant c_7 \lceil n/m' \rceil^2}^{4\binom{m'}{2}} < 2^{c_6 n^2/4}.$$

By multiplying these four bounds, we obtain the required. □

The number of options in Step (7) can be bounded from above by

$$(2^{e(R'_2)} \cdot 3^{e(R'_3)} \cdot 4^{e(R'_4)})^{\lceil n/m' \rceil^2} \leqslant (2^{r'_2} \cdot 3^{r'_3} \cdot 4^{r'_4})^{n^2/2+O(n)} = 2^{Ln^2/2+O(n)}, \tag{3.6}$$

where $L = r'_2 + \log_2(3) r'_3 + 2r'_4$. One can easily show that the maximum of L, given (3.3) and (3.4) (and the non-negativity of $r'_1, \ldots, r'_4$), is $(\log_2 18)/4$, with the (unique) optimal assignment being $r'_1 = r'_4 = 0$, $r'_2 = 1/4$, and $r'_3 = 1/2$. When combined with Lemma 3.1, this allows one to conclude that, for example,

$$F(n, 4, 3) \leqslant 18^{n^2/8} \cdot 2^{2c_6 n^2}, \quad \text{for all } n \geqslant n_0. \tag{3.7}$$

This is essentially the argument from [**1**]. We need to take this argument further. As the first step, we derive some information about r_2 and r_3 for a typical coloring χ. We call a pair (G, χ) (or the coloring χ) *satisfactory* if

$$r_2 > 1/4 - c_5/2 \quad \text{and} \quad r_3 > 1/2 - c_5. \tag{3.8}$$

Otherwise, (G, χ) is *unsatisfactory*. Next, we establish some results about satisfactory colorings. Later, this will allow us to define two other important properties of colorings (namely, being *good* and being *perfect*).

LEMMA 3.2. *For every graph G with $n \geqslant n_0$ vertices the number of unsatisfactory K_3-free edge 4-colorings is less than $18^{n^2/8} \cdot 2^{-c_6 n^2}$. In particular, if $G \in \mathcal{G}_n$, then almost every coloring is satisfactory.*

Proof. We use the Coloring Procedure and bound from above the number of outputs that give unsatisfactory colorings. By Lemma 3.1, the number of choices in Steps (1)–(6) is at most $2^{c_6 n^2}$. We use (3.6) to estimate the number of choices in Step (7).

The value of L under constraints (3.3), (3.4) and

$$r'_3 \leqslant 1/2 - c_5 \tag{3.9}$$

(as well as the non-negativity of the variables r'_i) is at most

$$L_{\max} = (1/4 + 3c_5/2) + (1/2 - c_5)\log_2 3 < (1/4 - c_5^2)\log_2 18.$$

This can be seen by multiplying (3.3), (3.4) and (3.9) by, respectively, $y_1 = 1/2$, $y_2 = 0$ and $y_3 = \log_2 3 - 3/2 > 0$, and adding these inequalities. The obtained inequality has $L_{\max}$ in the right-hand side while each coefficient of the left-hand side is at least the corresponding coefficient of L, giving the required bound. (In fact, these reals y_i are the optimal dual variables for the linear program of maximizing L.)

Likewise, when we maximize L under constraints (3.3), (3.4) and

$$r'_2 \leqslant 1/4 - c_5/2, \tag{3.10}$$

then we have the same upper bound $L_{\max}$ (with the optimal dual variables for (3.3), (3.4) and (3.10) being, respectively, $y_1 = 2 - \log_2 3 > 0$, $y_2 = 4\log_2 3 - 6 > 0$ and $y_3 = 2\log_2 3 - 3 > 0$). Since in Step (7) we have only two (possibly overlapping) cases depending on which of (3.10) or (3.9) holds, the total number of choices in Step (7) is by (3.6) at most

$$2 \cdot 2^{L_{\max} n^2/2 + O(n)} < 18^{(1/8 - c_5^2/3)n^2}.$$

By multiplying this by $2^{c_6 n^2}$, we obtain the required upper bound on the number of unsatisfactory colorings. □

For each satisfactory coloring of $G \in \mathcal{G}$, we record the vector $\nu(\chi) = (m, V_i, H_i)$ of parameters. Call a vector (m, V_i, H_i) *popular* if

$$|\nu^{-1}((m, V_i, H_i))| \geqslant 18^{n^2/8} \cdot 2^{-3c_8 n^2},$$

that is, if it appears for at least $18^{n^2/8} \cdot 2^{-3c_8 n^2}$ satisfactory colorings, where $n = v(G)$. As the number of possible choices of vectors is bounded by (3.5), the number of satisfactory colorings for which the corresponding vector is not popular is at most

$$2^{O(\log n)} \cdot 18^{n^2/8} \cdot 2^{-3c_8 n^2} \leqslant 18^{n^2/8} \cdot 2^{-2c_8 n^2},$$

that is, $o(1)$-fraction of all colorings. Let $\mathrm{Pop}(G)$ be the set of all popular vectors and let

$$\mathcal{S}(G) = \nu^{-1}(\mathrm{Pop}(G)) \tag{3.11}$$

be the set of satisfactory K_3-free edge 4-colorings of G for which the corresponding vector is popular. By Lemma 3.2, $\mathcal{S}(G)$ is non-empty.

Our next goal is to exhibit a stability property, namely, that every graph $G \in \mathcal{G}$ is almost complete 4-partite. First, we show that, for every popular vector $(m, V_i, H_i) \in \mathrm{Pop}(G)$, the cluster graph R is almost complete 4-partite. Then we extend this result to G.

LEMMA 3.3. *Let $n \geqslant n_0$, $G \in \mathcal{G}_n$ and $(m, V_i, H_i) \in \mathrm{Pop}(G)$. Then there exist equitable partitions $[m] = A \cup B$, $A = U_1 \cup U_2$ and $B = U_3 \cup U_4$ such that*

$$|R_3 \triangle K(A, B)| < c_4 m^2, \tag{3.12}$$

$$|R_2[A] \triangle K(U_1, U_2)| < 2c_3 m^2, \tag{3.13}$$

$$|R_2[B] \triangle K(U_3, U_4)| < 2c_3 m^2, \tag{3.14}$$

$$|R \triangle K(U_1, U_2, U_3, U_4)| < 5c_3 m^2. \tag{3.15}$$

Proof. We have already proved that R_3 is triangle-free. As (m, V_i, H_i) is associated with a satisfactory coloring, (3.8) is satisfied; in particular, $r_3 > 1/2 - c_5$. Therefore, $e(R_3) = r_3 m^2/2 > t_2(m) - c_5 m^2/2$. As $c_5 \ll c_4$, we can apply Lemma 2.3 to partition $V(R_3) = [m]$ into two sets A and B such that $|A| = \lfloor m/2 \rfloor$, $|B| = \lceil m/2 \rceil$ and (3.12) holds.

Since $R_2 \cap R_3 = \emptyset$, we have $|R_2 \cap K(A, B)| \leqslant |K(A, B) \setminus R_3| < c_4 m^2$. This and (3.8) imply that

$$e(R_2[A]) + e(R_2[B]) > e(R_2) - c_4 m^2 = r_2 m^2/2 - c_4 m^2 > m^2/8 - 2c_4 m^2. \tag{3.16}$$

What we show in the following sequence of claims is that $R_2[A]$ and $R_2[B]$ are both close to being triangle-free and have roughly $m^2/16$ edges each; then we can apply Lemma 2.3 to these graphs, obtaining the desired partitions of A and B.

For a vertex $a \in A$, let $B_a = N_{R_3}(a) \cap B$ be the set of R_3-neighbors of a that lie in B. Similarly, for a vertex $b \in B$, let $A_b = N_{R_3}(b) \cap A$.

CLAIM 3.4. *For every $a \in A$ the graph $R_2[B_a]$ is 4-partite.*

Proof of Claim 3.4. Each pair ab with $b \in B_a$ is contained in R_3 and, by definition, is labeled with a 3-element subset X_b of $[4]$. Color b by the unique element of $[4] \setminus X_b$. If two adjacent vertices b and b' of $R_2[B_a]$ receive identical color c, then the label of $bb' \in R_2$ (a 2-element subset of $[4]$) has a non-empty intersection with $[4] \setminus \{c\}$, which is the label of both $ab, ab' \in R_3$. This implies the existence of a triangle in some H_i, which is a contradiction. □

CLAIM 3.5. *If $a_1a_2 \in E(R_2[A])$, then $K_3 \not\subseteq R_2[B_{a_1} \cap B_{a_2}]$.*

Proof of Claim 3.5. Suppose, on the contrary, that we have an edge a_1a_2 in $R_2[A]$ and a triangle in $R_2[B_{a_1} \cap B_{a_2}]$ with vertices b_1, b_2 and b_3. Let S be the multiset produced by the union of the labels of the edges a_1a_2, a_ib_j and b_ib_j. As each edge a_ib_j is labeled with a 3-element subset of [4] and the remaining four edges are labeled with a 2-element subset of [4], we have $|S| = 6 \cdot 3 + 4 \cdot 2 = 26$. By the Pigeonhole Principle, some member of [4] belongs to S with multiplicity at least 7. But this corresponds to some H_i having at least seven edges among the five vertices a_1, a_2, b_1, b_2, b_3. By Turán's result (1.1), this implies that H_i has a triangle, which is a contradiction. □

Define

$$B' = \{b \in B : |A_b| > |A| - \sqrt{c_4}m\}.$$

As each vertex of $B \setminus B'$ contributes at least $\sqrt{c_4}m$ to $|K(A,B) \setminus R_3|$, there are less than $\sqrt{c_4}m$ such vertices by (3.12). Thus, $|B'| > |B| - \sqrt{c_4}m \geqslant (1/2 - \sqrt{c_4})m$. Similarly, we can define A' to be the set of vertices $a \in A$ for which $|B_a| > |B| - \sqrt{c_4}m$ and note that $|A'| > |A| - \sqrt{c_4}m > 0$.

