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Decomposable graphs and definitions with no quantifier alternation

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Abstract

Let $D(G)$ be the minimum quantifier depth of a first order sentence Φ that defines a graph G up to isomorphism. Let $D_0(G)$ be the version of $D(G)$ where we do not allow quantifier alternations in Φ. Define $q_0(n)$ to be the minimum of $D_0(G)$ over all graphs G of order n.

We prove that for all n we have

$$\log^* n - \log^* \log^* n - 2 \le q_0(n) \le \log^* n + 22,$$

where $\log^* n$ is equal to the minimum number of iterations of the binary logarithm needed to bring n to 1 or below. The upper bound is obtained by constructing special graphs with modular decomposition of very small depth.

1. Introduction

We are interested in defining a given graph G in first order logic, being as succinct as possible. In order to state this problem formally, we have to specify what we mean by the terms *defining*, *succinct*, etc.

The vocabulary consists of the following symbols:

- *variables* (x, y, y_1, etc);
- the *relations* $=$ (equality) and $\sim$ (graph adjacency);

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- the *quantifiers* $\forall$ (universality) and $\exists$ (existence);
- the usual Boolean *connectives* ($\vee$, $\wedge$, and $\neg$);
- *parentheses* (to indicate or change the precedence of operations).

These can be combined into *first order formulas* accordingly to the standard rules. The term *first order* means that the variables represent vertices so the quantifiers apply to vertices only. In this paper, a *sentence* is a first order formula without free variables. On the intuitive level it is perfectly clear what we mean when we say that a sentence Φ *is true* on a graph G. This is denoted by $G \models \Phi$; we write $G \not\models \Phi$ for its negation (Φ *is false* on G). We do not formalize these notions. A more detailed discussion can be found in e.g. [15, Section 1].

Of course, if $G \models \Phi$ and $H \cong G$ (i.e. H is isomorphic to G), then $H \models \Phi$. On the other hand, for any finite graph G it is possible to find a sentence Φ which *defines* G, that is, $G \models \Phi$ while $H \not\models \Phi$ for any $H \not\cong G$. Indeed, let $V(G) = \{v_1, \ldots, v_n\}$ be the vertex set of G and $E(G)$ be its edge set. The required sentence could read:

$$\begin{aligned}\Phi = {} & \exists x_1 \cdots \exists x_n \; (\mathrm{Distinct}(x_1, \ldots, x_n) \wedge \mathrm{Adj}(x_1, \ldots, x_n)) \\ & \wedge\; \forall x_1 \cdots \forall x_{n+1} \neg\, \mathrm{Distinct}(x_1, \ldots, x_{n+1}),\end{aligned} \tag{1}$$

where, for the notational convenience, we use the following shorthands:

$$\mathrm{Distinct}(x_1, \ldots, x_k) = \bigwedge_{1 \le i < j \le k} \neg\,(x_i = x_j)$$

$$\mathrm{Adj}(x_1, \ldots, x_n) = \bigwedge_{\{v_i, v_j\} \in E(G)} x_i \sim x_j \;\wedge \bigwedge_{\{v_i, v_j\} \notin E(G)} \neg\,(x_i \sim x_j).$$

In other words, we first specify that there are n distinct vertices, list the adjacencies and non-adjacencies between them, and then state that the total number of vertices is at most n.

A defining sentence Φ is not unique, so we are interested in finding one which is as succinct as possible. All natural succinctness measures of Φ are of interest:

- the *length* $L(\Phi)$ which is the total number of symbols in Φ (each variable symbol contributes 1);
- the *quantifier depth* $D(\Phi)$ which is the maximum number of nested quantifiers in Φ;
- the *width* $W(\Phi)$ which is the number of variables used in Φ (different occurrences of the same variable are not counted).

For example, for the sentence in (1) we have $L(\Phi) = \Theta(n^2)$ and $D(\Phi) = W(\Phi) = n+1$. All three characteristics inherently arise in the analysis of the computational problem of checking if a Φ is true on a given graph, see e.g. Grädel [8]. They give us a small hierarchy of descriptive complexity measures for graphs: $L(G)$ (resp. $D(G)$, $W(G)$) is the minimum of $L(\Phi)$ (resp. $D(\Phi)$, $W(\Phi)$) over all sentences Φ defining G. These graph invariants will be referred to as the *logical length, depth*, and *width* of G. We have

$$W(G) \le D(G) \le L(G).$$

The former number is of relevance for graph isomorphism testing, see Cai et al. [4]. The parameters $W(G)$ and $D(G)$ admit a purely combinatorial characterization in terms of the Ehrenfeucht game, see [4,15].

