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On the quadratic 8-edge case of the Brown–Erdős–Sós problem

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ABSTRACT

Let $f^{(r)}(n; s, k)$ be the maximum number of edges in an n-vertex r-uniform hypergraph containing no k edges on at most s vertices. Brown, Erdős and Sós conjectured in 1973 that the limit $\lim_{n\to\infty} n^{-2} f^{(3)}(n; k+2, k)$ exists for all k. Recently, Delcourt and Postle settled the conjecture and their approach was generalised by Shangguan to every uniformity $r \geq 4$: the limit $\lim_{n\to\infty} n^{-2} f^{(r)}(n; rk-2k+2, k)$ exists for all $r \geq 3$ and $k \geq 2$.

The value of the limit is currently known for $k \in \{2, 3, 4, 5, 6, 7\}$ due to various results authored by Glock, Joos, Kim, Kühn, Lichev, Pikhurko, Rödl and Sun. In this paper we consider the case $k = 8$, determining the value of the limit for each $r \geq 4$ and presenting a lower bound for $r = 3$ that we conjecture to be sharp.

1. Introduction

For an integer $r \geq 2$, an *r-uniform hypergraph* (in short, *r-graph*) H consists of a vertex set $V(H)$ and an edge set $E(H) \subseteq \binom{V(H)}{r}$, that is, every edge is an r-element subset of $V(H)$. Given a family $\mathcal{F}$ of r-graphs, the *Turán number* of $\mathcal{F}$, denoted by $\mathrm{ex}(n; \mathcal{F})$, is defined as the maximum number of edges in an n-vertex r-graph containing no element of $\mathcal{F}$ as a subgraph.

In this paper, we focus on the family $\mathcal{F}^{(r)}(s, k)$, consisting of all r-graphs with k edges and at most s vertices. Brown, Erdős and Sós [2] initiated the systematic investigation of the function

$$f^{(r)}(n; s, k) := \mathrm{ex}(n; \mathcal{F}^{(r)}(s, k)).$$

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They showed that

$$\Omega(n^{(rk-s)/(k-1)}) = f^{(r)}(n; s, k) = O(n^{\lceil (rk-s)/(k-1) \rceil}).$$

If $t := (rk-s)/(k-1)$ is an integer, i.e. $s = rk - tk + t$, then $f^{(r)}(n; rk-tk+t, k) = \Theta(n^t)$. In the sequel, we are mainly interested in the case when $t = 2$; thus $s = rk - 2k + 2$ and the magnitude of the function is $\Theta(n^2)$. A natural question here is whether the limit

$$\pi(r, k) := \lim_{n\to\infty} n^{-2} f^{(r)}(n; rk - 2k + 2, k)$$

exists, which was originally conjectured by Brown, Erdős and Sós for $r = 3$.

In their initial paper [2], Brown, Erdős and Sós confirmed the conjecture for $k = 2$ by observing that $\pi(3, 2) = 1/6$. Many years later, Glock [6] solved the case $k = 3$ and showed that $\pi(3, 3) = 1/5$. In a recent work, Glock, Joos, Kim, Kühn, Lichev and Pikhurko [8] solved the case $k = 4$ by showing that $\pi(3, 4) = 7/36$. Building on their work, Delcourt and Postle [3] finally resolved the Brown–Erdős–Sós conjecture, namely, $\pi(3, k)$ exists for any $k \geq 2$, without determining its value.

For the more general case $\pi(r, k)$ with uniformity $r \geq 4$, the existence of the limit (without the explicit value) was shortly confirmed by Shangguan [12], following the approach of Delcourt and Postle. The natural remaining question is to determine the limits. Recently, a range of results regarding this direction has been established. Apart from values for $r = 3$ mentioned above, the celebrated work of Rödl [11] on the existence of approximate Steiner systems implies that $\pi(r, 2) = \frac{1}{r^2-r}$ for every $r \geq 3$. Moreover, Glock, Joos, Kim, Kühn, Lichev and Pikhurko [8] proved that for every $r \geq 3$,

$$\pi(r, 3) = \frac{1}{r^2 - r - 1} \quad \text{and} \quad \pi(r, 4) = \frac{1}{r^2 - r}.$$

Very recently, Glock, Kim, Lichev, Pikhurko and Sun [9] obtained the limits for $k \in \{5, 7\}$, which are the same as for $k = 3$:

$$\pi(r, 5) = \pi(r, 7) = \frac{1}{r^2 - r - 1} \quad \text{for every } r \geq 3.$$

They also resolved the case $k = 6$. Curiously, in this case, the value behaves differently when $r = 3$ and $r \geq 4$ as follows:

$$\pi(3, 6) = \frac{61}{330} \quad \text{and} \quad \pi(r, 6) = \frac{1}{r^2 - r} \text{ for every } r \geq 4.$$

Meanwhile, Letzter and Sgueglia [10] provided the exact value

$$\pi(r, k) = \frac{1}{r^2 - r} \tag{1}$$

for even integer k and $r \geq r_0(k)$ sufficiently large. In their paper, they asked for the smallest r such that (1) holds.

In this paper, we determine the limit for $k = 8$ and $r \geq 4$ as follows.

Theorem 1.1. *For every $r \geq 4$, we have $\pi(r, 8) = \frac{1}{r^2-r}$.*

Moreover, we provide a lower bound for $r = 3$, which implies that $r = 4$ is the smallest uniformity such that $\pi(r, 8) = \frac{1}{r^2-r}$.

Theorem 1.2. $\pi(3, 8) \geq \frac{3}{16}$.

We conjecture that this lower bound is sharp.

Conjecture 1.3. $\pi(3, 8) = \frac{3}{16}$.

A connection to generalised Ramsey numbers.

In a recent work, Bennett, Cushman and Dudek [1] found a connection between the Brown–Erdős–Sós function for 4-graphs and generalised Ramsey numbers, that were introduced by Erdős and Shelah [4] and were first systematically studied by Erdős and Gyárfás [5].

For integers p, q such that $p \geq 3$ and $2 \leq q \leq \binom{p}{2}$, a (p, q)-*colouring* of K_n is a colouring of the edges of K_n such that every clique of size p receives at least q colours. The *generalised Ramsey number* $\mathrm{GR}(n, p, q)$ is the minimum number of colours such that K_n has a (p, q)-colouring.

Erdős and Gyárfás [5] proved among other results that for arbitrary $p \geq 3$ and $q_{\mathrm{quad}} := \binom{p}{2} - \lfloor p/2 \rfloor + 2$, it holds that

$$\mathrm{GR}(n, p, q_{\mathrm{quad}}) = \Omega\left(n^2\right) \quad \text{and} \quad \mathrm{GR}(n, p, q_{\mathrm{quad}} - 1) = o\left(n^2\right).$$

Thus q_{quad} is the threshold for quadratic growth.

Bennett, Cushman and Dudek [1] showed the following connection between generalised Ramsey numbers and the Brown–Erdős–Sós function.

Theorem 1.4 (*[1, Theorem 3]*)**.** *For all even $p \geq 6$, we have*

$$\lim_{n\to\infty} \frac{\mathrm{GR}(n, p, q_{\mathrm{quad}})}{n^2} = \frac{1}{2} - \pi\left(4, \frac{p}{2} - 1\right).$$

In particular, the limit on the left exists by [12].

From Theorem 1.1, together with Theorem 1.4, we directly obtain the following asymptotic value of $\mathrm{GR}(n, 18, 146)$.

Theorem 1.5.

$$\lim_{n\to\infty} \frac{\mathrm{GR}(n, 18, 146)}{n^2} = \frac{5}{12}.$$

Organisation. The remainder of this paper is organised as follows. We introduce some necessary definitions and notation in Section 2. Section 3 provides the proofs of lower bounds in Theorems 1.1 and 1.2. The proof of upper bound in Theorem 1.1 can be found in Section 4.

2. Preliminaries

Throughout the paper, we shall use the following notation and terminology. For integers m and n, we denote by $[n]$ the set $\{1, \dots, n\}$ and by $[m, n]$ the set $[n] \setminus [m-1] = \{m, \dots, n\}$. For a set X, we define $\binom{X}{m} := \{Y \subseteq X : |Y| = m\}$ to be the family of all m-subsets of X. For simplicity, we often denote the unordered pair $\{x, y\}$ (resp. triple $\{x, y, z\}$) by xy (resp. xyz).

We will often identify an r-graph G with its edge set. In particular, if we specify only the edge set $E(G)$ then the vertex set is assumed to be the union of these edges, that is, $V(G) = \bigcup_{e\in E(G)} e$. We let $|G|$ be the number of edges of G and $v(G)$ be the number of vertices of G. For two r-graphs G and H, we define their *union* $G \cup H$ by $E(G \cup H) := E(G) \cup E(H)$, and their *difference* $G \setminus H$ by $E(G \setminus H) := E(G) \setminus E(H)$. By a *graph*, we will mean a 2-graph.

A *diamond* is an r-graph consisting of two edges sharing exactly two vertices. For positive integers s and k, an (s, k)-*configuration* is an r-graph with k edges and at most s vertices, that is, an element of $\mathcal{F}^{(r)}(s, k)$. In particular, if $s = rk - 2k + 2$, then we usually omit s and refer to it as a k-*configuration*. Moreover, if $s = rk - 2k + 1$, we refer to it as k^--*configuration*. We say that an r-graph is k-*free* (resp. k^--*free*) if it contains no k-configuration (resp. k^--configuration). Let $\mathcal{G}_k^{(r)}$ denote the family of all k-configurations and all ℓ^--configurations with $\ell \in [2, k-1]$, namely,

$$\mathcal{G}_k^{(r)} := \mathcal{F}^{(r)}(rk - 2k + 2, k) \cup \left(\bigcup_{\ell=2}^{k-1} \mathcal{F}^{(r)}(r\ell - 2\ell + 1, \ell)\right).$$

Note that $\mathcal{G}_k^{(r)}$ not only contains k-configurations, which is the primary topic of this paper, but also includes "denser" r-graphs of smaller sizes. In the following sections, we will see that this family is closely related to the lower and upper bounds on $\pi(r, k)$.

We use the following definitions introduced in [9]. For an r-graph G, a pair xy of distinct vertices (not necessarily in $\binom{V(G)}{2}$) and $A \subseteq \mathbb{N} \cup \{0\}$, we say that G *A-claims* the pair xy if, for every $i \in A$, there are i distinct edges $e_1, \ldots, e_i \in E(G)$ such that $|\{x, y\} \cup (\bigcup_{j=1}^{i} e_j)| \le ri - 2i + 2$. In particular, if $xy \in \binom{V(G)}{2}$, this is equivalent to the existence, for every $i \in A$, of an i-configuration $J \subseteq G$ such that $\{x, y\} \subseteq V(J)$. Let $P_A(G)$ be the set of all pairs in $\binom{V(G)}{2}$ that are A-claimed by G. If $A = \{i\}$ is a singleton, we simply write *i-claims* (resp. $P_i(G)$) instead of $\{i\}$-claims (resp. $P_{\{i\}}(G)$). For $i = 1$, $P_1(G)$ is the usual 2-*shadow* of G consisting of all pairs uv of vertices such that there exists some edge $e \in E(G)$ with $u, v \in e$. Let $C_G(xy)$ be the set of those $i \ge 0$ such that the pair xy is i-claimed by G, that is,

$$C_G(xy) := \left\{ i \ge 0 : \exists \text{ distinct } e_1, \ldots, e_i \in E(G) \text{ such that } \left|\{x, y\} \cup (\textstyle\bigcup_{j=1}^{i} e_j)\right| \le ri - 2i + 2 \right\}. \tag{2}$$

More generally, for disjoint subsets $A, B \subseteq \mathbb{N}$, we say that G *$\overline{A}B$-claims* a pair xy if $A \cap C_G(xy) = \varnothing$ and $B \subseteq C_G(xy)$. For simplicity, we often omit curly brackets. For example, when $A = \{1\}$ and $B = \{i\}$ we just say *$\overline{1}i$-claims*; also, we let $P_{\overline{1}i}(G) := P_i(G) \setminus P_1(G)$ denote the set of pairs in $\binom{V(G)}{2}$ that are $\overline{1}i$-claimed by G, and similarly let $P_{\overline{12}i}(G) := P_i(G) \setminus (P_1(G) \cup P_2(G))$.

3. Lower bounds

In order to prove lower bounds in Theorems 1.1 and 1.2, we need the following result proved by Glock, Joos, Kim, Kühn, Lichev and Pikhurko [8].

Theorem 3.1 ([8, Theorem 3.1]). *Fix $k \ge 2$ and $r \ge 3$. Let F be a $\mathcal{G}_k^{(r)}$-free r-graph. Then,*

$$\liminf_{n \to \infty} \frac{f^{(r)}(n; rk - 2k + 2, k)}{n^2} \ge \frac{|F|}{2\,|P_{\le \lfloor k/2 \rfloor}(F)|},$$

where we define $P_{\le t}(F) := \{xy \in \binom{V(F)}{2} \mid C_F(xy) \cap [t] \ne \varnothing\}$ to consist of all pairs xy of vertices of F such that $C_F(xy)$ contains some i with $1 \le i \le t$.

In brief, Theorem 3.1 is proved by finding, for any large n, an almost optimal edge-packing of copies of the graph $J := P_{\le \lfloor k/2 \rfloor}(F)$ in the n-clique, and putting a copy of F "on top" of each copy of J. Since F has no k-configuration, it remains to prevent any k-configurations that use at least two different copies of J. For this, the packing has to be chosen carefully, using the general theory of conflict-free hypergraph matchings developed independently by Delcourt and Postle [3] and Glock, Joos, Kim, Kühn and Lichev [7].