CLAIM 3.6. *The graph $R_2[B']$ is triangle-free.*

Proof of Claim 3.6. Suppose, on the contrary, that b_1, b_2, b_3 form a K_3 in $R_2[B']$. Let $X = A_{b_1} \cap A_{b_2} \cap A_{b_3}$. By definition, $|A \setminus A_{b_i}| < \sqrt{c_4}m$. So $|X| > |A| - 3\sqrt{c_4}m$. By Claim 3.5, there are no edges within X. So $e(R_2[A]) \leqslant |A \setminus X| \cdot |A| < 3\sqrt{c_4}m^2$.

Let us estimate $e(R_2[B])$ from above. Consider B_a for some $a \in A'$. By definition $|B_a| > |B| - \sqrt{c_4}m$ and, by Claim 3.4, B_a is 4-partite. By (2.1) the number of edges in $R_2[B]$ is at most $(3/4)|B_a|^2/2 + \sqrt{c_4}m|B|$.

However, these upper bounds on $e(R_2[A])$ and $e(R_2[B])$ contradict (3.16). □

In particular, $R_2[B]$ may be made triangle-free by the removal of at most $|B \setminus B'| \cdot |B| < \sqrt{c_4}m^2$ edges. Hence, we have in fact that

$$e(R_2[B]) < (1 - \tfrac{1}{2})|B|^2/2 + \sqrt{c_4}m^2 \leqslant m^2/16 + 2\sqrt{c_4}m^2. \tag{3.17}$$

By (3.16) and (3.17), $e(R_2[A]) > m^2/16 - 2c_4m^2 - 2\sqrt{c_4}m^2$. As above, by removing at most $\sqrt{c_4}m^2$ edges, we can form a graph R_2' on vertex set A, which is triangle-free. We can now apply Lemma 2.3 to R_2', to find a partition $A = U_1 \cup U_2$ such that $|R_2' \triangle K(U_1, U_2)| < c_3m^2$. As R_2' and $R_2[A]$ differ in at most $\sqrt{c_4}m^2$ edges, we derive (3.13). The existence of an equitable partition $B = U_3 \cup U_4$ satisfying (3.14) is proved similarly.

By (3.12)–(3.14), we have $|(R_2 \cup R_3) \triangle K(U_1, U_2, U_3, U_4)| < 4c_3m^2 + c_4m^2$. Also, by (3.1) and (3.8), we have $r_1 + r_4 \leqslant 4c_5$ and $|R_1 \cup R_4| \leqslant 2c_5m^2$. Now (3.15) follows, completing the proof of Lemma 3.3. □

For a graph $G \in \mathcal{G}$ and a popular vector $(m, V_i, H_i) \in \mathrm{Pop}(G)$, fix the sets $A, B, U_1, \ldots, U_4$ given by Lemma 3.3. For $i \in [4]$, let $\tilde{U_i} = \bigcup_{j \in U_i} V_j$ be the *blow-up* of U_i. Let $\tilde{F} = K(\tilde{U_1}, \tilde{U_2}, \tilde{U_3}, \tilde{U_4})$.

LEMMA 3.7. *For every $n \geqslant n_0$, $G \in \mathcal{G}_n$ and $(m, V_i, H_i) \in \mathrm{Pop}(G)$, we have $|G \triangle \tilde{F}| < 12c_3n^2$.*

Proof. It routinely follows that the size of $G \setminus \tilde{F}$ is at most the sum of the following terms:

(i) $m\binom{\lceil n/m \rceil}{2}$, the number of edges of G inside parts V_i;

(ii) $4c_8\binom{\lceil m \rceil}{2} \cdot \lceil n/m \rceil^2$, edges between parts that are not c_8-regular for at least one color graph;

(iii) $4c_7\binom{n}{2}$, edges between parts of density at most c_7 for at least one color; and

(iv) $|R \setminus K(U_1, U_2, U_3, U_4)| \cdot \lceil n/m \rceil^2 \leqslant 5c_3m^2 \cdot \lceil n/m \rceil^2$, where we used (3.15).

Adding up, this gives less than $6c_3n^2$.

Next, we estimate $|\tilde{F} \setminus G|$ by bounding the number of satisfactory colorings of G that give our fixed vector (m, V_i, H_i). Again, we use the Coloring Procedure to generate all such colorings, where m, V_i, H_i are fixed in advance. By Lemma 3.1, we have at most $2^{c_6n^2}$ options in Steps (4)–(6). Once we have fixed the choices in these steps, the remaining uncolored edges of G are restricted to those between the parts while the graphs $R_1, \dots, R_4$ specify how many choices of color each edge has. Thus, the number of options in Step (7) is at most

$$\prod_{f=2}^{4} \prod_{ij \in R_f} f^{\lceil n/m \rceil^2 - |K(V_i,V_j) \setminus G|} \leqslant (2^{2c_6n^2} \cdot 18^{n^2/8}) \prod_{ij \in R_2 \cup R_3} 2^{-|K(V_i,V_j) \setminus G|}, \tag{3.18}$$

where we used the bound in (3.7) together with the maximization result mentioned immediately after (3.6). Let us look at the last factor in (3.18). If we replace the range of ij in the product by $K(U_1, U_2, U_3, U_4)$ instead of $R_2 \cup R_3$, this will affect at most $(c_4 + 4c_3)m^2$ pairs ij by (3.12)–(3.14) and we get an extra factor of at most $2^{5c_3n^2}$. Thus,

$$\prod_{ij \in R_2 \cup R_3} 2^{-|K(V_i,V_j) \setminus G|} \leqslant 2^{-|\tilde{F} \setminus G|} \cdot 2^{5c_3n^2}.$$

Since the vector (m, V_i, H_i) is popular, we conclude that

$$|\tilde{F} \setminus G| \leqslant c_6n^2 + 5c_3n^2 + 2c_6n^2 + 3c_8n^2 \leqslant 6c_3n^2,$$

giving the required bound on $|G \triangle \tilde{F}|$. □

Now, for every input graph G we fix a *max-cut* 4-partition $V(G) = W_1 \cup W_2 \cup W_3 \cup W_4$, that is, one that maximizes the number of edges of G across the parts.

LEMMA 3.8 (Stability Property). *Let $n \geqslant n_0$, $G \in \mathcal{G}_n$ and $W_1' \cup W_2' \cup W_3' \cup W_4'$ be a partition of $V(G)$ with*

$$|G \cap K(W_1', W_2', W_3', W_4')| \geqslant |G \cap K(W_1, W_2, W_3, W_4)| - c_3n^2.$$

Then we have

$$|G \triangle K(W_1', W_2', W_3', W_4')| \leqslant 15c_3n^2, \tag{3.19}$$

and, for every popular vector $(m, V_i, H_i) \in \mathrm{Pop}(G)$, there is a relabeling of $W_1', \dots, W_4'$ such that, for each $i \in [4]$,

$$|W_i' \triangle \tilde{U}_i| \leqslant 2000c_3n. \tag{3.20}$$

Also, we have $|\,|W_i| - n/4| \leqslant c_2n$ for each $i \in [4]$ and $|G \triangle K(W_1, W_2, W_3, W_4)| \leqslant 15c_3n^2$.

Proof. Let $F' = K(W_1', W_2', W_3', W_4')$ and $F = K(W_1, W_2, W_3, W_4)$. As the max-cut partition $W_1 \cup \ldots \cup W_4$ maximizes the number of edges across parts, we have $|F' \cap G| + c_3n^2 \geqslant |F \cap G| \geqslant |\tilde{F} \cap G|$. Since the partitions $[m] = U_1 \cup \ldots \cup U_4$ and $[n] = V_1 \cup \ldots \cup V_m$ are equitable, we have

$$|\,|\tilde{U}_i| - n/4| \leqslant m + n/m. \tag{3.21}$$

Thus, we have $|\tilde{F}| \geqslant |F'| - c_7 n^2$ and, by Lemma 3.7,

$$\begin{aligned} |F' \triangle G| &= |F'| + |G| - 2|F' \cap G| \\ &\leqslant (|\tilde{F}| + c_7 n^2) + |G| - 2(|\tilde{F} \cap G| - c_3 n^2) \\ &= |\tilde{F} \triangle G| + c_7 n^2 + 2c_3 n^2 \leqslant 15 c_3 n^2, \end{aligned} \tag{3.22}$$

proving the first part of the lemma.

We look for a relabeling of $W_1', \ldots, W_4'$ such that $|\tilde{U}_i \setminus W_i'| < 500 c_3 n$ for each $i \in [4]$. Suppose that no such relabeling exists. Then, since $c_3 \ll 1$ and, for example, each $|W_i'| \leqslant n/3$, there is $i \in [4]$ such that, for every $j \in [4]$, we have that $|\tilde{U}_i \setminus W_j'| \geqslant 500 c_3 n$. Take $j \in [4]$ such that $|\tilde{U}_i \cap W_j'| \geqslant |\tilde{U}_i|/4$ and let $X = \tilde{U}_i \cap W_j'$ and $Y = \tilde{U}_i \setminus W_j'$. However, $X, Y \subseteq \tilde{U}_i$ and Lemma 3.7 imply that $e(G[X,Y]) < 12 c_3 n^2$, whereas $X \subseteq W_j'$, $Y \cap W_j' = \emptyset$, (3.21) and (3.22) imply that

$$e(G[X,Y]) \geqslant |X|\,|Y| - 15 c_3 n^2 \geqslant (n/16 - c_7 n) \cdot 500 c_3 n - 15 c_3 n^2 > 12 c_3 n^2,$$

which is a contradiction. So take the stated relabeling. Now (3.20) follows from the observation that

$$W_i' \setminus \tilde{U}_i \subseteq \bigcup_{j \in [4] \setminus \{i\}} (\tilde{U}_j \setminus W_j').$$

Alternatively, one could use Lemma 3.7 that G is $12 c_3 n^2$-close to the complete 4-partite graph $\tilde{F}$ whose part sizes are close to $n/4$ by (3.21). One would get a weaker upper bound on $|W_i' \triangle \tilde{U}_i|$ (of order $\sqrt{c_3} n$) but which would also be sufficient for our proof.