Here, we address the logical depth of graphs which was recently studied in Bohman et al. [1, 9,11–13,16,18]. We focus on the following general question: How do restrictions on logic affect the descriptive complexity of a graph? Call a sentence Φ *a-alternating* if it contains negations only in front of relation symbols and every sequence of nested quantifiers in Φ has at most a quantifier alternations, that is, the occurrences of $\forall\exists$ and $\exists\forall$. Let $D_a(G)$ denote the variant of $D(G)$ for a-alternating defining sentences. Clearly, for any integer $a \geq 0$ we have

$$D(G) \leq D_{a+1}(G) \leq D_a(G).$$

For example, the sentence in (1) has no alternations. Thus it shows that for any graph G we have

$$D_0(G) \leq v(G) + 1, \tag{2}$$

where $v(G)$ denotes the number of vertices in G. This bound is in general best possible: for example, $D_0(K_n) = D(K_n) = n + 1$. In Kim et al. [9] we proved that $D(G) = \log_2 n - \Theta(\log_2 \log_2 n)$ and $D_0(G) \leq (2 + o(1)) \log_2 n$ for almost all graphs G of order n.

In the above results, the functions $D(G)$ and $D_0(G)$ are the same or differ by at most a constant factor. However, they can be very far apart in general. In [11, Corollary 5.7] we demonstrated a *superrecursive gap* between $D(G)$ and $D_0(G)$: namely, we proved that for any total recursive function f there is a graph G with $D_0(G) > f(D(G))$. This is not too surprising, since the logic of 0-alternating sentences is very restrictive and provably weaker than the unbounded first order logic. Whereas the problem of deciding if a first order sentence is satisfiable by some finite graph is unsolvable, it becomes solvable if restricted to 0-alternating sentences. The former of these facts is the content of Trakhtenbrot's theorem [17], while the latter dates from Ramsey's logical work [14] founding the combinatorial Ramsey theory (see Nešetřil [10, pp. 1336–1337] for historical comments on the relations between Ramsey theory and logic).

Given Ramsey's decidability result, it is reasonable to concentrate on the first order definability with no quantifier alternation. As our main result here (Theorem 1), we determine the asymptotic behavior of the *succinctness function* $q_0(n)$, where for an integer $a \geq 0$ we define

$$q_a(n) = \min \{D_a(G) : v(G) = n\}.$$

Let *log-star* $\log^* n$ be equal to the minimum number of iterations of the binary logarithm needed to bring n to 1 or below.

Theorem 1. *For all n we have*

$$\log^* n - \log^* \log^* n - 2 \leq q_0(n) \leq \log^* n + 22. \tag{3}$$

The estimates (3) are in sharp contrast to the result in [11, Corollary 9.1] which shows a superrecursive gap between

$$q(n) = \min \{D(G) : v(G) = n\}$$

and n. Thus Theorem 1, besides being an interesting result on its own, implies that we cannot have $q_0(n) \leq f(q(n))$ for some total recursive f and all n. This implies, again, a superrecursive gap between the graph invariants $D(G)$ and $D_0(G)$.

The upper bound in (3) improves a couple of earlier results. In [11, Theorem 7.1] a weaker bound

$$q_0(n) \leq 2 \log^* n + O(1) \tag{4}$$

is proved for only an infinite sequence of values of n. The best bound of this kind that was known to hold for all n is $q_3(n) \leq \log^* n + O(1)$. It is a direct corollary of [9, Theorem 20] saying that for the Erdős–Rényi evolutional random graph model $G(n, p)$ we have $D_3(G(n, n^{-1/4})) = \log^* n + O(1)$.

Note that (4) is proved in [11] by inductively constructing large asymmetric trees and estimating $D_0(G)$ in terms of their (very small) radius. Here, our construction produces a graph of large order that has very short *modular decomposition* (as defined in Brandstädt et al. [3, Section 1.5]), starting with small complement-connected graphs. It seems feasible that many other recursively defined constructions of graphs (see Borie et al. [2] and Brandstädt et al. [3, Section 11] for surveys) may lead to upper bounds on $q_0(n)$ compatible with (3). However, the proof of the upper bound in (3) required from us many delicate auxiliary lemmas, even though we chose a construction which is, in our opinion, most suitable for our purposes. So, a general theorem would probably be very messy and difficult to prove.