We note that Theorem 3.1 was used to derive lower bounds of $\pi(r, k)$ for $k \in [4, 7]$ in [8,9], as well as to prove the existence of $\pi(r, k)$ in [3,12]. In the following, we will also apply this theorem to determine the lower bounds in Theorems 1.1 and 1.2.

Proof of the lower bound in Theorem 1.1. For $r \ge 4$, the lower bound $\pi(r, 8) \ge 1/(r^2 - r)$ follows from Theorem 3.1 with the r-graph F being a single edge e (as then $P_{\le 4}(F) = \binom{e}{2}$ consists of all pairs inside e). $\square$

Let us informally describe the construction of F used to prove Theorem 1.2 via an application of Theorem 3.1. Let R be the 5-vertex 3-edge 3-graph obtained from an edge abc by adding a diamond $\{buv, cuv\}$, see Fig. 1. This 3-graph $\overline{1}3$-claims the pairs au and av. Our goal is to construct a 4-free and 8-free 3-graph F consisting of many copies of R such that the number of $\overline{1}3$-claimed pairs in F is much smaller than $|F|$. We fix a large integer m and a bipartite graph G with $2m$ vertices such that G does not contain a 4-cycle as a subgraph and each vertex has degree $\Theta(\sqrt{m})$. We take a random

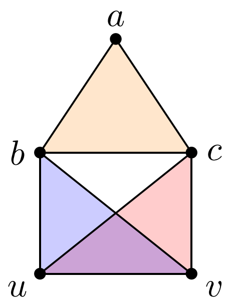

Fig. 1. An illustration of R.

collection $\mathcal{P}$ of 2-paths (that is, paths consisting of 2 edges) where each 2-path of G is included into $\mathcal{P}$ with probability $p := (\log m)/\sqrt{m}$, except we remove some (negligibly many) paths to satisfy the property that for any $i \in [8]$, every i-subset $\{P_1, \ldots, P_i\}$ of $\mathcal{P}$ satisfies $v\left(\bigcup_{j\in[i]} P_j\right) \geq i+2$. We construct F from $\mathcal{P}$ as follows. For each path $P_i \in \mathcal{P}$, say $P_i = \{u_ia_i, a_iv_i\}$, we add two vertices b_i, c_i to $V(F)$ and add all edges from $R_i := \{a_ib_ic_i, b_iu_iv_i, c_iu_iv_i\}$ to $E(F)$. Thus R_i is a copy of R on top of P_i such that the two edges of P_i are 13-claimed by R_i.

We split the proof of Theorem 1.2 into two main parts. Lemma 3.2 returns a collection $\mathcal{P}$ of 2-paths with the required properties. Lemma 3.5 verifies that the constructed F satisfies all conditions of Theorem 3.1, which will give us the desired lower bound of Theorem 1.2.

First, we provide the lemma finding a desired family of 2-paths.

Lemma 3.2. *For any sufficiently large prime power q, there exists a family $\mathcal{P}$ of 2-paths satisfying the following properties with $m := q^2 + q + 1$:*

(i) $|\mathcal{P}| = \Omega(m^{3/2} \log m)$,
(ii) *the union graph* $\bigcup_{P\in\mathcal{P}} P$ *has* $O(m^{3/2})$ *edges,*
(iii) *the union graph* $\bigcup_{P\in\mathcal{P}} P$ *is triangle-free,* C_4*-free and* C_5*-free,*
(iv) *for any* $i \in [8]$, *every* i*-subset* $\{P_1, \ldots, P_i\}$ *of* $\mathcal{P}$ *satisfies* $v\left(\bigcup_{j\in[i]} P_j\right) \geq i+2$.

Proof. Let G' be the incidence bipartite graph of the Desarguesian projective plane over the finite field $\mathbb{F}_q$. In more detail, the parts of G' are the sets V_i with $i = 1, 2$, consisting of i-dimensional linear subspaces of $\mathbb{F}_q^3$, where we have an edge between $x \in V_1$ and $X \in V_2$ if $x \subseteq X$. As it is well-known (and easy to check directly), we have $|V_i| = m$ and each vertex has degree $q+1$ in G'.

Let $\mathcal{P}(G')$ be the set of all 2-paths of G'. We choose every 2-path in $\mathcal{P}(G')$ randomly and independently with probability $(\log m)/m^{1/2}$. Denote by $\mathcal{P}_0 \subseteq \mathcal{P}(G')$ the set of all chosen 2-paths. Since

$$|\mathcal{P}(G')| = \sum_{v\in V(G')} \binom{d(v)}{2} = \Theta\left(m^2\right),$$

we have

$$\mathbb{E}\big[\,|\mathcal{P}_0|\,\big] = \Theta\left(m^2\right) \cdot \frac{\log m}{m^{1/2}} = \Theta\left(m^{3/2}\log m\right), \tag{3}$$

which is exactly the order of magnitude in Condition (i). Furthermore, since $\bigcup_{P\in\mathcal{P}_0} P \subseteq G'$, $\mathcal{P}_0$ satisfies conditions (ii) and (iii) trivially. To meet condition (iv), we shall use the probabilistic deletion method to remove some 2-paths from $\mathcal{P}_0$ which form configurations prohibited in (iv) and show that the number of such removed 2-paths is $o(m^{3/2}\log m)$.

Given an integer i, an i-set $\{P_1, \ldots, P_i\} \subseteq \mathcal{P}(G')$ is called *dense* if $v\left(\bigcup_{j\in[i]} P_j\right) \leq i+1$, and *sparse* otherwise. We note that every 1-set $\{P_1\}$ is sparse as $v(P_1) = 3 > 1+1$. We say that a dense i-set

$\mathcal{S}$ is a *minimal dense i-set* if every j-subset of $\mathcal{S}$ is sparse for every $j \in [i-1]$. An easy induction argument gives the following proposition.

Proposition 3.3. *Every dense i-set with $i \geq 1$ contains a minimal dense j-set as a subset for some $1 \leq j \leq i$.*

Let us count the number of minimal dense i-sets of $\mathcal{P}(G')$ in the following claim.

Claim 3.4. *Given an integer $i \geq 2$, the number of minimal dense i-sets is $O\left(m^{1+\frac{i}{2}}\right)$.*

Proof of Claim 3.4. Fix an $i \geq 2$. Let N_i denote the number of minimal dense i-sets. For $k \leq i+1$, we define $N_{i,k}$ to be the number of minimal dense i-sets $\{P_1, \dots, P_i\} \subseteq \mathcal{P}(G')$ satisfying $v\left(\bigcup_{j\in[i]} P_j\right) = k$. Recalling the definition of dense i-sets, we have $N_i = \sum_{k\in[i+1]} N_{i,k}$. Hence, it suffices to prove that $N_{i,k} = O\left(m^{1+\frac{i}{2}}\right)$ for every $k \in [i+1]$.

Let $k \in [i+1]$ be fixed. Consider a minimal dense i-set $\mathcal{S} := \{P_1, \dots, P_i\} \subseteq \mathcal{P}(G')$ with $v\left(\bigcup_{j\in[i]} P_j\right) = k$. We claim that the union graph $\bigcup_{j\in[i]} P_j$ is connected. If not, then $\bigcup_{j\in[i]} P_j$ consists of several connected components, which will give a natural partition of $\mathcal{S}$ (since, for every path $P_j \in \mathcal{S}$, its vertex set $V(P_j)$ entirely lies in one component). Now assume that $\bigcup_{j\in[i]} P_j$ has $t \geq 2$ components $C_1, \dots, C_t$. For $\ell \in [t]$, let $\mathcal{S}_\ell \subseteq \mathcal{S}$ be the proper subset of paths of $\mathcal{S}$ which entirely lie in the component C_ℓ. Then there must exist some $\mathcal{S}_\ell \subseteq \mathcal{S}$ with $\ell \in [t]$ which is a dense $|\mathcal{S}_\ell|$-set, since otherwise we can derive the contradiction that

$$v\Big(\bigcup_{j\in[i]} P_j\Big) = \sum_{\ell\in[t]} v\left(C_\ell\right) = \sum_{\ell\in[t]} v\Big(\bigcup_{P\in\mathcal{S}_\ell} P\Big) \geq \sum_{\ell\in[t]} (|\mathcal{S}_\ell| + 2) = i + 2t > i + 2.$$

However, this contradicts to the fact that $\mathcal{S}$ is a minimal dense i-set.

Now we know that each minimal dense i-set $\mathcal{S} := \{P_1, \dots, P_i\} \subseteq \mathcal{P}(G')$ with $v\left(\bigcup_{j\in[i]} P_j\right) = k$ corresponds to a connected graph $\bigcup_{j\in[i]} P_j$ on k vertices in G'. On the other hand, a k-vertex connected subgraph of G' can be the union graph for at most $\binom{k^3}{i}$ minimal dense i-sets $\mathcal{S} := \{P_1, \dots, P_i\} \subseteq \mathcal{P}(G')$ with $v\left(\bigcup_{j\in[i]} P_j\right) = k$, since k^3 is a trivial upper bound on the number of 2-paths in this subgraph. Let $\mathcal{F}_k$ be the set of all k-vertex connected subgraphs of G'. Then $N_{i,k} = O\left(|\mathcal{F}_k|\right)$.

Let $\mathcal{T}_k$ be the set of all k-vertex trees in G'. For every connected k-vertex graph F in $\mathcal{F}_k$, F contains a spanning tree $T \subseteq F$ as a subgraph. On the other hand, every k-vertex tree $T \in \mathcal{T}_k$ can serve as a spanning tree for at most constant number of k-vertex connected subgraphs. Therefore, $|\mathcal{F}_k| = O(|\mathcal{T}_k|)$. By the regularity of G', we can bound $|\mathcal{T}_k|$ as follows:

$$|\mathcal{T}_k| \leq 2m \cdot (k-1)! \cdot \left(\Delta(G')\right)^{k-1} = O(m \cdot m^{\frac{k-1}{2}}) = O(m^{1+\frac{i}{2}}).$$

This bound is obtained by choosing a vertex in G' and then iteratively building a tree by choosing a selected vertex and adding one of its neighbours in G'. Thus we conclude that

$$N_{i,k} = O(|\mathcal{F}_k|) = O(|\mathcal{T}_k|) = O(m^{1+\frac{i}{2}}),$$

as claimed. □

Let X_i be the number of minimal dense i-sets in which every 2-path has been chosen. Then, by Claim 3.4,

$$\mathbb{E}\Big[\sum_{i\in[8]} X_i\Big] = \sum_{i\in[8]} \mathbb{E}[X_i] \leq \sum_{i\in[8]} \left(\frac{\log m}{m^{1/2}}\right)^i \cdot O\left(m^{1+\frac{i}{2}}\right) = O(m\log^8 m). \tag{4}$$

Let $\mathcal{P} \subseteq \mathcal{P}_0$ be the set obtained from $\mathcal{P}_0$ by removing one 2-path from each minimal dense i-set with $i \in [8]$. Then $\mathcal{P}$ inherits properties (*ii*) and (*iii*) from $\mathcal{P}_0$ as $\bigcup_{P\in\mathcal{P}} P$ is a subgraph of $\bigcup_{P\in\mathcal{P}_0} P$. Also, it follows from (3) and (4) that

$$\mathbb{E}\big[\,|\mathcal{P}|\,\big] \geq \mathbb{E}\Big[\,|\mathcal{P}_0| - \sum_{i\in[8]} X_i\Big] \geq \Theta(m^{3/2}\log m) - O(m\log^8 m) = \Theta(m^{3/2}\log m).$$

Take a deterministic outcome $\mathcal{P}$ with $|\mathcal{P}| = \Omega(m^{3/2}\log m)$. Then, $\mathcal{P}$ avoids all minimal dense i-sets with $i \in [8]$. By Proposition 3.3, $\mathcal{P}$ avoids all dense i-sets for $i \in [8]$, proving Item (iv). This finishes the proof. □

The construction of F to be utilised in Theorem 3.1 is presented below.

Lemma 3.5. *For an infinite sequence of m, there exists a 3-graph F satisfying the following properties:*

(a) F is 4-free and 8-free,
(b) F is k^--free for every $k \in [2, 7]$,
(c) $|F| = \Omega(m^{3/2}\log m)$,
(d) $|P_{\le 4}(F)| = \frac{8}{3}|F| + O(m^{3/2})$.

Proof. For $m = q^2 + q + 1$ with sufficiently large prime power q, let $\mathcal{P} = \{P_1, \dots, P_{|\mathcal{P}|}\}$ be the family of 2-paths returned by Lemma 3.2. For a 2-path $P_i \in \mathcal{P}$, we denote by a_i the internal vertex of P_i, and denote by u_i, v_i the two endpoints of P_i. Now we construct F, using $\mathcal{P}$. For each $P_i \in \mathcal{P}$, we add new vertices b_i and c_i to $V(F)$, and add all edges of $R_i := \{a_ib_ic_i, b_iu_iv_i, c_iu_iv_i\}$ to F. Thus R_i is a copy of R that sits on top of P_i; we also say that P_i *supports* R_i. Let F be the union of the 3-graphs R_i for $i \in [\,|\mathcal{P}|\,]$. We call each such copy of R a *block*.

Let us first calculate $|F|$. We claim that $|F| = 3|\mathcal{P}|$. By our construction of F, we know that each R_i contributes 3 edges to $E(F)$. It is enough to show that for distinct $i, j \in [\,|\mathcal{P}|\,]$, we have $E(R_i) \cap E(R_j) = \varnothing$. Indeed, each edge of R_i contains either b_i or c_i. Since $b_i, c_i \notin V(R_j)$, no edge of R_i belongs to $E(R_j)$. Thus, Item (c) holds.

To prove Item (b), it suffices to have the following claim.