Finally, the last two claims of Lemma 3.8 can be derived by taking $W_i' = W_i$ for $i \in [4]$ (and using (3.21)). □

Define a *pattern* as an assignment $\pi : \binom{[4]}{2} \to \binom{[4]}{2} \cup \binom{[4]}{3}$ (to every edge of K_4 we assign a list of two or three colors) such that $\pi^{-1}(c)$ forms a 4-cycle for every color $c \in [4]$. Up to isomorphism (of colors and vertices) there is only one pattern. We say that an edge 4-coloring χ of $G \in \mathcal{G}_n$ *follows the pattern* π if, for every $ij \in \binom{[4]}{2}$, we have

$$|\chi^{-1}([4] \setminus \pi(ij)) \cap G[W_i, W_j]| \leqslant c_2 n^2,$$

that is, at most $c_2 n^2$ edges of $G[W_i, W_i]$ get a color not in $\pi(ij)$.

Recall that the set $\mathcal{S}(G)$ consists of all satisfactory colorings whose associated vector is popular.

Lemma 3.9. *For every graph $G \in \mathcal{G}_n$ with $n \geqslant n_0$, every coloring $\chi \in \mathcal{S}(G)$ follows a pattern.*

Proof. Take any $\chi \in \mathcal{S}(G)$. Recall that $A, B, U_1, \ldots, U_4$ are the sets given by Lemma 3.3. Let

$$R' = (R_3 \cap K(A,B)) \cup (R_2 \cap K(U_1,U_2)) \cup (R_2 \cap K(U_3,U_4)).$$

Let the *label* of an edge uv in R be $\hat{\chi}(uv) = \{i \in [4] : uv \in E(H_i)\}$. So, for all edges $u_i u_j \in R'$ across $U_i \times U_j$, we have

$$|\hat{\chi}(u_i u_j)| = \begin{cases} 2, & \text{if } \{i,j\} \in \{\{1,2\},\{3,4\}\}, \\ 3, & \text{otherwise.} \end{cases} \tag{3.23}$$

We show next that $\hat{\chi}$ has a very simple structure: with the exception of a small fraction of edges, $\hat{\chi}$ behaves as the blow-up of some labeling on K_4. Furthermore, the latter labeling is isomorphic to some pattern π, as defined above.

CLAIM 3.10. *Let the sets* $\{v_1, v_2, v_3, v_4\}$ *and* $\{w, v_2, v_3, v_4\}$ *both span a* K_4*-subgraph in* R'*, where* $w \in U_1$ *and each* $v_i \in U_i$*. Then* $\hat{\chi}(v_1v_i) = \hat{\chi}(wv_i)$ *for all* $i \in \{2, 3, 4\}$*.*

Proof of Claim 3.10. First consider the restriction of $\hat{\chi}$ to $X = \{v_1, v_2, v_3, v_4\}$. Let S be the multiset produced by the union of $\hat{\chi}(v_iv_j)$, $1 \leqslant i < j \leqslant 4$. So $|S| = 2 \cdot 2 + 4 \cdot 3 = 16$. As each $H_t[X]$ is triangle-free, it follows by the uniqueness of the Turán graph that $\hat{\chi}^{-1}(t)$ forms a 4-cycle on X for each $t \in [4]$. When taking (3.23) into consideration, we see that there is only one possible configuration (up to isomorphism). A nice property of this configuration is that $\hat{\chi}(v_iv_j) = \hat{\chi}(v_kv_\ell)$ whenever $\{i, j, k, \ell\} = [4]$, that is, edges that form a matching on X receive identical labels. As $\{w, v_2, v_3, v_4\}$ also spans a copy of K_4, we have $\hat{\chi}(wv_j) = \hat{\chi}(v_kv_\ell) = \hat{\chi}(v_1v_j)$, where $\{j, k, \ell\} = \{2, 3, 4\}$, proving the claim. □

Now choose $X = \{v_1, v_2, v_3, v_4\}$, where $v_i \in U_i$, such that $R'[X] \cong K_4$ and, for each vertex $v_i \in X$, we have

$$|N_{R'}(v_i) \cap U_j| > |U_j| - 2\sqrt{c_3}m \quad \text{for all } j \in [4] \setminus \{i\}. \tag{3.24}$$

We may build such a set iteratively by picking $v_1 \in U_1$ satisfying (3.24), then $v_2 \in U_2 \cap N(v_1)$ satisfying (3.24), and so on. Such vertices exist as at most $2c_3m^2$ edges across a pair U_i, U_j are missing from R'. In fact, the number of vertices $u \in U_i$ that fail condition (3.24) is less than $3\sqrt{c_3}m$.

Let $A_i \subseteq U_i$ consist of those vertices that lie in $N_{R'}(v_j)$ for all $v_j \in X$ with $j \in [4] \setminus \{i\}$. As all vertices v_j satisfy (3.24), we have $|A_i| > |U_i| - 6\sqrt{c_3}m$. If $a_ia_j \in R'[A_i, A_j]$, then all three sets $X, \{a_i, v_j, v_k, v_\ell\}$, and $\{a_i, a_j, v_k, v_\ell\}$ form 4-cliques in R', where $\{i, j, k, \ell\} = [4]$. By Claim 3.10 we have that $\hat{\chi}(v_iv_j) = \hat{\chi}(a_iv_j) = \hat{\chi}(a_ia_j)$. Thus, the labeling on X determines the labeling on all edges of R' with the possible exception of at most $m \cdot 24\sqrt{c_3}m$ edges incident to vertices of $\bigcup_{i=1}^4 (U_i \setminus A_i)$. As $|R \setminus R'| < 5c_3m^2$, we have a pattern π such that $\hat{\chi}(u_iu_j) = \pi(ij)$ for all, but at most $25\sqrt{c_3}m^2$ edges in R.

Now Lemma 3.8 implies that, for some relabeling of $W_1, \ldots, W_4$, we have

$$|K(W_1, W_2, W_3, W_4) \setminus K(\tilde{U_1}, \tilde{U_2}, \tilde{U_3}, \tilde{U_4})| < 4n \cdot 2000c_3n.$$

Then, including at most $5c_7n^2$ edges that disappear without a trace in any H_i during the application of the Regularity Lemma and at most $12c_3n^2$ edges lost in Lemma 3.7, we have that $\chi(w_iw_j) \in \pi(ij)$ for all, but at most

$$5c_7n^2 + 12c_3n^2 + 25\sqrt{c_3}m^2 \cdot \lceil n/m \rceil^2 + 8000c_3n^2 < c_2n^2$$

edges w_iw_j in $G[W_i, W_j]$, proving the lemma. □

Since c_2 is small, Lemma 3.8 implies that the pattern π in Lemma 3.9 is unique. This allows us to make the following definition. A coloring $\chi \in \mathcal{S}(G)$ of a graph $G \in \mathcal{G}_n$ is *good* if, for every $ij \in \binom{[4]}{2}$, all subsets $X_i \subseteq W_i$ and $X_j \subseteq W_j$ with $|X_i| \geqslant c_1n$ and $|X_j| \geqslant c_1n$ and every color $c \in \pi(ij)$, there is at least one edge xy in $G[X_i, X_j]$ with $\chi(xy) = c$, where π is the pattern of χ. Otherwise $\chi \in \mathcal{S}(G)$ is called *bad*.

LEMMA 3.11. *The number of bad colorings of any* $G \in \mathcal{G}_n$*,* $n \geqslant n_0$*, is at most* $18^{n^2/8} \cdot 2^{-c_1^2n^2/8}$*.*

Proof. The following procedure generates each bad coloring of G at least once.

(i) Pick an arbitrary pattern π, a pair $ij \in \binom{[4]}{2}$ and a color $c \in \pi(ij)$.
(ii) Choose up to $6c_2n^2$ edges and color them arbitrarily.

(iii) Pick subsets $X_i \subseteq W_i$ and $X_j \subseteq W_j$ of size $\lceil c_1 n \rceil$ each.
(iv) Color edges inside a part W_i arbitrarily.
(v) Color all edges in $X_i \times X_j$ using the colors from $\pi(ij) \setminus \{c\}$.
(vi) For each $k\ell \in \binom{[4]}{2}$ color all remaining edges of $G[W_k, W_\ell]$ using colors in $\pi(k\ell)$.

The number of choices in Steps (i)–(iii) is bounded from above by

$$O(1)\binom{\binom{n}{2}}{\leqslant 6c_2n^2} 4^{6c_2n^2}\binom{|W_i|}{|X_i|}\binom{|W_j|}{|X_j|} < 2^{c_1^3 n^2}.$$

The number of choices at Step (iv) is at most $4^{15c_3n^2}$ by Lemma 3.8. The number of choices in Steps (v)–(vi) is at most

$$\left(\frac{|\pi(ij)|-1}{|\pi(ij)|}\right)^{|X_i|\,|X_j|} \prod_{k\ell\in\binom{[4]}{2}} |\pi(k\ell)|^{|W_k|\,|W_\ell|} \leqslant (2/3)^{c_1^2n^2}(2^2 3^4)^{n^2/16+c_2n^2},$$

where we used Lemma 3.8. We obtain the required result by multiplying the above bounds. □

Call a good coloring χ of a graph $G \in \mathcal{G}$ *perfect* if $\chi(v_iv_j) \in \pi(ij)$ for every $ij \in \binom{[4]}{2}$ and every edge $v_iv_j \in G[W_i, W_j]$, where π is the pattern of χ. Let $\mathcal{P}(G)$ denote the set of perfect colorings of G.