In [11, Theorem 9.3] we have shown that

$$\log^* n - \log^* \log^* n - 2 \le q(n) \le \log^* n + 4,$$

where the upper bound holds for all n while the lower bound holds, inevitably, for only infinitely many n. Combined with Theorem 1 and the obvious inequalities

$$q_0(n) \ge q_a(n) \ge q(n), \quad \text{for any integer } a \ge 1,$$

this implies that for any fixed a we have $q_a(n) = (1 + o(1)) \log^* n$ for infinitely many n. We do not even know if $q_1(n) = (1 + o(1)) \log^* n$ for all large n.

In fact, Theorem 1 holds also for digraphs, where instead of the adjacency relation $\sim$ we use the relation $x \mapsto y$ to denote that the ordered pair (x, y) is an arc. For example, the digraph version of the lower bound in (3) reads as follows.

Theorem 2. *For any digraph G on n vertices we have*

$$D_0(G) \ge \log^* n - \log^* \log^* n - 2. \tag{5}$$

Let us see how these results are related. Take any graph G and a 0-alternating sentence Φ defining it. Let the digraph G' be obtained from G by replacing each edge $\{x, y\} \in E(G)$ by a pair of arcs (x, y) and (y, x). Then the sentence

$$(\forall x\, \neg\, (x \mapsto x)) \wedge (\forall x \forall y\ \ ((x \mapsto y) \wedge (y \mapsto x)) \vee (\neg(x \mapsto y) \wedge \neg(y \mapsto x))) \wedge \Phi'$$

defines G', where Φ' is obtained from Φ by replacing each occurrence of $x \sim y$ by $x \mapsto y$. Thus

$$D_0(G') \le \max(2, D_0(G)) = D_0(G).$$

This shows that it is enough to prove the upper bound in Theorem 1 and the lower bound of Theorem 2. Our proofs in fact show that for any fixed $k \ge 2$ we have

$$\log^* n - \log^* \log^* n - O(1) \le q_0^{(k)}(n) \le \log^* n + O(1),$$

where $q_0^{(k)}(n)$ is the smallest quantifier depth of a 0-alternating sentence defining an n-element structure over a k-ary vocabulary.

2. Definitions

The abbreviation ‘iff’ means ‘if and only if.’ We denote $[m, n] = \{m, m + 1, \ldots, n\}$ and $[n] = [1, n]$. We define the *tower-function* by $\text{Tower}(0) = 1$ and $\text{Tower}(i) = 2^{\text{Tower}(i-1)}$ for each subsequent i. Note that $\log^* n \le i$ iff $\mathrm{Tower}(i) \ge n$. The notation $x \in^i X$ means $x \in X$ for odd i and $x \notin X$ for even i. (The mnemonic rule to remember which is which is $\in^1 = \in$.)

All graphs are supposed to be finite with non-empty vertex set. We use the following graph notation: $\overline{G}$ is the complement of G; $G \sqcup H$ is the vertex-disjoint union of graphs G and H; $G \subset H$ means that G is isomorphic to an induced subgraph of H (we will say that G is *embeddable* into H). For graphs (resp. sets) A and B the relation $A \subset B$ does not exclude the case of isomorphism $A \cong B$ (resp. equality $A = B$).

We call G *complement-connected* if both G and $\overline{G}$ are connected. An inclusion-maximal complement-connected induced subgraph of G will be called a *complement-connected component* of G or, for brevity, *cocomponent* of G. Cocomponents have no common vertices and their vertex sets partition $V(G)$.

The *decomposition* of G, denoted by $Dec\,G$, is the set of all connected components of G (this is a set of graphs, not just isomorphism types). Furthermore, given $i \ge 0$, we define the *depth i decomposition* $Dec_i\,G$ of G by

$$Dec_0\,G = Dec\,G \quad \text{and} \quad Dec_{i+1}\,G = \bigcup_{F \in Dec_i\,G} Dec\,\overline{F}.$$

Note that $Dec_i\,G$ consists of connected graphs, and distinct vertices x, y of an $F \in Dec_i\,G$ are adjacent in F if and only if $\{x, y\} \in^{i+1} E(G)$. Moreover,

$$P_i = \{V(F) : F \in Dec_i\,G\} \tag{6}$$

is a partition of $V(G)$ and P_{i+1} refines P_i. The *depth i environment* of a vertex $v \in V(G)$, denoted by $Env_i(v; G)$, is the graph F in $Dec_i\,G$ containing v. If the underlying graph G is clear from the context, we will usually write $Env_i(v)$.