Claim 3.6. *For every $k \in [2, 8]$, every k edges of F span at least $k + 2$ vertices.*

Proof of Claim 3.6. Consider an arbitrary set $S := \{e_1, \dots, e_k\}$ of k edges of F. We can think of S as a subgraph of F. By our construction of F, each edge of S belongs to some copy of R. Let us assume that the edges of S come from j blocks, say $R_1, \dots, R_j$. Then we have $j \le k$ trivially. Each R_i, with $i \in [j]$, contributes 1, 2 or 3 edges of S. In other words, one can obtain S from $\bigcup_{i\in[j]} R_i$ by trimming 0, 1 or 2 edges which do not belong to S from each R_i. Furthermore, when trimming an edge $e \notin S$, if some vertex v of e does not lie in $V(S) = \cup_{i=1}^{k} e_i$, then we remove it.

For every $i \in [j]$, let $t(R_i)$ be the number of trimmed edges of R_i, and let $r(R_i)$ be the number of removed vertices of R_i. Then $t(R_i) \in \{0, 1, 2\}$ for every $i \in [j]$. In the proposition below, we give a relation between $t(R_i)$ and $r(R_i)$.

Proposition 3.7. *For every $i \in [j]$, it holds that $r(R_i) \le t(R_i)$.*

Proof of Proposition 3.7. We split the proof into three cases, depending on whether $t(R_i) = 1$, 2 or 0. Recall that $R_i := \{a_ib_ic_i, b_iu_iv_i, c_iu_iv_i\}$. When $t(R_i) = 1$ and $a_ib_ic_i$ is removed, both $b_iu_iv_i$ and $c_iu_iv_i$ belong to S. Hence, $\{b_i, c_i, u_i, v_i\} \subseteq V(S)$ and a_i is the only possible vertex which may be removed from $V(R_i)$. If $b_iu_iv_i$ (resp. $c_iu_iv_i$) is removed and $a_ib_ic_i, c_iu_iv_i \in S$ (resp. $a_ib_ic_i, b_iu_iv_i \in S$), then $\{a_i, b_i, c_i, u_i, v_i\} \subseteq V(S)$ and no vertex of R_i can be removed. Thus $r(R_i) \le 1 = t(R_i)$.

When $t(R_i) = 2$, the conclusion follows by observing that at least 3 vertices of R_i remain so at most 2 vertices may be removed.

When $t(R_i) = 0$, meaning that no edge is trimmed from R_i, all three edges of R_i belong to S, and no vertex of R_i can be removed. □

Since $j \le k \le 8$, we may use Lemma 3.2(iv) for the j-set $\{P_1, \dots, P_j\}$ to obtain that $v(\bigcup_{i\in[j]} P_i) \ge j + 2$. Recall that R_i is a copy of R that is based on P_i and its vertex set is obtained by adding two new vertices. Thus, we have

$$v\Big(\bigcup_{i\in[j]} R_i\Big) = v\Big(\bigcup_{i\in[j]} P_i\Big) + 2j \ge 3j + 2. \tag{5}$$

Furthermore, the fact that $E(R_k) \cap E(R_\ell) = \varnothing$ for distinct $k, \ell \in [j]$ yields that $|\bigcup_{i\in[j]} R_i| = 3j$. Therefore, in total there are exactly $3j - k$ edges trimmed from $\bigcup_{i\in[j]} R_i$ to obtain S. On the other hand, by Proposition 3.7, the number of vertices removed from $\bigcup_{i\in[j]} R_i$ is

$$\sum_{i\in[j]} r(R_i) \leq \sum_{i\in[j]} t(R_i) = 3j - k. \tag{6}$$

Hence, by (5) and (6), the number of remaining vertices, which is exactly $|V(S)|$, is at least

$$3j + 2 - (3j - k) = k + 2, \tag{7}$$

as desired. This finishes the proof of Claim 3.6. □

Remark. In the proof of Claim 3.6, for $k \in [2, 8]$, if k edges span exactly $k + 2$ vertices, it means that all the inequalities in (5) and (6) are equalities. In other words, the corresponding 2-paths $\{P_1, \dots, P_j\}$ satisfy $v(\bigcup_{i\in[j]} P_i) = j + 2$. Also, for those R_i with $t(R_i) = 1$ (resp. $t(R_i) = 2$), we have $r(R_i) = 1$ (resp. $r(R_i) = 2$). In particular, by the proof of Proposition 3.7, if $t(R_i) = 1$ and $r(R_i) = 1$, then the unique edge we trim from R_i is $a_ib_ic_i$ and the removed vertex of R_i is a_i.

Now we turn to Item (a). First, we prove the following helpful proposition.

Proposition 3.8. *Suppose that S is a k-configuration of F with $3 \leq k \leq 8$ and S consists of edges from j blocks $R_1, \dots, R_j$. Let $I := \{i \in [j] : t(R_i) = 0\}$ and $R_I := \bigcup_{i\in I} R_i$. Then*

(A) $I \neq \varnothing$,
(B) R_I is a $3|I|$-configuration,
(C) R_I can be obtained from S by iteratively deleting edges $e_{k-3|I|}, \dots, e_2, e_1$, such that each e_ℓ with $\ell \in [\,k-3|I|\,]$ shares exactly two vertices with the 3-graph $R_I \cup \{e_1, \dots, e_{\ell-1}\}$. Furthermore, each e_ℓ comes from some block R_i with $t(R_i) \in \{1, 2\}$, and e_ℓ is $b_iu_iv_i$ or $c_iu_iv_i$.

Proof. By Claim 3.6, S spans exactly $k+2$ vertices. Consider any R_i that contributes 1 edge to S, that is, satisfies $t(R_i) = 2$. If $a_ib_ic_i \in S$ and $b_iu_iv_i, c_iu_iv_i \notin S$ then, since b_i, c_i cannot lie in other edges of S (by our construction of F), we can remove the edge $a_ib_ic_i$ from S together with two vertices b_i, c_i and obtain a k^--configuration, which contradicts Claim 3.6. Therefore, either $b_iu_iv_i \in S$ or $c_iu_iv_i \in S$.

For each R_i contributing 2 edges, by the remark after the proof of Claim 3.6, we have that $b_iu_iv_i, c_iu_iv_i \in S$ and $a_ib_ic_i \notin S$. For each such i, we remove first $b_iu_iv_i$ and then $c_iu_iv_i$ from S. Meanwhile, b_i and c_i can also be deleted since they do not belong to any other edges of S. Moreover, u_i, v_i must lie in some other edges of the remaining 3-graph, since otherwise after deleting $b_iu_iv_i$ (or $c_iu_iv_i$) we would gain a j^--configuration with $j \in [k]$, which contradicts Claim 3.6. Therefore, during each removal, we lose exactly one edge and one associated vertex. After cleaning all edges from R_i with $t(R_i) = 1$, we obtain a k_0-configuration S' with exactly $k_0 + 2$ vertices with $k_0 \in [k]$. Then we remove edges from blocks R_i with $t(R_i) = 2$ from S'. For each R_i with $t(R_i) = 2$, we remove $b_iu_iv_i$ and $c_iu_iv_i$ from S'. By similar arguments, we conclude that during each edge removal, we lose exactly one edge and one vertex. Indeed, if $|I| \geq 1$, then the remaining 3-graph is R_I with $3|I|$-edges and $3|I| + 2$ vertices. (If $I = \varnothing$, then R_I is the empty 3-graph). We proved Items (B) and (C), subject to proving Item (A).

Suppose on the contrary that Item (A) fails, that is $I = \varnothing$, which implies that every block R_i contributes one or two edges. By the argument above, we can enumerate $S = \{e_k, \dots, e_2, e_1\}$ so that each e_ℓ with $\ell \in [k] \setminus \{1\}$ shares exactly two vertices with $e_1 \cup \dots \cup e_{\ell-1}$. For e_2, we have $e_2 \cap e_1 = 2$. If e_1 and e_2 are from distinct blocks, say $e_1 \in R_1$ and $e_2 \in R_2$, then R_1 intersects R_2 on $\{u_1, v_1\} = \{u_2, v_2\}$ as $e_1 \in \{b_1u_1v_1, c_1u_1v_1\}$ and $e_2 \in \{b_2u_2v_2, c_2u_2v_2\}$. Then the paths P_1 and P_2, which support R_1 and R_2 respectively, share two endpoints, giving a C_4. The other case is that e_2, e_1 come from the same block, say R_1, and $e_1 = b_1u_1v_1$ and $e_2 = c_1u_1v_1$ or vice versa. Then for e_3, since $t(R_1) \in \{1, 2\}$, e_3 is from some other block, say R_2, and $e_3 \in \{b_2u_2v_2, c_2u_2v_2\}$ shares exactly two vertices with $e_1 \cup e_2$. Since all b_i, c_i cannot be shared by distinct blocks, we know that the shared vertices are $\{u_1, v_1\} = \{u_2, v_2\}$. It follows that P_1 and P_2 form a C_4, which is a contradiction to Lemma 3.2(iii). □

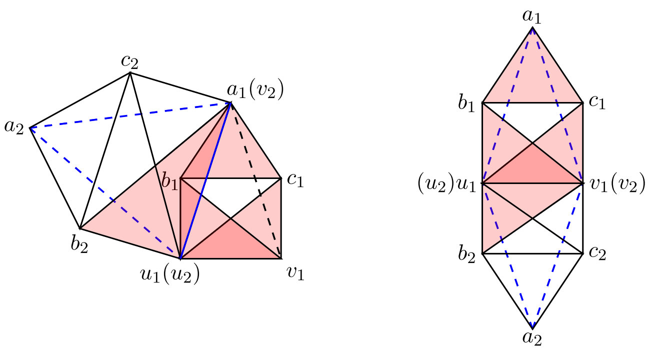

Fig. 2. An illustration of the proof that F is 4-free.

Assume that F contains a 4-configuration S coming from j blocks $R_1, \dots, R_j$. By Claim 3.6, S spans exactly 6 vertices. By Proposition 3.8, there exists some block contributing 3 edges to S and thus $j = 2$. Without loss of generality, we assume that $t(R_1) = 0$. Then, R_2 contributes one edge $e \in \{b_2u_2v_2, c_2u_2v_2\}$ to S, and e shares two vertices with R_1. Since b_2, c_2 cannot lie in other blocks, we have $e \cap V(R_1) = \{u_2, v_2\}$. On the other hand, since b_1, c_1 cannot lie in other blocks, we also have $e \cap V(R_1) \subseteq \{a_1, u_1, v_1\}$. If $e \ni a_1$ then $P_1 \cup P_2$ contains a triangle $a_2u_2v_2$, if $e \cap V(R_1) = \{u_1, v_1\}$ then $P_1 \cup P_2$ contains a 4-cycle $a_1u_1a_2v_1$, see Fig. 2. Both statements contradict Lemma 3.2(iii). Therefore, F is 4-free.

Suppose that F contains an 8-configuration Q consisting of edges from j blocks $R_1, \dots, R_j$. By Proposition 3.8, we have $1 \le |I| \le 2$. Let $e_1, \dots, e_{8-3|I|}$ be the sequence of edges returned by Proposition 3.8(C). First assume that $|I| = 1$ and $t(R_1) = 0$. Without loss of generality, suppose that e_1 belongs to R_2; thus $e_1 \in \{b_2u_2v_2, c_2u_2v_2\}$. Since b_2, c_2 cannot lie in other blocks, we derive that $e_1 \cap R_1 = \{u_2, v_2\}$. Similarly to the arguments in the proof of 4-freeness, we have $e_1 \cap V(R_1) \subseteq \{a_1, u_1, v_1\}$. If $e_1 \cap V(R_1) = \{a_1, u_1\}$ or $e_1 \cap V(R_1) = \{a_1, v_1\}$, then $P_1 \cup P_2$ contains a triangle $a_2u_2v_2$. If $e_1 \cap V(R_1) = \{u_1, v_1\}$, then $P_1 \cup P_2$ contains a 4-cycle $a_1u_1a_2v_1$. Both cases contradict Lemma 3.2(iii).

The remaining case is that $|I| = 2$. Assume that $t(R_1) = t(R_2) = 0$. Then, by Proposition 3.8 and Claim 3.6, $R_1 \cup R_2$ is a 6-configuration on 8 vertices. Thus, R_1 shares two vertices with R_2. Again b_1, c_1, b_2, c_2 do not belong to other blocks. Therefore, $V(R_1) \cap V(R_2) \subseteq \{a_1, u_1, v_1\}$ and $V(R_1) \cap V(R_2) \subseteq \{a_2, u_2, v_2\}$. Note that $V(R_1) \cap V(R_2)$ cannot be $\{u_1, v_1\}$ or $\{u_2, v_2\}$, as otherwise $P_1 \cup P_2$ contains a triangle or C_4. This means that P_1 and P_2 share one edge, and thus $P_1 \cup P_2$ is a 3-path or a star. In the notation of Proposition 3.8(C), assume that e_1 comes from R_3. By Proposition 3.8(C), we have $e_1 \in \{b_3u_3v_3, c_3u_3v_3\}$. Then, e_1 shares two vertices u_3, v_3 with $R_1 \cup R_2$ and $u_3, v_3 \in \{a_1, a_2, u_1, v_1, u_2, v_2\}$. This implies that the endpoints of P_3 lie in $V(P_1 \cup P_2)$. One can easily check $P_1 \cup P_2 \cup P_3$ must contain a triangle, C_4 or C_5, which is a contradiction to Lemma 3.2(iii). Thus, F is 8-free. We proved Item (a) of Lemma 3.5.

Let us turn to Item (d). Observe that every 3-configuration in F comes from one block by Proposition 3.8(A). Let us show that the same also holds for any 2-configuration. Suppose on the contrary that a 2-configuration $\{e_1, e_2\}$ satisfies, say, $e_1 \in R_1$ and $e_2 \in R_2$ for distinct R_1, R_2. Then, $|e_1 \cap e_2| = 2$, which implies that $e_1 \in \{b_1u_1v_1, c_1u_1v_1\}$ and $e_2 \in \{b_2u_2v_2, c_2u_2v_2\}$. Again, since b_1, b_2, c_1, c_2 cannot lie in other blocks, we have $e_1 \cap e_2 = \{u_1, v_1\} = \{u_2, v_2\}$. This gives a C_4 in $P_1 \cup P_2$, a contradiction.