The following lemma provides a key step of the whole proof.

Lemma 3.12. *Let G be a graph of order $n \geqslant n_0 + 2$ such that $F(G,4,3) \geqslant 18^{n^2/8} \cdot 2^{-c_9n^2}$ and, for every distinct $v, v' \in V(G)$, we have*

$$\frac{F(G,4,3)}{F(G-v,4,3)} \geqslant (18-c_3)^{n/4}, \tag{3.25}$$

$$\frac{F(G,4,3)}{F(G-v-v',4,3)} \geqslant (18-c_3)^{(n+(n-1))/4}. \tag{3.26}$$

Then the following conclusions hold.

(i) *The graph G is 4-partite.*
(ii) *Almost every coloring of G is perfect; specifically,*

$$|\mathcal{P}(G)| \geqslant (1-2^{-c_9n})F(G,4,3).$$

(iii) *If $G \ncong T_4(n)$, then there is a graph G' of order n with $F(G',4,3) > F(G,4,3)$.*

Proof. Since $F(G-v-v',4,3) > F(G-v,4,3)/4^n > F(G,4,3)/16^n$ for any $v, v' \in V(G)$, we have $G-v, G-v-v' \in \mathcal{G}$ and the notion of a good coloring with respect to $G-v$ or $G-v-v'$ is well defined.

Claim 3.13. *For any distinct $v, v' \in V(G)$, there is a good coloring χ of $G-v$ or of $G-v-v'$, respectively, such that the number of ways to extend it to the whole of G is at least $(18-c_2)^{n/4}$ or at least $(18-c_2)^{n/2}$, respectively.*

Proof of Claim 3.13. By Lemma 3.11 the number of bad colorings of $G-v$ is at most $2^{-c_1^2n^2/9}F(G,4,3)$. If the claim fails for all good colorings of $G-v$, then

$$F(G,4,3) \leqslant 4^n \cdot 2^{-c_1^2n^2/9}F(G,4,3) + (18-c_2)^{n/4}F(G-v,4,3),$$

contradicting (3.25). The claim about $G-v-v'$ is proved in an analogous way. □

Claim 3.14. *For all $i \in [4]$ and $v \in W_i$, we have $|N(v) \cap W_i| < 8c_1 n$.*

Proof of Claim 3.14. Suppose, on the contrary, that some vertex v contradicts the claim. Take the good coloring χ of $G - v$ given by Claim 3.13.

For each class W_j (defined with respect to G), let $n_j = |N(v) \cap W_j|$. Note that

$$n_j \leqslant |W_j| \leqslant n/4 + c_2 n, \quad \text{for all } j \in [4], \tag{3.27}$$

by Lemma 3.8. Let $W_1' \cup W_2' \cup W_3' \cup W_4'$ be the selected max-cut partition of $G - v$. As

$$|G \cap K(W_1' \cup \{v\}, W_2', W_3', W_4')| > |G \cap K(W_1, W_2, W_3, W_4)| - n,$$

it follows again from Lemma 3.8 that, after a relabeling of $W_1', \dots, W_4'$, we have

$$|W_i \triangle W_i'| \leqslant 4000c_3 n + 1, \quad \text{for all } i \in [4]. \tag{3.28}$$

Also, let π be the pattern (with respect to $W_1', \dots, W_4'$) associated with the good coloring χ of $G - v$.

For each extension $\bar{\chi}$ of χ to G, record the vector $\mathbf{x}$ whose ith component is the number of colors c such that at least $2c_1 n$ edges of G between v and W_i get color c. Let $\mathbf{x} = (x_1, \dots, x_4)$ be a vector that appears most frequently over all extensions $\bar{\chi}$. Fix some $\bar{\chi}$ that gives this $\mathbf{x}$. For a color c and a class W_j, let

$$Z_{j,c} = \{u \in W_j : \bar{\chi}(uv) = c\}.$$

(Thus, x_j is the number of colors c with $|Z_{j,c}| \geqslant 2c_1 n$.) Analogously, for a color c, let y_c be the number of classes W_j for which $|Z_{j,c}| \geqslant 2c_1 n$. By (3.28) we have $|Z_{j,c} \cap W_j'| > c_1 n$ whenever $|Z_{j,c}| > 2c_1 n$.

Let us show $y_c \leqslant 2$ for each $c \in [4]$. Indeed, if some $y_c \geqslant 3$, then among the three corresponding indices we can find two, say p and q, such that $c \in \pi(pq)$. Since χ is good, there is an edge $uw \in (Z_{p,c} \cap W_p') \times (Z_{q,c} \cap W_q')$ such that $\chi(uw) = c$, giving a $\bar{\chi}$-monochromatic triangle on $\{u, v, w\}$, which is a contradiction. In particular, we have

$$x_1 + x_2 + x_3 + x_4 = y_1 + y_2 + y_3 + y_4 \leqslant 8. \tag{3.29}$$

Since there are at most 5^4 choices of $(x_1, \dots, x_4)$ and we fixed a most frequent vector, the total number of extensions of χ to G is at most

$$5^4 \prod_{j \in [4]} \binom{4}{x_j} \binom{n_j}{\leqslant 2c_1 n}^{4 - x_j} \max(x_j, 1)^{n_j} < 2^{c_0 n} \prod_{\substack{j \in [4] \\ x_j \neq 0}} x_j^{n_j}. \tag{3.30}$$

As $W_1 \cup W_2 \cup W_3 \cup W_4$ is a max-cut partition, we have $|N(v) \cap W_j| \geqslant 8c_1 n$ for all $j \in [4]$. By the pigeonhole principle, we have that $x_j \geqslant 1$ for all $j \in [4]$. This and (3.29) imply that $x_1 x_2 x_3 x_4 \leqslant 16$. By (3.27) and (3.30), the total number of extensions of χ is at most

$$2^{c_0 n} \cdot (x_1 x_2 x_3 x_4)^{n/4} \cdot 4^{4c_2 n} < 2^{2c_0 n} \cdot 16^{n/4} < (18 - c_2)^{n/4},$$

contradicting the choice of χ. □

We now strengthen Claim 3.14 and prove part (i) of the lemma.

Claim 3.15. *For all $i \in [4]$ and distinct $v, v' \in W_i$, we have $vv' \notin E(G)$.*

Proof of Claim 3.15. Suppose on the contrary that the claim fails for some v and v'. Assume without loss of generality that $v, v' \in W_1$.

Let χ be the good coloring of $G - v' - v \in \mathcal{G}_{n-2}$ with at least $(18-c_2)^{n/2}$ extensions to G given by Claim 3.13. Let us recycle the definitions of Claim 3.14 that formally remain unchanged even though χ is undefined on edges incident to v'. On top of them, we define a few more parameters.

Specifically, we look at all extensions $\bar{\chi}$ that give rise to the fixed most frequent vector $\mathbf{x}$. For each such $\bar{\chi}$, we define $Z'_{j,c} = \{u \in W_j : \bar{\chi}(uv') = c\}$ and let x'_j be the number of colors c such that $|Z'_{j,c}| \geqslant 2c_1 n$. Then we fix a most popular vector $\mathbf{x}' = (x'_1, \ldots, x'_4)$ and take any extension $\bar{\chi}$ that gives both $\mathbf{x}$ and $\mathbf{x}'$ and, conditioned on this, such that the color $\bar{\chi}(vv')$ assumes its most frequent value, which we denote by s. We define y_c as before and let y'_c be the number of $j \in [4]$ such that $|Z'_{c,j}| \geqslant 2c_1 n$. This is consistent with the definitions of Claim 3.14 because there we did not have any restriction on $\bar{\chi}$ except that it gives the vector $\mathbf{x}$.

Claim 3.14, the upper bounds on n_i and $n'_i = |N(v') \cap W_i|$ of Lemma 3.8, and the argument leading to (3.30) show that the total number of extensions of χ to G is at most

$$(5^4)^2 \cdot 4 \cdot 2^{c_0 n} \cdot (4^{8c_1 n + 3c_2 n})^2 \cdot \prod_{i=2}^{4} (\max(x_i, 1) \cdot \max(x'_i, 1))^{n/4}. \tag{3.31}$$

If some $|Z_{j,c}| \geqslant 2c_1 n$, but $c \notin \pi(\{1, j\})$, say $j = 3$, then the 4-cycle formed by color c visits indices $1, 2, 3, 4$ in this order and, since χ is good, we have $|Z_{2,c}| < 2c_1 n$ and $|Z_{4,c}| < 2c_1 n$ (otherwise $\bar{\chi}$ contains a color-c triangle via v). Thus, y_c contributes at most 1 to $x_2 + x_3 + x_4$. Since each $y_i \leqslant 2$, we have that $x_2 + x_3 + x_4 \leqslant 7$. It follows that $\prod_{i=2}^4 \max(x_i, 1) \leqslant 12$. Since $x'_2 + x'_3 + x'_4 \leqslant 8$, we have $\prod_{i=2}^4 \max(x'_i, 1) \leqslant 18$. Thus, the expression in (3.31) is at most $2^{2c_0 n} \cdot (12 \cdot 18)^{n/4}$, contradicting the choice of χ.