We define the *rank* of a graph G, denoted by $rk\,G$, inductively as follows:

- If G is complement-connected, then $rk\,G = 0$.
- If G is connected but not complement-connected, then $rk\,G = rk\,\overline{G}$.
- If G is disconnected, then $rk\,G = 1 + \max\{rk\,F : F \in Dec\,G\}$.

Note that for connected graphs $rk\,G$ is equal to the smallest k such that $P_{k+1} = P_k$ or, equivalently, such that P_k consists of $V(F)$ for all cocomponents F of G.

Let G be a connected graph and let $k = rk\,G$. We call G *uniform* if $Dec_{k-1}\,G$ contains no complement-connected graph, that is, every cocomponent appears in $Dec_k\,G$ and no earlier. We call G *inclusion-free* if the following two conditions are true for every $0 \le i \le k$:

1. For any $K \in Dec_i\,G$, $\overline{K}$ contains no isomorphic connected components.
2. Of any two elements $K, M \in Dec_i\,G$ none is properly embeddable into the other, that is, either $K \cong M$ or none is an induced subgraph of the other.

Let us now describe the *Ehrenfeucht game* $\mathrm{Ehr}_k(G, H)$ which will be our tool for studying the logical depth of graphs. The board consists of two vertex-disjoint graphs G and H. There are k rounds. The graphs G, H and the number k are known to both players, *Spoiler* and *Duplicator* (or *he* and *she*). In each round Spoiler selects one vertex in either G or H; then Duplicator must choose a vertex in the other graph. Let $x_i \in V(G)$ and $y_i \in V(H)$ denote the vertices selected by the players in the ith round, irrespectively of who selected them. Duplicator wins the game if the componentwise correspondence between the ordered k-tuples $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$ is a partial isomorphism from G to H. Otherwise the winner is Spoiler. In the 0-*alternation game* Spoiler must play all the game in the same graph he selects in the first round.

Assume that $G \ncong H$. Let $D(G, H)$ (resp. $D_0(G, H)$) denote the minimum of $D(\Phi)$ over all (resp. 0-alternating) sentences Φ that are true on one of the graphs and false on the other. The Ehrenfeucht theorem [6] (see also Fraïssé [7]) relates $D(G, H)$ and the length of the Ehrenfeucht game on G and H. We will use the following version of the theorem: $D_0(G, H)$ is equal to the minimum k such that Spoiler has a winning strategy in the k-round 0-alternation Ehrenfeucht game on G and H. We will also use the fact (see [11, Proposition 3.6]) that

$$D_0(G) = \max\left\{D_0(G, H) : H \ncong G\right\}.$$

We refer the reader to [15, Section 2] which contains a detailed discussion of the Ehrenfeucht game.

3. Proof of the upper bound in Theorem 1

3.1. Preliminaries

Lemma 1. *Every complement-connected graph G of order at least 5 has a vertex v such that $G - v$ is still complement-connected.*

Proof. Suppose that the claim is false. Take an arbitrary $v \in V(G)$. This vertex does not work so assume that, for example, $G - v$ is disconnected. Choose a proper partition $V(G) \setminus \{v\} = A_1 \cup A_2$ such that no edge of G connects A_1 to A_2. Assume that $|A_1| \ge |A_2|$. Since G is connected, the graph $G_i = G[A_i \cup \{v\}]$ is connected, $i = 1, 2$. This implies that $U_i \neq \emptyset$ for $i = 1, 2$, where

$$U_i = \{u \in A_i \mid G_i - u \text{ is connected}\}.$$

Let $u \in U_1$. The graph $G - u$ is connected because any vertex of $A_1 \setminus \{u\}$ can be connected (in $G_1 - u$) to v and then connected (in G_2) to any vertex of A_2. Since $\overline{G}$ contains all edges between A_1 and A_2 (and $|A_1| \ge 2$), the graph $\overline{G} - u - v$ is connected. Thus the only way that u can fail to satisfy the conclusion of the lemma is that v is adjacent (in G) to every other vertex except u (the vertex v cannot be adjacent to u too because G is complement-connected). The latter condition determines u uniquely and therefore $U_1 = \{u\}$. If $|A_2| \ge 2$, then the same argument shows that U_2 should consist of the unique neighbor u of v in $\overline{G}$, which is impossible. Thus, $|A_2| = 1$ and hence $|A_1| \ge 3$. Let $w \in A_1$ be some neighbor of u and let $z \in A_1 \setminus \{u, w\}$. Then $G_1 - z$ is still connected: u is connected to v via w while any other vertex is directly adjacent to v. Hence, $z \in U_1$. This contradiction finishes the proof. ■

Now we come to two strategic lemmas. The arguments of each lemma are listed in square brackets. This is convenient when we refer back to these results and, hopefully, makes the dependences between the lemmas easier to verify.