Recall that F consists of $|\mathcal{P}| = |F|/3$ copies of R. Thus, $P_1(F) = \bigcup_{i \in [\,|\mathcal{P}|\,]} P_1(R_i)$. Each block R satisfies $|P_1(R)| = 8$. Furthermore, $P_1(R_i) \cap P_1(R_j) = \emptyset$ for every distinct $i, j \in [\,|\mathcal{P}|\,]$, since otherwise we would obtain a 2-configuration from distinct blocks. Hence, we have $|P_1(F)| = 8|\mathcal{P}| = \frac{8}{3}|F|$. Moreover, we claim that $P_{\bar{1}2}(F) = \emptyset$. Indeed, every 2-configuration lies in a single block and by our construction, every 2-claimed pair is also 1-claimed. Now let us consider $P_{\bar{1}3}(F)$. By the arguments above, we know that every 3-configuration is a block. For a block R_i, only a_iu_i and a_iv_i are $\overline{1}3$-claimed, and these two pairs are edges of the union graph $\bigcup_{P\in\mathcal{P}} P$. Consequently, $|P_{\overline{1}3}(F)| \le |\bigcup_{P\in\mathcal{P}} P| \le O(m^{3/2})$ by Lemma 3.2. At the end, since F is 4-free, there is no pair 4-claimed by F. We conclude that $P_{\le 4}(F) = P_{\le 3}(F) = \frac{8}{3}|F| + O(m^{3/2})$.

This finishes the proof of Lemma 3.5. □

Proof of Theorem 1.2. Let F be the 3-graph given by Lemma 3.5 for some m. Then F is $\mathcal{G}_8^{(3)}$-free. By Theorem 3.1, we obtain

$$\liminf_{n\to\infty} \frac{f^{(3)}(n;\,10,8)}{n^2} \ge \frac{|F|}{2\,|P_{\le 4}(F)|} \ge \frac{3}{16}\cdot\frac{|F|}{|F| + O(m^{3/2})}.$$

Since $|F| = \Omega(m^{3/2}\log m) = \omega(m^{3/2})$, we have $\pi(3,8) \ge \frac{3}{16}$ by taking $m \to \infty$. □

4. Proof of the upper bound in Theorem 1.1

For the upper bound, we will use the following lemma, proved by Delcourt and Postle [3, Theorem 1.7] for $r = 3$ and by Shangguan [12, Lemma 5] for $r \ge 4$, which enables us to exclude smaller "denser" structures when considering the limit.

Lemma 4.1 (*[12, Lemma 5]*). *For all fixed $r \ge 3$ and $k \ge 3$,*

$$\limsup_{n\to\infty} \frac{f^{(r)}(n;\, rk-2k+2, k)}{n^2} \le \limsup_{n\to\infty} \frac{\mathrm{ex}(n, \mathcal{G}_k^{(r)})}{n^2}.$$

Lemma 4.1 gives that, when proving an upper bound on $\pi(r,8)$, it suffices to consider only $\mathcal{G}_8^{(r)}$-free r-graphs.

Let us now briefly outline our proof strategy, which was inspired by [8,9]. Assume that G is an n-vertex $\mathcal{G}_8^{(r)}$-free r-graph. Starting with the trivial edge partition of $E(G)$ into single edges, we apply some merging rules and iteratively merge parts into larger clusters. The $\mathcal{G}_8^{(r)}$-freeness allows us to describe the possible structure of each final cluster. We then assign weights to some vertex pairs within each final cluster (which is a subgraph of G). During this step, we ensure that every pair in $\binom{V(G)}{2}$ receives a total weight at most 1. This method enables us to derive an upper bound on $|G|$ by upper bounding the ratio of the number of edges to the sum of weights for each possible cluster.

We split the proof of the upper bound in Theorem 1.1 into two parts. First, we will introduce the merging operation and analyse the structure of the final edge partition. In the second part, we assign weights and finalise the proof.

4.1. Merging and analysing

4.1.1. Some common definitions and results

Recall that for an r-graph F and a pair uv, the set $C_F(uv)$ consists of all integers $j \ge 0$ such that F has j edges that together with uv include at most $rj - 2j + 2$ vertices. Note that, by definition, 0 always belongs to $C_G(uv)$, which is notationally convenient in the statement of the following easy but very useful observation.

Lemma 4.2 (*[9, Lemma 5.1]*). *For any $\mathcal{F}^{(r)}(rk-2k+2, k)$-free r-graph G, any $uv \in \binom{V(G)}{2}$ and any edge-disjoint subgraphs $F_1, \ldots, F_s \subseteq G$, the sum-set $\sum_{i=1}^{s} C_{F_i}(uv) = \left\{\sum_{i=1}^{s} m_i \mid m_i \in C_{F_i}(uv)\right\}$ does not contain k.*

As we mentioned in the introduction, our proof strategy to bound the size of an $(rk-2k+2, k)$-free r-graph G from above is to analyse possible isomorphism types of the parts of some partition of $E(G)$ which is obtained from the trivial partition into single edges by iteratively applying some merging rules. We build the final partition in stages (with each stage having a different merging rule) as the intermediate families are also needed in our analysis. Let us now develop some general notation and prove some basic results related to merging.

Let G be an arbitrary r-graph. When dealing with a partition $\mathcal{P}$ of $E(G)$, we will view each element $F \in \mathcal{P}$ as an r-graph whose vertex set is the union of the edges in F. Let $A, B \subseteq \mathbb{N}$ be any (not necessarily disjoint) sets of positive integers. For two subgraphs $F, H \subseteq G$, if they are edge disjoint and there is a pair uv such that $A \subseteq C_F(uv)$ and $B \subseteq C_H(uv)$, then we say that F and H are $(A|B)$-*mergeable (via uv)*. Note that this relation is not symmetric in F and H: the first (resp. second) r-graph A-claims (resp. B-claims) the pair uv. When the ordering of the two r-graphs does not matter, we use the shorthand $A|B$-*mergeable* to mean $(A|B)$-mergeable or $(B|A)$-mergeable. For a partition $\mathcal{P}$ of $E(G)$, its $A|B$-*merging* is the partition $\mathcal{M}_{A|B}(\mathcal{P})$ of $E(G)$ obtained from $\mathcal{P}$ by iteratively and as long as possible taking a pair of distinct $A|B$-mergeable parts in the current partition and replacing them by their union. Note that the final partition $\mathcal{M}_{A|B}(\mathcal{P})$ is a coarsening of $\mathcal{P}$ and contains no $A|B$-mergeable pairs of r-graphs. When $\mathcal{P}$ is clear from the context, we may refer to the elements of $\mathcal{M}_{A|B}(\mathcal{P})$ as $A|B$-*clusters*. Likewise, a subgraph F of G that can appear as a part in some intermediate stage of the $A|B$-merging process starting with $\mathcal{P}$ is called a *partial $A|B$-cluster* and we let $\mathcal{M}'_{A|B}(\mathcal{P})$ denote the set of all partial $A|B$-clusters. In other words, $\mathcal{M}'_{A|B}(\mathcal{P})$ is the smallest family of r-graphs which contains $\mathcal{P}$ as a subfamily and is closed under taking the union of $A|B$-mergeable elements. Note that, if F and H are $A|B$-mergeable via uv, then

$$A \cup B \subseteq C_{F\cup H}(uv) \quad \text{and} \quad A + B \subseteq C_{F\cup H}(uv). \tag{8}$$

The first inclusion implies that the merging rule is monotone: merging parts preserves their mergeability with others. Consequently, this implies that $\mathcal{M}_{A|B}(\mathcal{P})$ is exactly the set of maximal (by inclusion) elements of $\mathcal{M}'_{A|B}(\mathcal{P})$ and that the final partition $\mathcal{M}_{A|B}(\mathcal{P})$ does not depend on the order in which we merge parts. Indeed, this process defines the closure of $\mathcal{P}$ under the union of $A|B$-mergeable parts, whose set of maximal elements is unique regardless of the merge order.

In the frequently occurring case when $A = \{1\}$ and $B = \{j\}$, we abbreviate $(\{1\}|\{j\})$ to (j) and $\{1\}|\{j\}$ to j in the above nomenclature. Thus, (j)-mergeable (resp. j-mergeable) means $(\{1\}|\{j\})$-mergeable (resp. $\{1\}|\{j\}$-mergeable).

As an example, let us look at the following merging rule that is actually used as the first step in our proof of the upper bound. Namely, given G, let

$$\mathcal{M}_1 := \mathcal{M}_{\{1\}|\{1\}}(\mathcal{P}_{\text{trivial}})$$

be the 1-merging of the trivial partition $\mathcal{P}_{\text{trivial}}$ of G into single edges. We call the elements of $\mathcal{M}_1$ 1-*clusters*. Here is an alternative description of $\mathcal{M}_1$. Call a subgraph $F \subseteq G$ *connected* if for any two edges $X, Y \in F$ there is a sequence of edges $X_1 = X, X_2, \ldots, X_m = Y$ in F such that, for every $i \in [m-1]$, we have $|X_i \cap X_{i+1}| \geq 2$. Then, 1-clusters are exactly maximal connected subgraphs of G (and partial 1-clusters are exactly connected subgraphs).

We will also use (often without explicit mention) the following result, which is a generalisation of the well-known fact that we can remove edges from any connected 2-graph one by one, down to any given connected subgraph, while keeping the edge set connected. The assumption of this result, roughly speaking, is that the merging process cannot create any new mergeable pairs.

Lemma 4.3 (*Trimming Lemma [9, Lemma 5.2]*). *Fix an r-graph G, a partition $\mathcal{P}$ of $E(G)$ and sets $A, B \subseteq \mathbb{N}$. Suppose that, for all $(A|B)$-mergeable (and thus edge-disjoint) $F, H \in \mathcal{M}'_{A|B}(\mathcal{P})$, there exist $(A|B)$-mergeable $F', H' \in \mathcal{P}$ such that $F' \subseteq F$ and $H' \subseteq H$.*

Then, for every partial $A|B$-clusters $F_0 \subseteq F$, there is an ordering $F_1, \ldots, F_s$ of the elements of $\mathcal{P}$ that lie inside $F \setminus F_0$ such that, for every $i \in [s]$, $\bigcup_{j=0}^{i-1} F_j$ and F_i are $A|B$-mergeable (and, in particular, $\bigcup_{j=0}^{i} F_j$ is a partial $A|B$-cluster for every $i \in [s]$).

In the special case $A = B = \{1\}$ (when partial clusters are just connected subgraphs), the assumption of Lemma 4.3 is vacuously true. Since we are going to use its conclusion quite often, we state it separately.

Corollary 4.4. *For every pair $F_0 \subseteq F$ of connected r-graphs, there is an ordering $X_1, \ldots, X_s$ of the edges in $F \setminus F_0$ such that, for every $i \in [s]$, the r-graph $F_0 \cup \{X_1, \ldots, X_i\}$ is connected.*

We say that an r-graph is a 1-*tree* if it contains only one edge. For $i \geq 2$, we recursively define an i-*tree* as any r-graph that can be obtained from an $(i-1)$-tree T by adding a new edge that consists of a pair ab in the 2-shadow $P_1(T)$ of T and $r-2$ new vertices (not present in T). Clearly, every i-tree is connected. Like the usual 2-graph trees, i-trees are the "sparsest" connected r-graphs of given size. Any i-tree T satisfies

$$|P_1(T)| = i\binom{r}{2} - i + 1 \quad \text{and} \quad |P_{\overline{1}2}(T)| \geq (i-1)(r-2)^2. \tag{9}$$

(Recall that $P_{\overline{1}2}(T)$ is the set of pairs which are 2-claimed but not 1-claimed by T.) Note that the second inequality in (9) is equality if, for example, T is an i-*path*, meaning that we can order the edges of the i-tree T as $X_1, \ldots, X_i$ so that, for each $j \in [i-1]$, the intersection $X_{j+1} \cap (\bigcup_{s=1}^{j} X_s)$ consists of exactly one pair of vertices and this pair belongs to $P_1(X_j) \setminus P_1(\bigcup_{s=1}^{j-1} X_s)$.

The following result shows that the 1-clusters of any $\mathcal{G}_k^{(r)}$-free graph have a very simple structure: namely, they are all small trees.

Lemma 4.5 ([9, Lemma 5.4]). *With the above notation, if G is $\mathcal{G}_k^{(r)}$-free, then every $F \in \mathcal{M}_1$ is an m-tree for some $m \in [k-1]$.*

4.1.2. Structural lemma for $\mathcal{M}_2$

Given a $\mathcal{G}_8^{(r)}$-free r-graph G, we consider another edge partition

$$\mathcal{M}_2 := \mathcal{M}_{\{1\}|\{2\}}(\mathcal{M}_1),$$

which is obtained from $\mathcal{M}_1$ by iteratively merging (2)-mergeable pairs. Let $\mathcal{M}_2' := \mathcal{M}_{\{1\}|\{2\}}'(\mathcal{M}_1)$ be the set of all partial $\{1\}|\{2\}$-clusters. For simplicity, we call the elements of $\mathcal{M}_2$ (resp. $\mathcal{M}_2'$) *2-clusters* (resp. *partial 2-clusters*).

In Lemma 4.7 below, we provide some combinatorial properties of 2-clusters, which will allow us to assign appropriate weights and then achieve correct upper bound in the next section. Before moving to the structural lemma, we first introduce some definitions.