Thus, $x_i \leqslant |\pi(\{1, i\})|$ for each $i \in \{2, 3, 4\}$ and all these inequalities are in fact equalities (otherwise $\prod_{i=2}^4 \max(x_i, 1) \leqslant 12$, giving a contradiction as before). We conclude, for $j \in \{2, 3, 4\}$, that $|Z_{j,c}| \geqslant 2c_1 n$ if and only if $c \in \pi(\{1, j\})$. The same applies to the parameters x'_i and $Z'_{j,c}$.

Let the special color $s = \bar{\chi}(vv')$ appear in, say, $\pi(\{1, 2\})$ with $|\pi(\{1, 2\})| \geqslant 3$. Then, for all $w \in W_2 \cap N(v) \cap N(v')$, there are at most $x_2 x'_2 - 1$ choices for the colors of vw and vw' when extending χ to G because s cannot occur on both edges. Also, if $w \in W_2 \setminus (N(v) \cap N(v'))$, then trivially there are at most four choices for this vertex w. This allows us to reduce the bound in (3.31) by factor $(8/9)^{n/4}$, giving the desired contradiction. □

Thus, we have proved part (i) of the lemma. Next, we prove part (ii). If it is false, then, by Lemma 3.11, there are at least $(1/2) \cdot 2^{-c_9 n} \cdot F(G, 4, 3)$ colorings of G that are good, but not perfect. For each such coloring there is a *wrong* edge vv' whose color does not conform to the pattern. Pick an edge vv' that appears most frequently this way, say $v \in W_1$ and $v' \in W_4$, and then a most frequent wrong color s of vv'.

By a version of (3.28), it is not hard to show that the number of good colorings χ of $G - v - v'$ for which there is an extension $\bar{\chi}$ which is a good coloring of G, but with a *different* pattern from that of χ is at most, for example, $2^{-c_1^2 n^2/9} \cdot F(G, 4, 3)$. Indeed, the pattern π may change only when some color $c \in \pi(ij)$ appears very infrequently in $G[W_i, W_j]$ and the proportion of such (degenerate) colorings can be bounded by, for example, $(2/3 + c_0)^{(n/4)^2}$.

It follows that there is a good coloring χ of $G - v - v'$ that has at least $(18 - c_2)^{n/2}$ pattern-preserving extensions to G with vv' getting the wrong color s. Indeed, if this is false, then we would get a contradiction to (3.26) by an argument of Claim 3.13:

$$\frac{(1/2) \cdot 2^{-c_9 n} \cdot F(G, 4, 3)}{4 \cdot \binom{n}{2}} \leqslant 2 \cdot 16^n \cdot 2^{-c_1^2 n^2/9} \cdot F(G, 4, 3) + (18 - c_2)^{n/2} F(G - v - v', 4, 3).$$

Here, we use the trivial upper bound 16^n on the number of extensions for colorings of $G - v - v'$ that are bad or are good, but admit an extension with a different pattern. On the other hand, a good, but not perfect coloring of G may be overcounted as there may be more than one choice of vv' and s; we use the trivial upper bound of $4 \cdot \binom{n}{2}$ here.

Defining $\pi, x_i, x_i', Z_{j,c}, Z_{j,c}', y_i, y_i'$ as in Claim 3.15, one can argue similarly to (3.31) that the number of pattern-preserving extensions of χ is at most

$$2^{c_0 n} \left(\prod_{j=2}^{4} \max(x_j, 1) \cdot \prod_{j=1}^{3} \max(x_j', 1) \right)^{n/4}, \tag{3.32}$$

where all smaller terms are swallowed by $2^{c_0 n}$. Moreover, as before, $|Z_{j,c}| \geqslant 2c_1 n$ if and only if $c \in \pi(\{1, j\})$ while $|Z_{j,c}'| \geqslant 2c_1 n$ if and only if $c \in \pi(\{j, 4\})$.

Since $s \notin \pi(\{1, 4\})$, we have $s \in \pi(\{1, 3\}) \cap \pi(\{3, 4\})$. But then the number of choices per $w \in W_3 \cap N(v) \cap N(v')$ is at most $x_3 x_3' - 1$ (instead of $x_3 x_3'$) because we cannot assign color s to both vw and vw'. Also, if vw or vw' is not an edge, then we have at most four choices per w (while $|\pi(\{1, 3\})| \cdot |\pi(\{3, 4\}| \geqslant 6)$. This allows us to improve (3.32) by factor $(8/9)^{n/4}$. This contradicts the choice of χ and proves part (ii) of Lemma 3.12.

Let $H = K(W_1, \ldots, W_4)$. Suppose first that $G \not\cong H$, that is, G is not complete 4-partite. We know that almost every coloring χ of G is perfect. Moreover, if we start with a perfect coloring χ of G and color all remaining edges in $E(H) \setminus E(G)$ according to the pattern of χ, then we get at least $2^{|H \setminus G|} \geqslant 2$ extensions to H, none containing a monochromatic K_3. Thus, $|\mathcal{P}(H)| \geqslant 2|\mathcal{P}(G)| > F(G, 4, 3)$ and we can take $G' = H$.

Finally, suppose that $G = H$, but $G \not\cong T_4(n)$. Let $d_i = |W_i|$ for $i \in [4]$. Assume, without loss of generality, that $d_1 \geqslant d_2 \geqslant d_3 \geqslant d_4$ with $d_1 \geqslant d_4 + 2$. Let G' be the complete 4-partite graph with parts of size $d_1 - 1, d_2, d_3, d_4 + 1$. We already know that almost every coloring of G is perfect. Thus, in order to finish the proof it is enough to show that, for example, $|\mathcal{P}(G')| > 1.1|\mathcal{P}(G)|$.

The number of perfect colorings of G is given by the following expression:

$$|\mathcal{P}(G)| = (12 + o(1))(S_1 + S_2 + S_3), \tag{3.33}$$

where

$$\begin{aligned} S_1 &= 2^{d_1 d_2 + d_3 d_4} 3^{d_1 d_3 + d_1 d_4 + d_2 d_3 + d_2 d_4}, \\ S_2 &= 2^{d_1 d_3 + d_2 d_4} 3^{d_1 d_2 + d_1 d_4 + d_2 d_3 + d_3 d_4}, \\ S_3 &= 2^{d_1 d_4 + d_2 d_3} 3^{d_1 d_2 + d_1 d_3 + d_2 d_4 + d_3 d_4}. \end{aligned}$$

Note that we have an error term in (3.33) because some (degenerate) colorings are overcounted in the right-hand side. Also,

$$\begin{aligned} |\mathcal{P}(G')| = (12 + o(1))(&2^{-d_2 + d_3} 3^{d_1 - d_4 - 1 + d_2 - d_3} \cdot S_1 \\ &+ 2^{d_2 - d_3} 3^{d_1 - d_4 - 1 - d_2 + d_3} \cdot S_2 + 2^{d_1 - d_4 - 1} \cdot S_3). \end{aligned}$$

But, as $d_1 - d_4 \geqslant \max\{2, d_2 - d_3\}$, the coefficient in front of each S_i is at least $4/3$. Therefore, $|\mathcal{P}(G')| > 1.1|\mathcal{P}(G)|$, completing the proof of Lemma 3.12. □

Routine calculations (omitted) show that

$$|\mathcal{P}(T_4(n))| = (C + o(1)) \cdot 18^{t_4(n)/3}, \tag{3.34}$$

where $C = (2^{14} \cdot 3)^{1/3}$ if $n \equiv 2 \pmod 4$ and $C = 36$ otherwise.

Proof of Theorem 1.1. Let, for example, $N = n_0^2$. Let G be an extremal graph on $n \geqslant N$ vertices. Suppose on the contrary that $G \not\cong T_4(n)$. Let $G_n = G$.

We iteratively apply the following procedure. Given a current graph G_m on $m \geqslant n_0 + 2$ vertices with $F(G_m, 4, 3) \geqslant 18^{m^2/8} \cdot 2^{-c_9 m^2}$ we apply Lemma 3.12. If (3.25) fails for some vertex $v \in V(G_m)$, then we let $G_{m-1} = G_m - v$, decrease m by 1, and repeat. Note that

$$F(G_{m-1}, 4, 3) \geqslant F(G_m, 4, 3)/(18 - c_3)^{m/4} \geqslant 18^{(m-1)^2/8} \cdot 2^{-c_9(m-1)^2}.$$

Likewise, if (3.26) fails for some distinct $v, v' \in V(G_m)$, we let $G_{m-2} = G - v - v'$, decrease m by 2, and repeat. If both (3.25) and (3.26) hold and $G_m \ncong T_4(m)$, then replace G_m by the graph G' returned by Lemma 3.12 and repeat the step (without decreasing m).