Lemma 2. *[x, x', y, y', G, H, l] Consider the Ehrenfeucht game on graphs G and H. Let $x, x' \in V(G)$, $y, y' \in V(H)$ and assume that the pairs x, y and x', y' were selected by the players in the same rounds. Furthermore, assume that all the following properties hold.*

1. $Env_l(x) \neq Env_l(x')$.
2. $Env_l(y) = Env_l(y')$.
3. $V(Env_{l+1}(y)) \neq V(Env_l(y))$.

Then Spoiler can win in at most $l + 1$ extra rounds, playing all the time in H.

Proof. We proceed by induction on l. The induction step takes care of the base case $l = 0$ too. Observe that, for every $0 \le i \le l$, we have $V(Env_{i+1}(y)) \ne V(Env_i(y))$ so we do not have to worry about Assumption 3 when using induction.

Let $m \in [0, l]$ be the minimum number such that $x' \notin Env_m(x)$. If $m < l$, Spoiler wins in $m + 1 \le l$ moves by induction. So suppose that $m = l$. Assume that y and y' are not adjacent in $Env_l(y)$ for otherwise Duplicator has already lost. By Assumption 3 the graph $Env_l(y)$ is connected but not complement-connected, so its diameter is at most 2. Spoiler selects any y'' adjacent to both y and y' in $Env_l(y)$. If Duplicator does not lose in this round, it means that her reply x'' lies outside $Env_{l-1}(x)$ (and that $l \ge 1$). We have $Env_{l-1}(x) \ne Env_{l-1}(x'')$ and $Env_{l-1}(y) = Env_{l-1}(y'')$. By the induction hypothesis applied to $[x, x'', y, y'', G, H, l - 1]$, Spoiler can win in at most l extra moves. ■

Lemma 3 *($[x_1, y_1, G, H, l]$). Suppose that $x_1 \in V(G)$ and $y_1 \in V(H)$ were selected in some round of the Ehrenfeucht game on (G, H) so that there is an $l \ge 0$ satisfying the following Assumptions 1–3.*

1. *$G_1 = Env_l(x_1)$ is not isomorphic to $H_1 = Env_l(y_1)$.*
2. *H_1 is a uniform inclusion-free graph such that every cocomponent of H_1 has at most c vertices.*
3. *For any $i \ge 0$, no member $A \in Dec_i\, H_1$ is embeddable as a proper subgraph into some $B \in Dec_i\, G_1$.*

Then Spoiler can win the game in at most $k + c - 1$ extra moves, playing all the time inside H, where $k = rk\, H_1 + l$.

Proof. Suppose that it is Spoiler's turn to move and, in addition to x_1 and y_1, we have the following configuration. Spoiler has already selected vertices $y_2, \dots, y_s \in V(H_1)$, Duplicator has selected $x_2, \dots, x_s \in V(G_1)$, and all of the following *Properties* 1–4 hold, where, for $j \in [s]$, we let $H_j = Env_{j+l-1}(y_j; H)$ and $G_j = Env_{j+l-1}(x_j; G)$.

1. For $i \in [2, s]$ we have $y_i \in V(H_{i-1})$.
2. For $i \in [2, s]$ we have $x_i \in V(G_{i-1})$.
3. For every $i \in [s]$ we have $H_i \not\cong G_i$.
4. For every $i \in [2, s]$ the vertices y_i and y_{i-1} belong to different components of $\overline{H_{i-1}}$. (Note that $y_i \in V(H_{i-1})$ by Property 1.)

Let us make a few remarks. Property 1 implies that

$$V(H_1) \supset \cdots \supset V(H_s),$$

and for $1 \le i \le j \le s$ we have $y_j \in V(H_i)$. Likewise by Property 2,

$$V(G_1) \supset \cdots \supset V(G_s), \tag{7}$$

and for $1 \le i \le j \le s$ we have $x_j \in V(G_i)$. Note also that $H_j = Env_{j-1}(y_j; H_1)$ and $G_j = Env_{j-1}(x_j; G_1)$. Property 1 and 4 imply that $y_j \notin V(H_i)$ for any $1 \le j < i \le s$. We stated Properties 1–4 this way in order to reduce the number of checks needed to verify them. Also, note that we do not require that the vertices x_i satisfy the analog of Property 4.