For $F \in \mathcal{M}_2'$ which is made by merging 1-clusters $F_1, \ldots, F_m$ in this order as in Lemma 4.3, we call the sequence $(|F_1|, \ldots, |F_m|)$ of sizes a *composition* of F. Its non-increasing reordering is called *the (non-increasing) composition* of F. Given an r-graph F, we say that an edge $e \in E(F)$ is *flexible* in F if there are $r-2$ distinct vertices $v_1, \ldots, v_{r-2} \in e$ such that, for each $i \in [r-2]$, v_i does not belong to any other edge of F. We also call the set of these $r-2$ vertices $\{v_1, \ldots, v_{r-2}\}$ a *flexible set*. This means that if we delete a flexible set from $V(F)$, then F will lose only one edge. In the following, we will refer to "removing/deleting" a flexible edge e from F as removing edge e and all $r-2$ vertices in its flexible set. We denote by $Q(F) \subseteq E(F)$ the set of flexible edges of F, and let $q(F) := |Q(F)|$.

Lemma 4.6. *For every m-tree F with $m \geq 2$, we have $q(F) \geq 2$.*

Proof. We use induction on m. For $m = 2$, a 2-tree is a pair of edges sharing exactly two vertices. Thus, both edges are flexible. Now consider an m-tree F with $m \geq 3$. Recall that F can be obtained from an $(m-1)$-tree, say F', by adding a new edge e which consists of a pair ab in the $P_1(F')$ and $r-2$ new vertices. Therefore, $e \in Q(F)$. On the other hand, by induction hypothesis, F' contains two flexible edges e_1, e_2. Let $S_1 \subseteq e_1$, and $S_2 \subseteq e_2$ be the flexible sets of e_1 and e_2 respectively. Then $S_1 \cap e_2 = S_2 \cap e_1 = \emptyset$, by the definition of flexible set. We claim that the new added edge e cannot intersect both S_1 and S_2. If it intersects both, then one of the two intersection vertices a, b lies in S_1 and the other in S_2. Since $ab \in P_1(F')$, the pair ab is covered by some edge $e_3 \in E(F') \setminus \{e_1, e_2\}$, which contradicts that S_1, S_2 are flexible sets of F'. Hence, without loss of generality, we can assume that $e \cap S_1 = \emptyset$. As a consequence, e_1 is also a flexible edge of F and $q(F) \geq 2$. □

One big challenge in proving tight upper bounds for $k = 8$ (also present for smaller k) is that there may be *exceptional* 2-clusters, that is, 2-clusters that contain more than k edges (while being

$\mathcal{G}_k^{(r)}$-free). We will now describe some families of 2-clusters that, as claimed by the first part of Lemma 4.7, include every exceptional one. Recall that if we do $(A|B)$-merging then A refers to the r-graph mentioned first (and B to the r-graph mentioned later).

Let $\mathcal{A}$ be the family of 9-edge 2-clusters with a composition $(5, 2, 2)$ which can be obtained as follows. Take any 5-tree S_1 in $\mathcal{M}_1$ with exactly two flexible $(r-2)$-sets A_1 and A_2. (In fact, every such tree S_1 can be shown to be a path but we will not need this.) Then $(1|2)$-merge S_1 with diamonds $D_1, D_2 \in \mathcal{M}_1$ so that the pair x_iy_i in $P_1(S_1)$ 2-claimed by the diamond D_i intersects A_i for $i = 1, 2$.

Let $\mathcal{B}$ be the family of 9-edge 2-clusters B with a composition $(1, 2, 4, 2)$ that can be obtained as follows. We start with a single 1-tree S_1, that is, with a single edge. Then we $(1|2)$-merge S_1 with a 2-tree (diamond) S_2 and with a 4-tree S_3 via any two distinct pairs in $P_1(S_1)$. Then we $(1|2)$-merge S_3 with a diamond S_4.

Let us remark that, since a 9-edge 2-cluster in a $\mathcal{G}_8^{(r)}$-free r-graph cannot have a flexible edge, we can describe the structure of $B \in \mathcal{B}$ more precisely. For example, it must hold that $q(S_3) \le 3$ and each flexible $(r-2)$-set of S_3 is made inflexible as the result of the two mergings involving S_3. However, here (and later) we prefer to give simple descriptions that, even if less precise, are nonetheless enough for our proof.

Let $\mathcal{C}_1$ be the family of 9-edge 2-clusters with a composition $(3, 2, 2, 2)$, which can be obtained from a 3-tree S by 3 times iteratively $1|2$-merging the current r-graph and a new diamond.

Let $\mathcal{C}_2$ be the family of 11-edge 2-clusters with a composition $(3, 2, 2, 2, 2)$ which can be obtained by taking one element F of $\mathcal{C}_1$ and $(1|2)$-merging it with a new diamond.

Let $\mathcal{E}$ be the set of 9-edge 2-clusters with a composition $(1, 2, 3, 1, 2)$ that can be obtained as follows. We start with a 1-tree S_1 and $(1|2)$-merge S_1 via distinct pairs of $P_1(S_1)$ with a diamond S_2 and a 3-tree S_3. Then we $(2|1)$-merge $S_1 \cup S_2 \cup S_3$ with another 1-tree S_4. A last diamond S_5 is $(2|1)$-merged with S_4.

Let $\mathcal{F}$ be the set of 9-edge 2-clusters with a composition $(1, 2, 2, 2, 2)$ which can be obtained as follows. We can start with a single 1-tree S_1 and $(1|2)$-merge S_1 via distinct pairs of $P_1(S_1)$ with two diamonds D_2, D_3. Then $(2|1)$-merge $S_1 \cup D_2 \cup D_3$ with a new diamond D_4 so that the common pair is in $P_{\overline{1}2}(D_2) \cup P_{\overline{1}2}(D_3)$ and belongs to exactly one edge of D_4. The last diamond D_5 is $(2|1)$-merged with D_4 via a pair in $P_1(e)$, where e is the unique flexible edge of $S_1 \cup D_2 \cup D_3 \cup D_4$ (which belongs to D_4).

Let $\mathcal{S}_i$ be the family of $(2i + 1)$-edge 2-clusters obtained from a single 1-tree S_1 by i times iteratively $(1|2)$-merging the current r-graph with diamonds $D_1, \ldots, D_i$. Thus each element of $\mathcal{S}_i$ has a composition with one entry being 1 and i entries being 2.

See Fig. 3 for some illustrations of the above families.

Lemma 4.7. *The following two statements hold.*

- *For any $F \in \mathcal{M}_2$ with $|F| \ge 9$, F belongs to $\mathcal{A}, \mathcal{B}, \mathcal{C}_1, \mathcal{C}_2, \mathcal{E}, \mathcal{F}$ or $\mathcal{S}_i$ with $i \in [4, 3\binom{r}{2}]$.*
- *For any $F \in \mathcal{M}_2$ with a composition $(e_1, \ldots, e_m)$, we have*

$$|P_1(F)| = \sum_{i\in[m]} \left(e_i \binom{r}{2} - e_i + 1 \right) \quad \text{and} \quad |P_{\overline{1}2}(F)| \ge 1 - m + \sum_{i=1}^{m} (e_i - 1)(r-2)^2. \qquad (10)$$

Proof. Consider $F \in \mathcal{M}_2$ with $|F| \ge 9$ and assume that F is obtained by merging m 1-clusters $T_1, \ldots, T_m \in \mathcal{M}_1$ in this order.

Let $s \in [m]$ be the first index such that $|\bigcup_{i=1}^{s} T_i| \ge 9$ and let $H := \bigcup_{i=1}^{s-1} T_i$. Then $|H| \le 7$. As F is $(8r - 14, 8)$-free, we have $|T_s| \ge 2$. If T_s 1-claims a pair which is $\overline{1}2$-claimed by H, that is, T_s and H are (2)-mergeable, then by Corollary 4.4 we can remove some edges from T_s to get an $(8r - 14, 8)$-configuration inside $H \cup T_s$, a contradiction. Thus T_s must $\overline{1}2$-claim some pair xy 1-claimed by H. Let $D \subseteq T_s$ be the diamond $\overline{1}2$-claiming xy. Note that $|H \cup D| \ge 9$, as otherwise using Corollary 4.4, we can remove some edges from $T_s \setminus D$ one by one to obtain an $(8r - 14, 8)$-configuration. Thus $|H| = 7$.

Let $T_1' := T_s$ and let $T_2' \in \{T_1, \ldots, T_{s-1}\}$ be a 1-cluster 2-mergeable with T_s via xy. Let $(T_2', \ldots, T_s')$ be the ordering of $\{T_1, \ldots, T_{s-1}\}$ returned by Lemma 4.3 for the partial 2-clusters $T_2' \subseteq \bigcup_{i=1}^{s-1} T_i$.

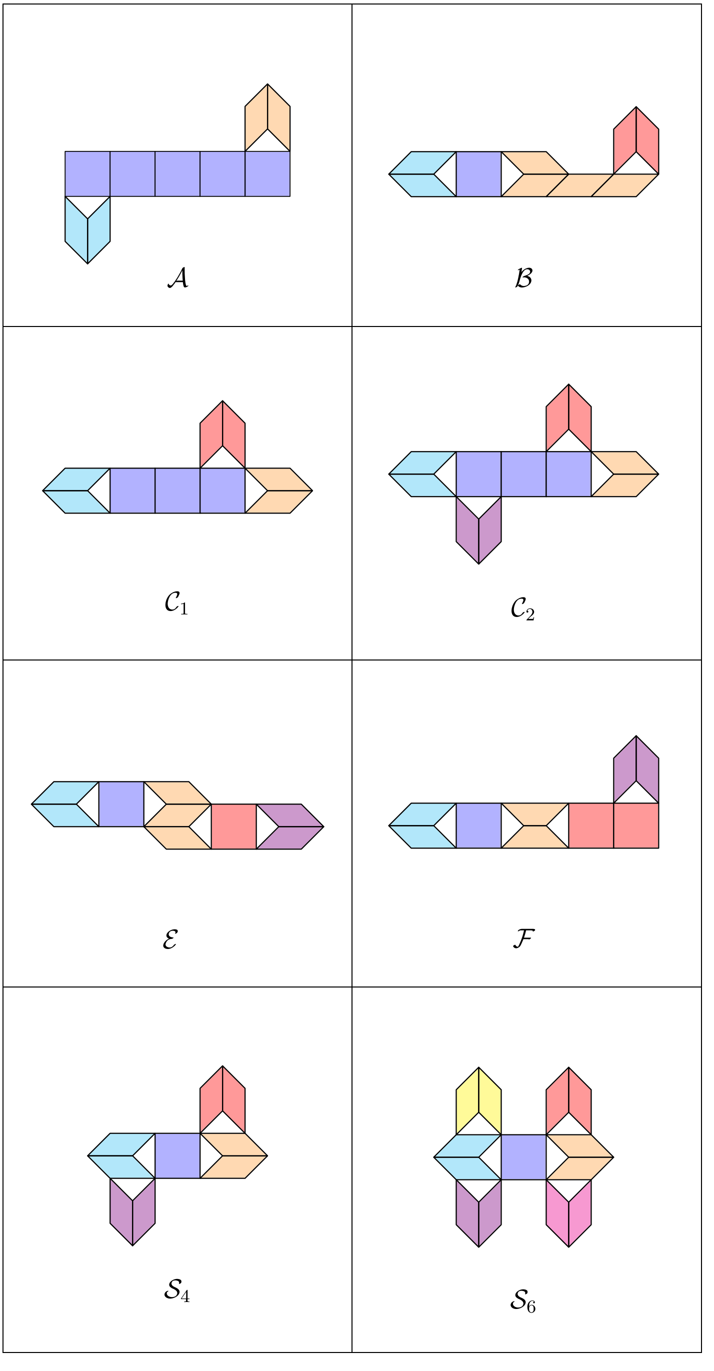

Fig. 3. Examples of 2-clusters in $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{S}_4$ and $\mathcal{S}_6$ for $r = 4$. Here distinct colours represent different 1-clusters.

Therefore, for each $i = 2, \dots, s$ the 1-cluster T'_i is 2-mergeable with the partial 2-cluster $\bigcup_{j=1}^{i-1} T'_j$. Let $t \in [s]$ be the first index such that $|\bigcup_{i=1}^{t} T'_i| \geq 9$. Set $H' := \bigcup_{i=1}^{t-1} T'_i$. By the same argument as in the previous paragraph, we have that $|H'| = 7$ and there is a diamond $D' \subseteq T'_t$ such that H' and D' are (2)-mergeable.

Thus, we have a partial 2-cluster $F' := \bigcup_{i=1}^{t} T'_i$ with at least 9 edges built via the sequence $(T'_1, T'_2, \dots, T'_t)$ so that the first 1-cluster $T'_1 = T_s$ (resp. the last 1-cluster T'_t) is merged with the rest through a pair which is $\overline{1}2$-claimed by the diamond $D \subseteq T_s$ (resp. $D' \subseteq T'_t$). Here we have the freedom to trim one or both of these two clusters, leaving any number of edges in each except exactly 1 edge. It routinely follows that $|T'_1| = |T'_t| = 2$ (that is, $T'_1 = D$ and $T'_t = D'$) and the 1-clusters T'_i with $2 \leq i \leq t-1$ contain exactly 5 edges in total.

Furthermore, we note that F', with $|F'| = 9$, cannot contain flexible edges, as otherwise we can remove such an edge to get an $(8r-14, 8)$-configuration. This allows us to prove the following claim about the partial 2-cluster F'.