Note that, for every m for which G_m is defined, we have

$$F(G_m, 4, 3) \geqslant F(G, 4, 3) \cdot (18 - c_3)^{-(n+(n-1)+\ldots+(m+1))/4}. \tag{3.35}$$

It follows that we never reach $m < n_0 + 2$, for otherwise, when this happens for the first time, we get the contradiction

$$F(G_m, 4, 3) \geqslant \frac{18^{n^2/8} \cdot 2^{-c_9 n^2}}{(18 - c_3)^{(\binom{n}{2} - \binom{m}{2})/4}} > 4^{\binom{m}{2}}.$$

Thus, we stop for some $m \geqslant n_0 + 2$, having $G_m \cong T_4(m)$. We cannot have $m = n$, for otherwise $T_4(n)$ strictly beats G. By Lemma 3.12, almost every coloring of $G_m \cong T_4(m)$ is perfect. Thus, by (3.35),

$$2 \cdot |\mathcal{P}(T_4(m))| > F(T_4(m), 4, 3) \geqslant F(G, 4, 3) \cdot (18 - c_3)^{-(n+(n-1)+\ldots+(m+1))/4}. \tag{3.36}$$

Also, note that $t_4(\ell) - t_4(\ell - 1) = \lfloor 3\ell/4 \rfloor$. Thus, (3.34) implies that, for example, $|\mathcal{P}(T_4(\ell))| \geqslant 18^{\ell/4-1}|\mathcal{P}(T_4(\ell - 1))|$ for all $\ell \geqslant n_0$. By the extremality of G, we conclude that

$$F(G, 4, 3) \geqslant F(T_4(n), 4, 3) \geqslant |\mathcal{P}(T_4(n))| \geqslant \frac{18^{(n+\ldots+(m+1))/4}}{18^{n-m}} |\mathcal{P}(T_4(m))|. \tag{3.37}$$

But (3.36) and (3.37) give a contradiction to $n > m$, proving Theorem 1.1. □

Remark 1. If we set $G = T_4(n)$ with $n \geqslant N$ in the above argument, then we conclude that $m = n$ (otherwise we get a contradiction as before). Thus, we do not perform any iterations at all, which implies that (3.25) and (3.26) hold for $T_4(n)$. By part (ii) of Lemma 3.12 almost every coloring of $T_4(n)$ is perfect. Thus, the estimate (1.5) that was claimed in Section 1 follows from (3.34).

4. *Proof of Theorem* 1.2

In this section, we prove Theorem 1.2. Some parts of the proof closely follow those of Theorem 1.1. We omit many details that have already been presented or are obvious modifications of those in Section 3. We start by fixing positive constants

$$c_0 \gg c_1 \gg \ldots \gg c_{10}.$$

Let $M = 1/c_9$ and $n_0 = 1/c_{10}$. Define

$$\mathcal{G}_n = \{G : v(G) = n,\ F(G, 4, 4) \geqslant 3^{4n^2/9} \cdot 2^{-c_8 n^2}\},$$

and let $\mathcal{G} = \bigcup_{n \geqslant n_0} \mathcal{G}_n$. The lower bound in (1.4) shows that $\mathcal{G}_n$ is non-empty for every $n \geqslant n_0$.

Using the obvious analogs of the previous definitions, we define the parameters $(m, V_i, H_i, R_i, R, r_i)$ arising from an arbitrary graph G and a K_4-free 4-coloring χ of the edges of G and fix one such vector for each pair (G, χ).

By Lemma 2.2, each cluster graph H_i is K_4-free and, by Turán's theorem (1.1), has at most $t_3(m)$ edges. Thus, by (2.1),

$$r_1 + 2r_2 + 3r_3 + 4r_4 = \frac{e(H_1) + e(H_2) + e(H_3) + e(H_4)}{m^2/2} \leqslant \frac{8}{3}. \tag{4.1}$$

We also have a procedure for generating all K_4-free edge 4-colorings of G at least once. This procedure is identical to the Coloring Procedure provided in Section 3 with the only difference being that in Step (3) the parameters r_i (where we omit primes for convenience) now satisfy (4.1) instead of (3.3) and (3.4). So, Lemma 3.1, which bounds the number of choices in Steps (1)–(6), still holds.

The number of options in Step (7) is again bounded by (3.6), that is, the expression $2^{Ln^2/2+O(n)}$, where $L = r_2 + \log_2(3)r_3 + 2r_4$. Under the constraint (4.1) and the non-negativity of the reals r_i, the maximum of L is $(8/9)\log_2 3$ with the (unique) optimal assignment being $r_1 = r_2 = r_4 = 0$ and $r_3 = 8/9$. We conclude that

$$F(n, 4, 4) \leqslant 3^{4n^2/9} \cdot 2^{2c_6 n^2},$$

as it was also shown in [**1**].

We shall now obtain structural information about the cluster graphs (and, indirectly, about G). We call a pair (G, χ) (or the coloring χ) *unsatisfactory* if

$$r_3 \leqslant 8/9 - c_4. \tag{4.2}$$

Otherwise, (G, χ) is *satisfactory*.

LEMMA 4.1. *For every graph G with $n \geqslant n_0$ vertices the number of unsatisfactory K_4-free edge 4-colorings is less than $3^{4n^2/9} \cdot 2^{-c_6 n^2}$.*

Proof. The maximum of L under constraints (4.1) and (4.2) (and the non-negativity of the variables r_i) is

$$L_{\max} = (8/9 - c_4)\log_2(3) + 3c_4/2 < (8/9)\log_2(3) - c_5,$$

with the optimal dual variables for (4.1) and (4.2) being $y_1 = 1/2$ and $y_2 = \log_2(3) - 3/2 > 0$, respectively. Therefore, the total number of choices is at most $2^{c_6 n^2} \cdot 2^{L_{\max} n^2/2+O(n)}$, giving the required upper bound on the number of unsatisfactory colorings. □

Call a vector (m, V_i, H_i) *popular* if it appears for at least $3^{4n^2/9} \cdot 2^{-3c_8 n^2}$ satisfactory K_4-free edge 4-colorings of G. As before, (3.5) guarantees that the number of colorings for which the associated vector is not popular is at most $3^{4n^2/9} \cdot 2^{-2c_8 n^2}$. Let $\mathrm{Pop}(G)$ be the set of all popular vectors and let $\mathcal{S}(G)$ consist of all satisfactory colorings for which the associated vector is popular.

LEMMA 4.2. *For any $n \geqslant n_0$, a graph $G \in \mathcal{G}_n$, and a popular vector $(m, V_i, H_i) \in \mathrm{Pop}(G)$, there exists an equitable partition $[m] = U_1 \cup \ldots \cup U_9$ such that*

$$|R_3 \triangle K(U_1, \ldots, U_9)| < c_3 m^2, \tag{4.3}$$

$$|R \triangle K(U_1, \ldots, U_9)| < 2c_3 m^2. \tag{4.4}$$

Proof. Suppose that some $Y \subseteq [m]$ induces a clique of order 10 in R_3. Then $R_3[Y]$ contains $\binom{10}{2} = 45$ edges, each of which, by definition, belongs to exactly three cluster graphs H_i. Each H_i is K_4-free and so, by Turán's Theorem (1.1), $H_i[Y]$ has at most $t_3(10) = 33$ edges. But $4 \cdot 33 < 3 \cdot 45$, which is a contradiction.

Thus, $K_{10} \not\subseteq R_3$. Since $e(R_3) \geqslant (8/9 - c_4)m^2/2$, Lemma 2.3 gives an equitable partition $[m] = U_1 \cup \ldots \cup U_9$ satisfying (4.3). This partition also satisfies (4.4) because $r_1 + r_2 + r_4 \leqslant 3c_4$ by (4.1) and the negation of (4.2). □

For a graph G and a popular vector $(m, V_i, H_i) \in \mathrm{Pop}(G)$, fix the equitable 9-partition $[m] = U_1 \cup \ldots \cup U_9$ given by Lemma 4.2. For $i \in [9]$, let $\tilde{U_i} = \bigcup_{j\in U_i} V_j$ be the *blow-up* of U_i. Let $\tilde{F} = K(\tilde{U_1}, \ldots, \tilde{U_9})$.

LEMMA 4.3. *For any $n \geqslant n_0$, $G \in \mathcal{G}_n$ and $(m, V_i, H_i) \in \mathrm{Pop}(G)$, we have $|G \bigtriangleup \tilde{F}| < 6c_3n^2$.*

Proof. First consider $G \setminus \tilde{F}$. Up to $5c_7n^2$ edges may be lost by application of the Regularity Lemma. In addition, at most $|R \setminus K(U_1, \ldots, U_9)| \cdot \lceil n/m\rceil^2$ edges are missing in $\tilde{F}$. Overall, $|G \setminus \tilde{F}| < 3c_3n^2$.

On the other hand, we may estimate $|\tilde{F} \setminus G|$ by bounding the number of colorings of G associated with our vector (m, V_i, H_i). We revert to the Coloring Procedure and compute the number of options in Step (7):

$$\prod_{f=2}^{4} \prod_{ij\in R_f} f^{\lceil n/m\rceil^2 - |K(V_i,V_j)\setminus G|} \leqslant (3^{4n^2/9} \cdot 2^{2c_6n^2}) \prod_{ij\in R_3} 2^{-|K(V_i,V_j)\setminus G|}$$
$$\leqslant (3^{4n^2/9} \cdot 2^{2c_6n^2}) \cdot 2^{-|\tilde{F}\setminus G|+2c_3n^2+O(n)}.$$

Since the vector (m, V_i, H_i) is popular, we have

$$|\tilde{F} \setminus G| \leqslant c_6n^2 + 2c_3n^2 + 2c_6n^2 + 3c_8n^2 + O(n) \leqslant 3c_3n^2,$$

as required. □

For each graph G fix a max-cut partition $V(G) = W_1 \cup \ldots \cup W_9$.

LEMMA 4.4 (Stability Property). *Let $n \geqslant n_0$, $G \in \mathcal{G}_n$ and $V(G) = W_1' \cup \ldots \cup W_9'$ be a partition with*

$$|G \cap K(W_1', \ldots, W_9')| \geqslant |G \cap K(W_1, \ldots, W_9)| - c_3n^2.$$

Then $|G \bigtriangleup K(W_1', \ldots, W_9')| \leqslant 9c_3n^2$ and, for any $(m, V_i, H_i) \in \mathrm{Pop}(G)$, there is a relabeling of $W_1', \ldots, W_9'$ such that

$$|W_i' \bigtriangleup \tilde{U_i}| \leqslant 12000c_3n, \quad \text{for each } i \in [9]. \tag{4.5}$$

Also we have $||W_i| - n/9| \leqslant c_2n$ for each $i \in [9]$ and $|G \bigtriangleup K(W_1, \ldots, W_9)| \leqslant 9c_3n^2$.