Claim 4.8. *Let $(S_1, S_2, \dots, S_t)$ be an arbitrary sequence of 1-clusters that can be merged in this order to give F'. For any $i \in [t]$ and any flexible edge e of $\bigcup_{j=1}^{i} S_j$, there exists a tree S_k with $k \in [i+1, t]$ such that $\bigcup_{j=1}^{i} S_j$ 1|2-merges S_k via some pair xy intersecting the flexible set of e.*

Proof. For $i \in [t]$, let $u_i v_i$ be a pair such that $\bigcup_{j=1}^{i-1} S_j$ and S_i are 1|2-mergeable via $u_i v_i$. Due to $\mathcal{G}_8^{(r)}$-freeness, one can easily prove by induction that for each $i \in [t-1]$, the partial 2-cluster $\bigcup_{j=1}^{i} S_j$ (having $\sum_{j=1}^{i} |S_j|$ edges by definition) has exactly $(r-2)\sum_{j=1}^{i} |S_j| + 2$ vertices. Therefore the following holds:

$$\text{for } i \in [t-1], \text{ we have } V(\textstyle\bigcup_{j=1}^{i-1} S_j) \cap V(S_i) = \{u_i, v_i\}. \tag{11}$$

Given a flexible edge $e \in \bigcup_{j=1}^{i} S_j$, the flexible set W of e must intersect some edge of $\bigcup_{j=i+1}^{t} S_j$, as otherwise e is a flexible edge of the whole F' and we can remove it to get an 8-configuration. Let $k \in [i+1, t]$ be the smallest integer such that $V(S_k)$ and W intersect. Let x be a vertex in $V(S_k) \cap W$. We prove that S_k satisfies the property in Claim 4.8. First, if $k \leq t-1$, then by (11) we have that $V(\bigcup_{j=1}^{k-1} S_j) \cap V(S_k) = \{u_k, v_k\}$ and then $x \in \{u_k, v_k\}$. Suppose to the contrary that S_k 1|2-merges with some S_ℓ with $i+1 < \ell \leq k-1$ rather than with $\bigcup_{j=1}^{i} S_j$. Then it means that $x \in \{u_k, v_k\} \subseteq V(S_\ell)$ and $x \in V(S_\ell) \cap W$, which contradicts to the fact that k is smallest integer such that $V(S_k)$ and W intersect.

Assume now that $k = t$ as otherwise we are done by the above argument. If there exists $x \in V(S_t) \cap W$ such that x belongs to $\{u_t, v_t\}$, then by a similar argument, S_t is as desired. Otherwise $\{u_t, v_t\} \cap W = \varnothing$, and $F' = \bigcup_{j=1}^{t} S_j$ has exactly 9 edges with at most

$$(r-2)\sum_{j=1}^{t-1} |S_j| + 2 + (r-2)|S_t| - |V(S_t) \cap W| = 9(r-2) + 2 - |V(S_t) \cap W|$$

vertices. Since $k = t$ is the smallest integer such that $V(S_k)$ and W intersect, if we remove the edge e and the vertex set $W \setminus V(S_t)$ from F' then we obtain an r-graph with 8 edges and at most

$$9(r-2) + 2 - |V(S_t) \cap W| - |W \setminus V(S_t)| = 9(r-2) + 2 - |W| = 8(r-2) + 2$$

vertices. This contradicts the 8-freeness of F'. □

For $i \in [t]$, define $e'_i := |T'_i|$. We know that $e'_1 = e'_t = 2$ and $\sum_{i=1}^{t} e'_i = 9$. This leaves us with several possibilities for the sequence $(e'_1, \dots, e'_t)$.

Case 1. (2, 5, 2)

We know that the unique 5-tree T in F' (1|2)-merges with two diamonds D, D' via two merging pairs, say from e, e', respectively. Then by $\mathcal{G}_8^{(r)}$-freeness, we have $q(T) = 2$, as otherwise we are able to remove one flexible edge not containing the merging pairs to obtain an 8-configuration. Thus the partial 2-cluster F' belongs to $\mathcal{A}$. Note that $|V(D) \cap V(D')| \le 1$, as otherwise we could trim one edge of T and get an 8-configuration. Also, this configuration F' cannot 2-merge with further 1-clusters. Indeed, if F' $(1|2)$-merges with some 1-cluster S_0 via a merging pair from a diamond $D_0 \subseteq S_0$, then we can trim either $D \cup \{e\}$ or $D' \cup \{e'\}$ from $F' \cup D_0$ to obtain an 8-configuration. On the other hand, if F' $(2|1)$-merges with some 1-cluster S_0 via a merging pair from an edge $e_0 \in S_0$, then we can trim either D or D' from $F' \cup \{e_0\}$ to obtain an 8-configuration. Therefore $F = F'$. Also, (10) follows easily from the above claims (combined with (11)).

Case 2. $(2, 4, 1, 2)$ and its permutations

By Lemma 4.3, there is a merging sequence starting with the 1-cluster S_1 which is a single edge. Then the only edge e of S_1 is a flexible edge, and by Claim 4.8, there exists a 1-cluster in the remaining part, say S_2, $(2|1)$-merging S_1. The 1-cluster S_2 is a 2-tree or 4-tree, and e is still a flexible edge of $S_1 \cup S_2$. Applying Lemma 4.3 with the partial 2-cluster $S_1 \cup S_2$ and Claim 4.8 with the flexible edge e of $S_1 \cup S_2$, we obtain another 1-cluster, say S_3, $(2|1)$-merging S_1. If S_2 and S_3 are both 2-trees then, when 1|2-merging the remaining 4-tree S_4, we can trim one edge from S_4 (using Corollary 4.4) to get an 8-configuration. Thus S_2 and S_3 are a 2-tree and a 4-tree in some order, also they attach to S_1 via different pairs as otherwise $S_1 \cup S_2 \cup S_3$ has at least 2 flexible edges. Assume by symmetry that S_3 is a 4-tree. Then the diamond S_4 must $(2|1)$-merge S_3 via a pair u_4v_4 intersecting a flexible set of S_3 that is still flexible in $S_1 \cup S_2 \cup S_3$. This means that $F' \in \mathcal{B}$.

Let us argue that no pair in $P_{12}(S_4) \setminus \{u_4v_4\}$ is 1-claimed or 2-claimed by $S_1 \cup S_2 \cup S_3$. Indeed, suppose on the contrary that there exists a pair $ab \in P_{12}(S_4) \setminus \{u_4v_4\}$ 1-claimed or 2-claimed by $S_1 \cup S_2 \cup S_3$. If $ab \in P_{12}(S_1 \cup S_3)$, then $S_1 \cup S_3 \cup S_4$ forms a configuration of $\mathcal{G}_8^{(r)}$. Otherwise we have $ab \in P_{12}(S_2)$ (since $P_{12}(S_1 \cup S_2 \cup S_3) = P_{12}(S_1 \cup S_3) \cup P_{12}(S_2)$ in this case). Then we can trim S_1 and get an 8-configuration $S_2 \cup S_3 \cup S_4$.

Also, F' cannot 2-merge with further 1-clusters. Indeed, if F' $(1|2)$-merges with some 1-cluster S_0 via a merging pair from a diamond $D_0 \subseteq S_0$, then we can trim either $S_1 \cup S_2$ or S_4 together with an edge from S_3 containing u_4v_4 from $F' \cup D_0$ to obtain an 8-configuration. On the other hand, if F' $(2|1)$-merges with some 1-cluster S_0 via a merging pair from an edge $e_0 \in S_0$, then we can trim either S_2 or S_4 from $F' \cup \{e_0\}$ to obtain an 8-configuration. It follows that $F = F'$ and (10) holds.

In the subsequent discussions, we always denote by u_iv_i a pair via which $\bigcup_{j=1}^{i-1} S_j$ and S_i are 1|2-mergeable; note that this pair is unique if $i \le t - 1$ by (11).

Case 3. $(2, 2, 3, 2)$ and its permutations

Using Lemma 4.3, we consider the sequence starting from the 3-tree S_1. If S_1 consists of three flexible edges e_1, e_2, e_3, i.e. S_1 is a 3-star, then by Claim 4.8, there exists a 2-tree S_2 1|2-merging S_1 via some pair xy where x belongs to the flexible set of an edge of S_1, say, e_1. If S_2 $(1|2)$-merges S_1 via xy, then y belongs to the flexible set of e_2 or e_3. Without loss of generality, let us assume that $y \in e_2$ and $e \in S_2$ is an edge containing xy. Then e_3 and the edge of S_2 different from e are two flexible edges of $S_1 \cup S_2$. By using Claim 4.8 twice, the remaining two diamonds S_3, S_4 will 1|2-merge with $S_1 \cup S_2$ via pairs in e_3 and in the edge of S_2 different from e respectively. We actually know that S_3, S_4 will $(2|1)$-merge with $S_1 \cup S_2$, since otherwise we can trim one edge from S_3 or S_4 to obtain an 8-configuration. By an argument similar to those in the previous two cases, no further 2-merging is possible, as otherwise we can trim a diamond (together with an edge if needed) to obtain an 8-configuration. Also, no pair in $P_{12}(S_4) \setminus \{u_4v_4\}$ is 1-claimed or 2-claimed by $S_1 \cup S_2 \cup S_3$ by $\mathcal{G}_8^{(r)}$-freeness. Thus $F = F' \in \mathcal{C}_1$ and (10) holds. On the other hand, if S_2 $(2|1)$-merges S_1, then both x and y belong to e_1. We still have two flexible edges $e_2, e_3 \in S_1 \cup S_2$. By Claim 4.8, either S_3 $(1|2)$-merges $S_1 \cup S_2$ via some pair intersecting both e_2 and e_3 (which is exactly as in the previous case), or S_3 $(2|1)$-merges $S_1 \cup S_2$ via a pair in e_2. In the latter case, S_4 will $(2|1)$-merge $S_1 \cup S_2 \cup S_3$ via a pair of e_3. By similar arguments, $F = F' \in \mathcal{C}_1$ and (10) holds.

Now let us assume that S_1 has exactly two flexible edges e_1, e_2. Note that in this case e_1, e_2 cannot form a diamond. By Claim 4.8, there exist two diamonds S_2, S_3 1|2-merging S_1 via two pairs intersecting the flexible sets of e_1, e_2 separately. If one of S_2, S_3 $(1|2)$-merges with S_1, say S_2, then $S_1 \cup S_2 \cup S_3$ would have a flexible edge from S_2. Then S_4 will $(2|1)$-merge with $S_1 \cup S_2 \cup S_3$ (specifically, via the flexible edge of S_2). Let us argue that no pair in $P_{12}(S_4) \setminus \{u_4v_4\}$ is 1-claimed or 2-claimed by $S_1 \cup S_2 \cup S_3$. Indeed, suppose on the contrary that there exists a pair $ab \in P_{12}(S_4) \setminus \{u_4v_4\}$ 1-claimed or 2-claimed by $S_1 \cup S_2 \cup S_3$. If $ab \in P_{12}(S_1 \cup S_2)$, then $S_1 \cup S_2 \cup S_4$ forms a configuration of $\mathcal{G}_8^{(r)}$. Otherwise we have $ab \in P_{12}(S_3)$, then we can trim an edge of S_2 containing u_4v_4 from $S_1 \cup S_2 \cup S_3 \cup S_4$ and get an 8-configuration. Similarly to the previous cases, no further 2-merging is possible. Again $F = F' \in \mathcal{C}_1$ and (10) is satisfied. Suppose that both S_2, S_3 (2|1)-merge with S_1 via distinct pairs from e_1, e_2 respectively. Then S_4 also (2|1)-merges with $S_1 \cup S_2 \cup S_3$, as otherwise we are able to trim an edge of S_4 using Corollary 4.4 and get an 8-configuration. Also due to $\mathcal{G}_8^{(r)}$-freeness, (10) holds. For further mergings, F' cannot (2|1)-merge other 1-clusters, as we could always trim S_2, S_3 or S_4 to find an 8-configuration. However it is possible for F' to (1|2)-merge a 1-cluster S_5. Then S_4 and S_5 must (2|1)-merge with $S_2 \cup \{e_1\}$ and $S_3 \cup \{e_2\}$ respectively. Indeed, if not, one can trim $S_2 \cup \{e_1\}$ or $S_3 \cup \{e_2\}$, obtaining an 8-configuration. Moreover S_5 is a diamond, as otherwise we can trim S_4 and S_2. Similarly, by $\mathcal{G}_8^{(r)}$-freeness, no pair in $P_2(S_5) \setminus \{u_5v_5\}$ is 1-claimed or 2-claimed by F' and no further 2-merging is possible. Hence $F = F' \cup S_5 \in \mathcal{C}_2$ and (10) holds for F.

Case 4. (2, 1, 1, 3, 2) and its permutations

By Lemma 4.3, let us re-order the sequence starting from a 1-tree $S_1 = \{e_1\}$. Notice that neither a 3-tree nor a 1-tree can be the last 1-cluster in the sequence, as otherwise we can trim one edge from F' and the remaining r-graph is an 8-configuration. By Claim 4.8, there exists a tree S_2 (2|1)-merging S_1. Since e_1 is still a flexible edge of $S_1 \cup S_2$, we use Claim 4.8 again and obtain a 1-cluster S_3 (2|1)-merging $S_1 \cup S_2$ via a pair in e_1. The next 1-cluster S_4 in the sequence must be the remaining 1-tree (1|2)-merging $S_1 \cup S_2 \cup S_3$. Since the 3-tree is not the last 1-cluster in the sequence, one of S_2 and S_3 is the 3-tree and the other is a diamond. Assuming S_2 is a diamond and S_3 is a 3-tree, then there is an edge e of S_3 such that e is a flexible edge of $S_1 \cup S_2 \cup S_3$. Since a 1-tree cannot be the last cluster, we derive that S_4 (1|2)-merges $S_1 \cup S_2 \cup S_3$ via a pair intersecting e, and the subsequent S_5 would (2|1)-merge with S_4. By $\mathcal{G}_8^{(r)}$-freeness, no pair in $P_2(S_5) \setminus \{u_5v_5\}$ is 1-claimed or 2-claimed by $S_1 \cup S_2 \cup S_3 \cup S_4$ and F' cannot 1|2-merge more trees. Therefore $F' = F \in \mathcal{E}$ and (10) holds.

Case 5. (2, 2, 1, 2, 2) and its permutations

Let S_1 be the unique 1-tree in the sequence, and let S_2, S_3, S_4, S_5 be the remaining 4 diamonds. As before, by Claim 4.8, some two diamonds, say S_2 and S_3, (2|1)-merge S_1 via two distinct pairs of $P_1(S_1)$.

If S_4 (1|2)-merges $S_1 \cup S_2 \cup S_3$ via a pair of $e \in S_4$, then the edge of S_4 different from e is a flexible edge of $S_1 \cup S_2 \cup S_3 \cup S_4$. The last 1-cluster S_5 would (2|1)-merge S_4, since a (1|2)-merging of S_5 and S_4 would enable us to trim one edge and this contradicts 8-freeness. Similarly, due to $\mathcal{G}_8^{(r)}$-freeness, no pair in $P_2(S_5) \setminus \{u_5v_5\}$ is 1-claimed or 2-claimed by $S_1 \cup S_2 \cup S_3 \cup S_4$ and F' admits no further 1|2-mergings. This case implies $F = F' \in \mathcal{F}$ and (10).

The other case is that S_4 (2|1)-merges $S_1 \cup S_2 \cup S_3$. Then the next merging is that S_5 (2|1)-merges $S_1 \cup S_2 \cup S_3 \cup S_4$ (as otherwise F' would contain a flexible edge coming from S_5). Thus $F' \in \mathcal{S}_4$. It remains to prove that $F \supseteq F'$ is in $\mathcal{S}_i$ for some $i \in [4, 3\binom{r}{2}]$ and (10) holds.

Let F be made by starting with F' and consecutively merging 1-clusters $S_6, S_7, \dots, S_m$ of $F \setminus F'$ in this order. Call a 1-cluster S_i for $i \geq 2$ of *type ab* if, in the merging chain from S_1 to the minimal partial 2-cluster containing $S_1 \cup S_i$, the first merging step is via a pair $ab \in P_1(S_1)$. (Note that the vertices a, b are not necessarily in S_i, for example, S_i can merge with a diamond 2-claiming ab.) By convention, assume that S_1 is of all $\binom{r}{2}$ types. Observe that at least two of the initial diamonds S_2, S_3, S_4, S_5 must be of different types (as otherwise $S_2 \cup S_3 \cup S_4 \cup S_5$ would be an 8-configuration).

Denote $H_i := S_1 \cup \cdots \cup S_{i-1}$. To prove that $F \in \mathcal{S}_i$ for some $i \in [4, 3\binom{r}{2}]$ and (10) holds, it suffices to prove the following.

Claim 4.9. *For every $i \in [2, m]$, S_i is a diamond that $\overline{1}2$-claims some previously 1-claimed pair $x_iy_i \in P_1(H_i)$, and no pair in $P_{\overline{1}2}(S_i) \setminus \{x_i, y_i\}$ is 1-claimed or 2-claimed by H_i. Also, $m \leq 1 + 3\binom{r}{2}$.*

Proof. We prove the first part by induction on $i \in [2, m]$. It is easy to check in the base case $i \in [5]$. Let $i \geq 6$ and let the 1-cluster S_i be of type ab. Note that we have at most 3 diamonds of each type (otherwise the first four diamonds of any fixed type would form a forbidden 8-configuration). If some edge $e' \in S_i$ 1-claims a pair $\overline{1}2$-claimed by H_i, then by keeping only this edge e' and removing one by one diamonds of types different from ab, we can reach a sub-structure with exactly 8 edges, a contradiction. So let the diamond $D_i \subseteq S_i$ $\overline{1}2$-claim a pair $x_iy_i \in P_1(H_i)$. There are at most two previous diamonds S_j, S_k of the same type as S_i (otherwise D_i with three such diamonds would give an 8-configuration). It follows that $D_i = S_i$ as otherwise a forbidden 8-edge configuration would be formed by S_1, D_i, some suitable edge of $S_i \setminus D_i$ plus diamonds S_j, S_k of type ab (if exist) or one or two diamonds $\overline{1}2$-claiming a pair in $P_1(S_1) \setminus \{ab\}$ (such diamonds exist among S_2, S_3, S_4). If S_i contains some other vertex $z_i \notin \{x_i, y_i\}$ from an earlier 1-cluster of the same type ab, then some edge e' of S_i shares at least two vertices with H_i, again leading to a forbidden 8-configuration in $H_i \cup \{e'\}$. Thus we are done (with proving the inductive statement and thus $F \in \mathcal{S}_i$ where $i \in [4, 3\binom{r}{2}]$ and (10) holds for this F) unless $P_{\overline{1}2}(S_i)$ contains a pair uv with both vertices in a 1-cluster S_ℓ of some different type $a'b'$ (where $\{u, v\}$ may possibly intersect $\{x_i, y_i\}$). If the number of earlier diamonds of types ab and $a'b'$ is at least 2 in total, then by trimming some diamonds if necessary, we can obtain exactly two diamonds that, together with S_i, S_ℓ, form an 8-configuration; otherwise we have at most 3 diamonds of type ab or $a'b'$ (including S_i, S_k) and these diamonds together with S_1 have $i \le 7$ edges and form an $(i(r-2)+1, i)$-configuration, a contradiction to $\mathcal{G}_8^{(r)}$-freeness.

The final inequality $m \le 1 + 3\binom{r}{2}$ follows from the fact that there are at most 3 diamonds of each type $ab \in P_1(S_1)$. □

Case 6. $(2, 2, 1, 1, 1, 2)$ and its permutations

This case is actually impossible. Indeed, let us consider the sequence starting from a 1-tree S_1. Using Claim 4.8 twice, note that some two diamonds, say S_2 and S_3, would $(2|1)$-merge S_1 via two distinct pairs of $P_1(S_1)$. Observe that a 1-tree cannot be the last 1-cluster in the sequence, as otherwise we can trim it from F' and obtain a forbidden 8-configuration. Therefore, the next two 1-clusters S_4, S_5 are 1-trees $(1|2)$-merging with the previous partial 2-clusters (and they cannot form a diamond by the merging rule). So S_4, S_5 are two flexible edges of $\cup_{i=1}^{5} S_i$. However, the subsequent merging of the diamond S_6 can eliminate only one flexible edge, say S_4, from $\cup_{i=1}^{5} S_i$, and we get a forbidden 8-configuration by removing S_5 from F'.

Case 7. $(2, 1, 1, 1, 1, 1, 2)$ and its permutations

This case is also impossible. Again consider the sequence starting from a 1-tree S_1. Using Claim 4.8 twice, the two diamonds, say S_2, S_3, would $(2|1)$-merge S_1 via two distinct pairs of $P_1(S_1)$. This contradicts to the fact that a 1-tree cannot be the last 1-cluster in the sequence.

It remains to prove (10) for a 2-cluster F with $|F| < 9$. Here, by 8-freeness, we have $|F| \le 7$. When we construct F by merging 1-clusters one by one, by $\mathcal{G}_8^{(r)}$-freeness each new 1-cluster shares exactly 2 vertices with the current configuration (namely, the pair of vertices through which the merging occurs). Thus (10) follows.

This finishes the proof of Lemma 4.7. □

4.2. Assigning weights

For every 2-cluster F and every pair $xy \in \binom{V(F)}{2}$, we will define some real $w_F(xy) \in [0, 1]$ which we will call the *weight* of xy given by F. Then we define

$$w(F) := \sum_{xy \in \binom{V(F)}{2}} w_F(xy), \quad \text{for } F \in \mathcal{M}_2,$$

and

$$w(xy) := \sum_{\substack{F \in \mathcal{M}_2 \\ xy \in \binom{V(F)}{2}}} w_F(xy), \quad \text{for } xy \in \binom{V(G)}{2}.$$

If the following inequalities hold for every $xy \in \binom{V(G)}{2}$ and $F \in \mathcal{M}_2$:

$$w(xy) \le 1, \tag{12}$$

$$w(F) \ge \binom{r}{2}|F|, \tag{13}$$

then we would have that

$$|G| = \sum_{F\in\mathcal{M}_2} |F| \le \sum_{F\in\mathcal{M}_2} \binom{r}{2}^{-1} w(F) = \binom{r}{2}^{-1} \sum_{xy\in\binom{V(G)}{2}} w(xy) \le \binom{r}{2}^{-1}\binom{n}{2}, \tag{14}$$

giving the desired upper bound.

We shall adopt two different weight assignment strategies for the cases when $r \ge 5$ and $r = 4$.

4.2.1. Weight functions for $r \ge 5$

For a 2-cluster F and a pair $uv \in \binom{V(F)}{2}$, we define the weight

$$w_F(uv) := \begin{cases} 1 & \text{if } 1 \in C_F(uv), \\ 1/3 & \text{if } 2 \in C_F(uv) \text{ and } 1 \notin C_F(uv), \\ 0 & \text{otherwise.} \end{cases}$$

The following claim shows that our weights satisfy (12).

Claim 4.10. *For every $uv \in \binom{V(G)}{2}$, it holds that $w(uv) \le 1$.*

Proof. Given $uv \in \binom{V(G)}{2}$, let $F_1, \ldots, F_s \in \mathcal{M}_2$ be all 2-clusters assigning positive weight to uv, ordered so that $w_{F_1}(uv) \ge w_{F_2}(uv) \ge \cdots \ge w_{F_s}(uv)$. By the definition of weights, each F_i satisfies either that $1 \in C_F(uv)$ or that $2 \in C_F(uv)$ and $1 \notin C_F(uv)$.

If $w_{F_1}(uv) = 1$, namely $1 \in C_{F_1}(uv)$, then $2 \notin C_F(uv)$ for any other 2-cluster $F \in \mathcal{M}_2 \setminus \{F_1\}$, as otherwise we would merge them together by the merging rule of $\mathcal{M}_2$. Therefore, $s = 1$ and $w(uv) = w_{F_1}(uv) = 1 \le 1$.

If $w_{F_1}(uv) = 1/3$ (which implies $2 \in C_{F_1}(uv)$), then by Lemma 4.2, the number of 2-clusters $F \in \mathcal{M}_2 \setminus \{F_1\}$ with $2 \in C_F(uv)$ is at most 2. Thus $s \le 3$ and $w(uv) \le 3 \cdot w_{F_1}(uv) \le 1$. □

Now, it is sufficient to verify (13) for every $F \in \mathcal{M}_2$.

Claim 4.11. *Let $r \ge 5$. For all $F \in \mathcal{M}_2$, we have $w(F) \ge \binom{r}{2}|F|$.*

Proof. Suppose that F is obtained by merging m 1-clusters $T_1, \ldots, T_m \in \mathcal{M}_1$ with $|T_i| = e_i$. Then by Lemma 4.7, we have

$$|F| = \sum_{i\in[m]} e_i, \quad |P_1(F)| = \sum_{i\in[m]} \left(e_i\binom{r}{2} - e_i + 1\right) \text{ and } |P_{\overline{1}2}(F)| \ge 1 - m + \sum_{i\in[m]} (e_i - 1)(r-2)^2.$$

Recall that

$$w(F) := \sum_{xy\in\binom{V(F)}{2}} w_F(xy) = |P_1(F)| + \frac{1}{3}|P_{\overline{1}2}(F)|.$$

By routine calculations, we have from the above that

$$\begin{aligned} 2w(F) - (r^2 - r)|F| &\ge (r^2 - r - 2)|F| + \frac{4m}{3} + \frac{2}{3} + \frac{2(r^2 - 4r + 4)}{3} \cdot (|F| - m) - (r^2 - r)|F| \\ &= \frac{2(r^2 - 4r + 1)}{3}|F| + \frac{4m}{3} + \frac{2}{3} - \frac{2(r^2 - 4r + 4)}{3} m \\ &= \frac{2}{3}(r^2 - 4r + 1) \cdot (|F| - m) - \frac{2}{3} m + \frac{2}{3}. \end{aligned} \tag{15}$$

To see that the right-hand side of (15) is at least 0, it suffices to verify the following:

$$|F| \ge m + \frac{m - 1}{r^2 - 4r + 1}. \tag{16}$$

Note that $|F| \geq m+1$, since there must exist an i-tree in the sequence with $i \geq 2$ according to the merging rule. Hence given that $r \geq 5$, if $m \leq 7$ then we derive that

$$|F| \geq m+1 \geq m+\frac{m-1}{6} \geq m+\frac{m-1}{r^2-4r+1}.$$

On the other hand, if $m \geq 8$, then $|F| \geq 9$. By Lemma 4.7, we conclude that $F \in \mathcal{S}_{m-1}$ (as any other family has $m < 8$). Thus,

$$|F| = 2(m-1)+1 \geq m+\frac{m-1}{r^2-4r+1}.$$

This finishes the proof of the upper bound for $r \geq 5$. □

4.2.2. Weight functions for $r = 4$

For a 2-cluster F and $uv \in \binom{V(G)}{2}$, we define the following functions h_i^F:

$$h_1^F(uv) := \begin{cases} 1 & \text{if } 1 \in C_F(uv), \\ 0 & \text{otherwise,} \end{cases} \qquad h_2^F(uv) := \begin{cases} 1/3 & \text{if } 2 \in C_F(uv), \\ 0 & \text{otherwise,} \end{cases}$$

$$h_3^F(uv) := \begin{cases} 1/2 & \text{if } 2,4 \in C_F(uv), \\ 0 & \text{otherwise,} \end{cases} \qquad h_4^F(uv) := \begin{cases} 1/2 & \text{if } 3,4,5 \in C_F(uv) \text{ and } uv \notin P_1(G), \\ 0 & \text{otherwise,} \end{cases}$$

$$h_5^F(uv) := \begin{cases} 1 & \text{if } 3,5,6 \in C_F(uv) \text{ and } uv \notin P_1(G), \\ 0 & \text{otherwise.} \end{cases}$$

We assign the weight

$$w_F(uv) := \max_{1 \leq i \leq 5} h_i^F(uv).$$

Claim 4.12. *For every $uv \in \binom{V(G)}{2}$, it holds that $w(uv) \leq 1$.*

Proof. Given $uv \in \binom{V(G)}{2}$, let $F_1, \ldots, F_s \in \mathcal{M}_2$ be all 2-clusters assigning positive weight to uv, ordered so that $w_{F_1}(uv) \geq w_{F_2}(uv) \geq \cdots \geq w_{F_s}(uv)$. In the following proof, let h_i mean $h_i^{F_1}$, the function coming from the 2-cluster F_1 (that assigns the maximum weight to uv).