Proof. Let $F' = K(W_1', \ldots, W_9')$ and $F = K(W_1, \ldots, W_9)$. As $W_1 \cup \ldots \cup W_9$ is a max-cut partition, we have $|F' \cap G| + c_3n^2 \geqslant |F \cap G| \geqslant |\tilde{F} \cap G|$. In addition, both $[m] = U_1 \cup \ldots \cup U_9$ and $[n] = V_1 \cup \ldots \cup V_m$ are equitable partitions, so $|\,|\tilde{U_i}| - n/9| < m + n/m$. It follows that $|\tilde{F}| \geqslant |F'| - c_7n^2$, and

$$|F' \bigtriangleup G| \leqslant |\tilde{F} \bigtriangleup G| + c_7n^2 + 2c_3n^2 \leqslant 9c_3n^2, \tag{4.6}$$

where we used Lemma 4.3. This proves the first part of the lemma.

To prove the next part, we look for a relabeling of $W_1', \ldots, W_9'$ such that $|\tilde{U_i} \setminus W_i'| < 1250c_3n$ for each $i \in [9]$. If no such relabeling exists, then we have some $i \in [9]$ such that $|\tilde{U_i} \setminus W_j'| \geqslant 1250c_3n$ for all $j \in [9]$. However, for some j, $|\tilde{U_i} \cap W_j'| \geqslant |\tilde{U_i}|/9$. Let $X = \tilde{U_i} \cap W_j'$ and $Y = \tilde{U_i} \setminus W_j'$. Then, by Lemma 4.3, we have $e(G[X, Y]) < 6c_3n^2$ while $X \subseteq W_j'$, $Y \cap W_j' = \emptyset$ and (4.6) imply that $e(G[X, Y]) > |X||Y| - 9c_3n^2 > 6c_3n^2$, which is a contradiction.

The desired estimate (4.5) follows from the observation that

$$W_i' \setminus \tilde{U_i} \subseteq \bigcup_{j\in[9]\setminus\{i\}} (\tilde{U_j} \setminus W_j').$$

The last two claims of the lemma follow by taking $W_i' = W_i$. □

A *pattern* is an assignment $\pi : \binom{[9]}{2} \to \binom{[4]}{3}$ (to every edge of K_9 we assign a list of three colors) such that $\pi^{-1}(i)$ is isomorphic to $T_3(9)$ for each $i \in [4]$. It is easy to check that up to isomorphism (of colors and vertices) there is only one pattern. It can be explicitly described as follows. Identify the 9-point vertex set with $(\mathbb{F}_3)^2$, the 2-dimensional vector space over the 3-element finite field $\mathbb{F}_3$. There are four possible directions of 1-dimensional subspaces. Let the color $i \in [4]$ be present in the pattern in those pairs whose difference is not parallel to the ith direction.

We say that an edge 4-coloring χ of $G \in \mathcal{G}_n$ *follows the pattern* π if, for every $ij \in \binom{[9]}{2}$, we have

$$|\chi^{-1}([4] \setminus \pi(ij)) \cap G[W_i, W_j]| \leqslant c_2 n^2.$$

LEMMA 4.5. *Let $n \geqslant n_0$ and $G \in \mathcal{G}_n$. Then every coloring $\chi \in \mathcal{S}(G)$ follows a pattern.*

Proof. Let $\chi \in \mathcal{S}(G)$ and (m, V_i, H_i) be the associated popular vector. Let $[m] = U_1 \cup \ldots \cup U_9$ be the partition given by Lemma 4.2.

Let the *label* of an edge $uv \in R_3$ be $\hat{\chi}(uv) = \{i \in [4] : uv \in E(H_i)\}$. So, $|\hat{\chi}(uv)| = 3$ for all edges $uv \in R_3$.

CLAIM 4.6. *Let $Y = \{v_1, \ldots, v_9\}$ be a subset of $[m]$ such that $R_3[Y] \cong K_9$ and $v_i \in U_i$ for each $i \in [9]$. Let $v_j' \in U_j$ be such that $Y' = (Y \setminus \{v_j\}) \cup \{v_j'\}$ also spans K_9 in R_3. Then $\hat{\chi}(v_j v_i) = \hat{\chi}(v_j' v_i)$ for all $i \in [9] \setminus \{j\}$.*

Proof of Claim 4.6. The identity $3 \cdot \binom{9}{2} = 4 \cdot t_3(9)$ and Turán's theorem imply that each K_4-free graph $H_i[Y]$ has exactly $t_3(9)$ vertices and thus is isomorphic to the Turán graph $T_3(9)$. Let $Y_{i,1}$, $Y_{i,2}$ and $Y_{i,3}$ be the parts of $H_i[Y]$. The family of 3-sets $\{Y_{i,j} : i \in [4],\ j \in [3]\}$ forms a Steiner triple system on Y, that is, every pair is covered exactly once. Thus, if we delete a vertex from Y, then the four triples that contain it are uniquely reconstructible. It follows that if we know $H_i[Y] - v_j$ for each $i \in [4]$, then the labels of the eight pairs containing v_j are uniquely determined. This and the analogous statement for Y' imply the claim. □

We can iteratively build a set $Y = \{v_1, \ldots, v_9\}$ such that $R_3[Y] \cong K_9$ and, for all $i \in [9]$, we have $v_i \in U_i$ and

$$|N_{R_3}(v_i) \cap U_j| > |U_j| - \sqrt{c_3} m \quad \text{for all } j \in [9] \setminus \{i\}. \tag{4.7}$$

Let $A_i \subseteq U_i$ consist of those vertices that lie in $N_{R_3}(v_j)$ for all $j \in [9] \setminus \{i\}$. As all $v_1, \ldots, v_9$ satisfy (4.7), we have $|A_i| > |U_i| - 8\sqrt{c_3} m$. Now, if $a_i a_j \in R_3[A_i, A_j]$ (without loss of generality assume that $(i, j) = (1, 2)$), then all three sets $\{v_1, v_2, \ldots, v_9\}, \{a_1, v_2, \ldots, v_9\}$ and $\{a_1, a_2, v_3, \ldots, v_9\}$ form 9-cliques. By Claim 4.6 we have $\hat{\chi}(v_i v_j) = \hat{\chi}(a_i v_j) = \hat{\chi}(a_i a_j)$. Therefore, the labeling on Y determines the labeling on all edges of R_3 with the possible exception of at most $72\sqrt{c_3} m^2$ edges incident to vertices of $\bigcup_{i=1}^{9} (U_i \setminus A_i)$. We, therefore, have a pattern π such that $\hat{\chi}(u_i u_j) = \pi(ij)$ for all, but at most $73\sqrt{c_3} m^2$ edges in R.

By applying (4.5) to $W_i' = W_i$ and arguing as in the proof of Lemma 3.9, one can show that χ follows the pattern π. □

A coloring $\chi \in \mathcal{S}(G)$ is called *good* if, for every distinct $i,j,k \in [9]$, all sets $X_i \subseteq W_i, X_j \subseteq W_j, X_k \subseteq W_k$ each of size at least $c_1 n$, and a color $c \in \pi(ij) \cap \pi(ik) \cap \pi(jk)$, we can find a monochromatic triangle in color c with one vertex in each of X_i, X_j, X_k. Otherwise, call χ *bad*.

We make use of the following result [**1**, Lemma 3.1] that is proved by the standard embedding argument; see, for example, [**13**, Theorem 5].

LEMMA 4.7. *Let G be a graph and let $V_1, \dots, V_k$ be subsets of vertices of G such that, for every $i \neq j$ and every pair of subsets $X_i \subseteq V_i$ and $X_j \subseteq V_j$ with $|X_i| \geqslant 10^{-k}|V_i|$ and $|X_j| \geqslant 10^{-k}|V_j|$, there are at least $\frac{1}{10}|X_i||X_j|$ edges between X_i and X_j in G. Then G contains a copy of K_k with one vertex in each set V_i.*

As a consequence of this lemma, a coloring fails to be good only if there are c, i, j such that $c \in \pi(ij)$, but for some sets $X_i \subseteq W_i$ and $X_j \subseteq W_j$ with, respectively, $|X_i|, |X_j| \geqslant c_1 n/1000$, $\chi^{-1}(c)$ has at most $|X_i||X_j|/10$ edges between X_i and X_j. The proof of Lemma 3.11 with obvious modifications gives the following.

LEMMA 4.8. *The number of bad colorings is at most $3^{4n^2/9} \cdot 2^{-c_1^2 n^2/10^7}$.*

A good coloring χ of G is *perfect* if $\chi(v_i v_j) \in \pi(ij)$ for every pair $ij \in \binom{[9]}{2}$ and every edge $v_i v_j \in G[W_i, W_j]$. Let $\mathcal{P}(G)$ consist of all perfect colorings of G.

LEMMA 4.9. *Let $G \in \mathcal{G}_n$ be a graph of order $n \geqslant n_0 + 2$ such that $F(G,4,4) \geqslant 3^{4n^2/9} \cdot 2^{-c_9 n^2}$ and, for every distinct $v, v' \in V(G)$, we have*

$$\frac{F(G,4,4)}{F(G-v,4,4)} \geqslant (3-c_3)^{8n/9}, \tag{4.8}$$

$$\frac{F(G,4,4)}{F(G-v-v',4,4)} \geqslant (3-c_3)^{(8/9)(n+(n-1))}. \tag{4.9}$$

Then the following conclusions hold:

(i) *G is 9-partite*;
(ii) $|\mathcal{P}(G)| \geqslant (1-2^{-c_9 n})F(G,4,4)$;
(iii) *if $G \not\cong T_9(n)$, then there is a graph G' with $v(G') = n$ and $F(G',4,4) > F(G,4,4)$.*

Proof. As in the proof of Lemma 3.12, the notion of a good coloring is well defined for $G - X$ provided $|X| \leqslant 2$.

CLAIM 4.10. *For each $i \in [9]$ and every $v \in W_i$, $|N(v) \cap W_i| < 8c_1 n$.*

Proof of Claim 4.10. Suppose that a vertex v violates the claim. Let $W_1' \cup \ldots \cup W_9'$ be the selected max-cut partition of $G - v$. Similarly to Claim 3.13, there is a good coloring χ of $G - v$ with at least $(3-c_2)^{8n/9}$ extensions to G. Let π be the pattern of χ (with respect to $W_1', \dots, W_9'$) and $n_i = |N(v) \cap W_i|$ for $i \in [9]$. As in the proof of Lemma 3.12, we take an extension $\bar{\chi}$ of χ that gives a most frequent vector $\mathbf{x} = (x_1, \dots, x_9)$, where x_i is the number of colors c such that $Z_{i,c} = \{u \in W_i : \bar{\chi}(uv) = c\}$ has at least $2c_1 n$ elements. Also, let y_c be the number of $j \in [9]$ such that $|Z_{j,c}| \geqslant 2c_1 n$. We have

$$x_1 + x_2 + \ldots + x_9 = y_1 + y_2 + y_3 + y_4. \tag{4.10}$$

By the max-cut property, each $x_i \geqslant 1$. The argument of (3.30) shows that the number of extensions of χ to G is at most $2^{c_0 n} \prod_{i=1}^{9} x_i^{n_i}$.

Suppose $y_c \geqslant 7$ for some color c. Any seven vertices of the color-c graph that is isomorphic to $T_3(9)$ span a triangle. The three c-neighborhoods of v in the corresponding parts W_i' have at least $|Z_{i,c}| - 24000c_3 n > c_1 n$ vertices each by (4.5). Since χ is good, this gives a copy of K_4 of color c in $\bar{\chi}$, which is a contradiction.

Thus, $y_c \leqslant 6$ for every $c \in [4]$ and the sum in (4.10) is at most 24. Since each x_i is a positive integer, their product is at most $2^3 3^6$ (it is clearly maximized when the factors are nearly equal). Also, each $n_i \leqslant n/9 + c_2 n$ by Lemma 4.4. Thus, the number of extensions of χ is at most $2^{2c_0 n}(2^3 3^6)^{n/9} < (3 - c_2)^{8n/9}$, which is a contradiction that proves the claim. □

Claim 4.11. *If $x_1, \dots, x_8$ are non-negative integers with sum* 24, *then $\prod_{i=1}^{8} \max(x_i, 1) \leqslant 3^8$ with equality if and only if each x_i equals* 3.

Proof of Claim 4.11. Clearly, the product of two positive integers k and l given their sum is maximum when $|k - l| \leqslant 1$. Thus, if exactly t values x_i are non-zero, then their product is maximized when each positive x_i is $\lfloor 24/t \rfloor$ or $\lceil 24/t \rceil$. Thus, for $t = 8, 7, \dots, 1$ the maximum of the product is, respectively, $3^8 = 6561$, $3^4 \cdot 4^3 = 5184$, $4^6 = 4096$, $4 \cdot 5^4 = 2500$, $6^4 = 1296$, $8^3 = 512$, $12^2 = 144$ and 24. Here, 3^8 is the largest entry. □

Claim 4.12. *For all $i \in [9]$ and all $v, v' \in W_i$, we have $vv' \notin E(G)$.*

Proof of Claim 4.12. Assume for a contradiction that $vv' \in E(G)$, where without loss of generality $v, v' \in W_9$. As in Claim 3.13, one can find a good coloring χ of $G - v - v' \in \mathcal{G}_{n-2}$ with at least $(3 - c_2)^{16n/9}$ extensions to G. Define the parameters $\pi, n_i, Z_{i,c}, x_i, y_i, n_i', Z_{i,c}', x_i', y_i', \bar{\chi}$ and a most frequent color s of vv', as it was done in Claim 3.15. Then a version of (3.31) states that the total number of extensions of χ is at most

$$(5^9)^2 \cdot 4 \cdot 2^{c_0 n} \cdot (4^{8c_1 n + 8c_2 n})^2 \cdot \prod_{i=1}^{8} (\max(x_i, 1) \cdot \max(x_i', 1))^{n/9}. \tag{4.11}$$

Since each $y_c \leqslant 6$, we have $\sum_{i=1}^{8} x_i \leqslant 24$. By Claim 4.11 we have that $x_i = x_i' = 3$ for each $i \in [8]$, for otherwise the bound in (4.11) is strictly less than $(3 - c_2)^{16n/9}$, which is a contradiction to the choice of χ.

Assume that the parts of $H_s \cong T_3(9)$ are $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5, 6\}$ and $A_3 = \{7, 8, 9\}$.

Suppose first that there is $j \in [8]$ such that $|Z_{j,s}| \geqslant 2c_1 n$ but $s \notin \pi(\{j, 9\})$, say $j = 8$. By (4.5), we have $|Z_{8,s} \cap W_8'| \geqslant c_1 n$. Since χ is good, in order to avoid a color-s K_4 in $\bar{\chi}$, we must have $|Z_{i,s}| < 2c_1 n$ for all $i \in A_1$ or for all $i \in A_2$. Thus, y_s contributes at most 5 to $\sum_{i=1}^{8} x_i$ and (since any other y_t is at most 6) this sum is at most 23, giving a contradiction by Claim 4.11 and (4.11).

Also, this implies that $|Z_{j,s}| \geqslant 2c_1 n$ for all $j \in [6]$ (for otherwise $y_s \leqslant 5$). The same claim applies to $|Z_{j,s}'|$. Let $y_1 z_1, \dots, y_m z_m$ be a maximal matching formed by color-s edges between W_1 and W_4. Since χ is good, we have that

$$m \geqslant \min(|W_1|, |W_4|) - c_1 n \geqslant n/9 - 2c_1 n.$$

When we extend the coloring χ to G, the number of choices to color the edges of $G[vv', y_i z_i]$ is at most $3^4 - 1$ for every $i \in [m]$ because, if all four pairs are present in G, then we are not allowed to color all of them with color s, while otherwise we have at most $4^3 < 3^4$ choices. This allows us to improve the bound in (4.11) by factor $(80/81)^{n/10}$, giving the desired contradiction. □

Thus, we have proved part (i) of the lemma.

Suppose on the contrary that the conclusion of part (ii) does not hold. As in the proof of Lemma 3.8, we can find an edge $vv' \in G$, say with $v \in W_1$ and $v' \in W_9$, a color s, and a good coloring χ of $G - v - v'$, such that there are at least $(3-c_2)^{16n/9}$ good extensions of χ to G that preserve the pattern π of χ and assign the 'wrong' color s to vv'. Defining $x_i, x_i', Z_{j,c}, Z_{j,c}', y_i, y_i'$ by the direct analogy with the definitions of Claim 3.15, one can argue similarly to (3.31) that the total number of extensions of χ is at most

$$2^{c_0 n} \cdot \left(\prod_{i=2}^{9} \max(x_i, 1) \cdot \prod_{i=1}^{8} \max(x_i', 1) \right)^{n/9}. \tag{4.12}$$

By Claim 4.11, we have $x_i = 3$ for each $2 \leqslant i \leqslant 9$ and $x_i' = 3$ for each $i \in [8]$. Thus, $y_i = y_i' = 6$ for all i. It follows that, for any $2 \leqslant j \leqslant 9$ and $c \in [4]$, we have $|Z_{j,c}| \geqslant 2c_1 n$ if and only if $c \in \pi(\{1, j\})$. Also, the analogous claim holds for $|Z_{j,c}'|$. Since $s \notin \pi(\{1, 9\})$, we can find distinct $i, j \in \{2, \ldots, 8\}$ such that s belongs to $\pi(ij)$ as well as to the label of each pair in $\{1, 9\} \times \{i, j\}$. As before, by considering a maximal color-s matching in $G[W_i, W_j]$, we can improve (4.12) by factor $(80/81)^{n/10}$, obtaining a contradiction and proving part (ii) of the lemma.

Let us prove part (iii). If G is not complete 9-partite, then by part (ii) we can take $G' = K(W_1, \ldots, W_9)$: indeed, $|\mathcal{P}(G')| \geqslant 3|\mathcal{P}(G)| > F(G, 4, 4)$. So suppose that G is complete 9-partite.

Let us determine the number of possible patterns (with distinguishable colors and vertices). For the color-1 graph we have $\binom{8}{2} \cdot \binom{5}{2}$ choices (there are $\binom{8}{2}$ choices for the part $A_1 \in \binom{[9]}{3}$ containing 1, then $\binom{5}{2}$ choices for the part A_2 containing the smallest element of $[9] \setminus A_1$). Then we have $9 \cdot 4$ choices for color 2, then two choices for color 3 and one choice for color 4. Thus, the total number of patterns is $20160 = 9!/18$. The same answer can be obtained by noting that when we permute [9], then we have a transitive action on patterns and every pattern is fixed by eighteen permutations.

It follows that G has $(9!/18 + o(1))3^{e(G)}$ perfect colorings in total, since every edge of G has exactly three choices for a given pattern. Since $G \not\cong T_9(n)$, we have $|\mathcal{P}(T_9(n))| \geqslant (3 + o(1))|\mathcal{P}(G)|$ and we can take $G' = T_9(n)$. This completes the proof of Lemma 4.9. ☐

Now Theorem 1.2 can be deduced from Lemma 4.9 in the same way (modulo some obvious modifications) as Theorem 1.1 was deduced from Lemma 3.12.

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