Case 1. $w_{F_1}(uv) = 1$.

There exists some $i \in \{1, 5\}$ such that $w_{F_1}(uv) = h_i(uv) = 1$.

If $w_{F_1}(uv) = h_1(uv) = 1$, then $1 \in C_{F_1}(uv)$, which means that $uv \in P_1(G)$. Thus, given an arbitrary 2-cluster $F \in \mathcal{M}_2 \setminus \{F_1\}$, it holds that $h_4^F(uv) = h_5^F(uv) = 0$. Note that $2 \notin C_F(uv)$, since otherwise F_1 would merge with F by our merging rule for $\mathcal{M}_2$. Thus $h_2^F(uv) = h_3^F(uv) = 0$. Analogously, by the merging rule for $\mathcal{M}_1$, we have $1 \notin C_F(uv)$ and $h_1^F(uv) = 0$. Hence, $w_F(uv) = 0$ and $w(uv) = w_{F_1}(uv) = 1$.

If $w_{F_1}(uv) = h_5(uv) = 1$, then $3, 5, 6 \in C_{F_1}(uv)$ and $uv \notin P_1(G)$. For every other 2-cluster $F \in \mathcal{M}_2 \setminus \{F_1\}$, we know by Lemma 4.2 that $2, 3 \notin C_F(uv)$ and thus $h_2^F(uv) = h_3^F(uv) = h_4^F(uv) = h_5^F(uv) = 0$. Since $uv \notin P_1(G)$, $h_1^F(uv) = 0$ as well. In total, we have $w_F(uv) = 0$ and $w(uv) = w_{F_1}(uv) = 1$.

Case 2. $w_{F_1}(uv) = 1/2$.

In this case, we know $w_{F_1}(uv) \in \{h_3(uv), h_4(uv)\}$. This means $4 \in C_{F_1}(uv)$. For every other 2-cluster $F \in \mathcal{M}_2 \setminus \{F_1\}$, we derive by Lemma 4.2 that $4 \notin C_F(uv)$, and thus $w_F(uv) \leq h_2^F(uv) = 1/3$. Again by Lemma 4.2, there is at most one 2-cluster of $\mathcal{M}_2 \setminus \{F_1\}$, say F_2, such that $2 \in C_{F_2}(uv)$ and $w_{F_2}(uv) = h_2^{F_2}(uv) = 1/3$. Therefore, $w(uv) \leq w_{F_1}(uv) + w_{F_2}(uv) \leq 1/2 + 1/3 \leq 1$.

Case 3. $w_{F_1}(uv) = 1/3$.

In this case, we know $w_{F_1}(uv) = h_2(uv)$ which implies $2 \in C_{F_1}(uv)$. By Lemma 4.2, there are at most two 2-clusters in $\mathcal{M}_2 \setminus \{F_1\}$ that 2-claim the pair uv. Each of these clusters (if exists) contributes at most 1/3 while all other clusters do not contribute anything to the weight of uv. Thus $w(uv) \leq w_{F_1}(uv) + 2/3 = 1$. □

Claim 4.13. *Let $r = 4$. For all $F \in \mathcal{M}_2$, we have $w(F) \geq 6\,|F|$.*

Proof. Let h_i refer to h_i^F. Assume that F is obtained by merging m 1-clusters $T_1, \ldots, T_m \in \mathcal{M}_1$ with $|T_i| = e_i$. Let us first focus on 1-claimed pairs $P_1(F)$ and $\overline{1}2$-claimed pairs $P_{\overline{1}2}(F)$. Each pair of $P_1(F)$ will contribute the weight 1 (by the definition of h_1), and each pair of $P_{\overline{1}2}(F)$ contribute at least weight $1/3$ (by the definition of h_2). We start with determining those 2-clusters F such that the weights contributed by $P_1(F) \cup P_{\overline{1}2}(F)$ is sufficient, i.e. $w(F) \geq |P_1(F)| + |P_{\overline{1}2}(F)|/3 \geq 6\,|F|$.

By Lemma 4.7, we have

$$|F| = \sum_{i\in[m]} e_i, \;\; |P_1(F)| = \sum_{i\in[m]} (5e_i + 1) \;\text{ and }\; |P_{\overline{1}2}(F)| \geq 1 - m + 4\sum_{i\in[m]} (e_i - 1).$$

Then,

$$\begin{aligned} w(F) = \sum_{xy\in\binom{V(F)}{2}} w_F(xy) &\geq |P_1(F)| + \frac{1}{3}\,|P_{\overline{1}2}(F)| \\ &\geq 5\,|F| + m + \frac{1}{3}\,(1 - m + 4|F| - 4m) \\ &= \frac{19}{3}\,|F| - \frac{2}{3}\,m + \frac{1}{3}. \end{aligned} \tag{17}$$

Thus if $|F| \geq 2m - 1$, then the expression in (17) is at least $6\,|F|$.

Now it is sufficient to consider the case where $|F| < 2m - 1$. We split the remaining discussion depending on the value of m.

Case 1. $m \leq 2$.

This case is impossible. Indeed, by the definition of 2-merging, we have at least one 1-cluster of size at least 2 in the sequence of forming F. However, then $|F| \geq m + 1 \geq 2m - 1$.

Case 2. $m = 3$.

We know $4 = m + 1 \leq |F| < 2m - 1 = 5$. Thus $|F| = 4$ and F has a composition $(2, 1, 1)$ where a diamond $D \subseteq F$ $\overline{1}2$-claims two pairs x_1y_1, x_2y_2 through which two 1-trees merge to D. These pairs are distinct as otherwise the two 1-trees would be in the same 1-cluster. Also, note that $2, 4 \in C_F(x_1y_1) \cap C_F(x_2y_2)$. Thus $w_F(x_iy_i) \geq h_3(x_iy_i) = 1/2$, and

$$w(F) \geq \sum_{xy\in P_1(F)} w_F(xy) + w_F(x_1y_1) + w_F(x_2y_2) \geq 23 + \frac{1}{2} + \frac{1}{2} = 24 = 6 \cdot 4.$$

Case 3. $m = 4$.

In this case, we have $5 = m+1 \leq |F| < 2m-1 = 7$. We split the proof into two cases depending on the size of F.

First, suppose that $|F| = 5$. Then F possesses a composition $(2, 1, 1, 1)$. Consider an arbitrary 1-tree T and the diamond D ($\overline{1}2$-claiming a pair in T) in the sequence. It is easy to see that $|P_{\overline{123}}(F)| \geq |P_{\overline{123}}(D \cup T)| = 8$. By 8-freeness, at most 2 pairs of $P_{\overline{123}}(F)$ lie in $P_1(G)$. Therefore, the remaining pairs $uv \in P_{\overline{123}}(D \cup T) \setminus P_1(G)$ satisfy $3, 4, 5 \in C_F(uv)$ and $w_F(uv) \geq h_4(uv) = 1/2$. Hence, by Lemma 4.7, we get

$$w(F) \geq \sum_{xy\in P_1(F)} w_F(xy) + \sum_{xy\in P_{\overline{123}}(D\cup T)\setminus P_1(G)} w_F(xy) \geq 29 + (8 - 2) \times \frac{1}{2} = 32 > 6 \cdot 5,$$

as desired.

Thus it remains to consider the case $|F| = 6$. Then the non-increasing composition of F is $(2, 2, 1, 1)$ or $(3, 1, 1, 1)$.

Let us show that there is at least one flexible 1-tree T_0 in F. Indeed, for $(2, 2, 1, 1)$, re-order the sequence starting from a 1-tree e; if e is not flexible edge of F, then the two diamonds $(2|1)$-merge with e and the other 1-tree is flexible. For $(3, 1, 1, 1)$, each 1-tree is flexible. So a flexible 1-tree T_0 exists.

Let $T \neq T_0$ be another 1-tree in F (which could be flexible), and let $D \subseteq F$ be a diamond 2-claiming a pair of $P_1(T)$. As before, $|P_{\overline{1}23}(D \cup T)| = 8$. At most one pair of $|P_{\overline{1}23}(D \cup T)|$ lies in $P_1(G)$ by $\mathcal{G}_8^{(4)}$-freeness. Moreover, since F has a flexible edge (namely, the one in T_0), each pair uv of $P_{\overline{1}23}(D \cup T) \setminus P_1(G)$ satisfies $5, 6 \in C_F(uv)$ and, hence, $w_F(uv) \geq h_5(uv) = 1$. Together with Lemma 4.7, this implies that

$$w(F) \geq \sum_{xy \in P_1(F)} w_F(xy) + \sum_{xy \in P_{\overline{1}23}(D \cup T) \setminus P_1(G)} w_F(xy) \geq 34 + 7 \times 1 = 41 > 6 \cdot 6.$$

Case 4. $m = 5$.

In this case, we have $6 = m + 1 \leq |F| < 2m - 1 = 9$. By 8-freeness, $|F| \neq 8$.

First suppose that $|F| = 6$. Its non-increasing composition must be (2, 1, 1, 1, 1). Here every 1-tree of F is flexible. Consider the diamond D and a 1-tree T with D 2-claiming a pair in T. As before, $|P_{\overline{1}23}(D \cup T)| = 8$ and, by $\mathcal{G}_8^{(4)}$-freeness, there is at most one pair in $P_{\overline{1}23}(D \cup T)$ that lies in $P_1(G)$. By trimming one flexible edge from F if needed, we know that each pair $uv \in P_{\overline{1}23}(D \cup T) \setminus P_1(G)$ satisfies $5, 6 \in C_F(uv)$. Hence, $w_F(uv) \geq h_5(uv) = 1$ and by Lemma 4.7,

$$w(F) \geq \sum_{xy \in P_1(F)} w_F(xy) + \sum_{xy \in P_{\overline{1}23}(D \cup T) \setminus P_1(G)} w_F(xy) \geq 35 + 7 \times 1 = 42 > 6 \cdot 6.$$

If $|F| = 7$ with a composition (2, 2, 1, 1, 1) or (3, 1, 1, 1, 1), then we obtain at least two flexible 1-trees T_0, T_0'. Indeed, for (3, 1, 1, 1, 1), every 1-tree is flexible. For (2, 2, 1, 1, 1), every non-flexible 1-tree would be 2-claimed by both diamonds. By $\mathcal{G}_8^{(4)}$-freeness, the number of non-flexible 1-trees is at most 1. Consider another 1-tree $T \neq T_0, T_0'$ (T can be flexible as well) and a diamond D 2-claiming a pair in T. None of the pairs in $P_{\overline{1}23}(D \cup T)$ lie in $P_1(G)$ due to $\mathcal{G}_8^{(4)}$-freeness. Each pair $uv \in P_{\overline{1}23}(D \cup T)$ satisfies $5, 6 \in C_F(uv)$ since F has two flexible edges T_0, T_0' not in $D \cup T$. Thus we have $w_F(uv) \geq h_5(uv) = 1$ and

$$w(F) \geq \sum_{xy \in P_1(F)} w_F(xy) + \sum_{xy \in P_{\overline{1}23}(D \cup T)} w_F(xy) \geq 40 + 8 \times 1 = 48 > 6 \cdot 7.$$

Case 5. $m \geq 6$.

In this case, we have $|F| \geq m + 1 \geq 7$. However when $|F| = 7$, the unique composition (2, 1, 1, 1, 1, 1) is impossible. Indeed, recall that each $\overline{1}2$-claimed pair in a 4-uniform diamond can be used for merging a 1-tree only once, as otherwise those 1-trees would already have been merged when constructing $\mathcal{M}_1$. On the other hand, the number of $\overline{1}2$-claimed pairs in a 4-uniform diamond is at most 4. So a diamond cannot be merged with five 1-trees.

Hence $|F| \geq 9$. By Lemma 4.7, we know that $F \in \mathcal{S}_i$ for some $i \in [4, 18]$ and $m = i + 1$, as any other configuration stated in Lemma 4.7 with more than 9 edges satisfies $m \leq 6$. However, in this case, we have

$$|F| = 2i + 1 = 2m - 1,$$

which is a contradiction to our assumption that $|F| < 2m - 1$.

This finishes the proof, since we considered every possible 2-cluster $F \in \mathcal{M}_2$. □

4.3. Putting all together

Proof of the upper bound in Theorem 1.1. By Lemma 4.1, it is enough to prove that the size of an arbitrary $\mathcal{G}_8^{(r)}$-free n-vertex r-graph G is at most $\binom{r}{2}^{-1}\binom{n}{2}$. As before, let $\mathcal{M}_1$ (resp. $\mathcal{M}_2$) denote the partition of $E(G)$ into 1-clusters (resp. 2-clusters).

For each 2-cluster F and every pair $xy \in \binom{V(F)}{2}$, we define weight functions $w_F(xy)$ as in Section 4.2.1 (for $r \geq 5$) or Section 4.2.2 (for $r = 4$). By Claims 4.10–4.13, the inequality in (14) proves the desired upper bound. □

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