The above properties determine all H-adjacencies between the vertices $y_1, \dots, y_s$. Indeed, take any $1 \le i < j \le s$. By Properties 1 and 4, y_i and y_j belong to different components of $\overline{H_i}$ so we have $\{y_i, y_j\} \in E(H_i)$. This means that $\{y_i, y_j\} \in^{i+l} E(H)$. In other words, the vertices y_i and y_j are adjacent in H if and only if $i + l$ is odd.

If $s = 1$, then Properties 1, 2, and 4 are vacuously true, while Property 3 is precisely Assumption 1 of the lemma.

We are going to show that Spoiler can either force the same situation after the next round (of course, with s increased by one) or win by making some extra moves.

Case 1. Suppose that $s \le k - l = rk\, H_1$.

As $H_s \not\cong G_s$, Assumption 3 (for $i = s - 1$, $A = H_s$, and $B = G_s$) implies that $H_s \not\subset G_s$. By Assumption 2, the connected graph $H_s \in Dec_{s-1}\, H_1$ is inclusion-free; in particular, its complement does not contain two isomorphic components. Hence, there is a component H_{s+1} of $\overline{H_s}$ which is not isomorphic to any component of $\overline{G_s}$.

Suppose first that $y_s \notin V(H_{s+1})$. Spoiler chooses an arbitrary $y_{s+1} \in V(H_{s+1})$. Properties 1 and 4 hold automatically. Let x_{s+1} be Duplicator's reply. Assume that x_{s+1} has the same adjacencies to the previously selected vertices as y_{s+1} for otherwise Spoiler has already won having made $s \le k - l$ moves. (Note that we do not count y_1 as a move, here or later in the proof.) Suppose that $x_{s+1} \notin V(G_s)$, for otherwise Properties 2 and 3 hold and we are done.

Claim 1. *We have $l \ge 1$ and x_{s+1} does not belong to $Env_{l-1}(x_1; G)$.*

Proof of Claim. First we argue that $x_{s+1} \notin V(G_1)$. Suppose that this is not true. In view of (7), take the largest $i \in [s - 1]$ such that $x_{s+1} \in V(G_i)$. By the definition of i, $x_{s+1} \notin V(G_{i+1})$, the latter being the component of $\overline{G_i}$ that contains x_{i+1}. Thus $\{x_{s+1}, x_{i+1}\} \in E(G_i)$. On the other hand, x_{s+1} is not adjacent to x_{i+1} in G_i because y_{s+1} is not adjacent to y_{i+1} in H_i, a contradiction.

Next, we have $\{y_1, y_{s+1}\} \in^{l+1} E(H)$, so $\{x_1, x_{s+1}\} \in^{l+1} E(G)$. Since $x_{s+1} \notin V(G_1) = V(Env_l(x_1))$, we have $l \ge 1$. For any vertex $z \in V(Env_{l-1}(x_1)) \setminus V(Env_l(x_1))$ we have $\{x_1, z\} \in^{l} E(G)$, so $x_{s+1} \notin V(Env_{l-1}(x_1))$, as required. ■

At this point it is possible to argue that, if $s \ge 2$, then Duplicator has already lost. However, we still have to deal with the case $s = 1$ (when we have just x_1 and x_2). Since ruling out the case $s \ge 2$ would not make the proof shorter, we do not do this.

We have $V(Env_{l+1}(y_1)) \ne V(Env_l(y_1))$ because the latter set contains y_{s+1} while the former does not (or because H_1 is uniform and $rk\, H_1 = k - l \ge s \ge 1$). Hence, Lemma 2 applies to $[x_1, x_{s+1}, y_1, y_{s+1}, G, H, l - 1]$, and Spoiler can win the game in at most l extra moves, having made at most $s + l \le k$ moves in total.

It remains to describe Spoiler's strategy if $y_s \in V(H_{s+1})$, when Spoiler cannot just choose some $y_{s+1} \in V(H_{s+1})$ as this would violate Property 4. Here, Spoiler first selects some $y_{s+1} \in V(H_s) \setminus V(H_{s+1})$. (This set is non-empty since $s \le rk\, H_1$.) Let Duplicator reply with x_{s+1}. If $x_{s+1} \notin V(G_s)$, then by the argument of Claim 1 we have that $l \ge 1$ and $Env_{l-1}(x_1) \ne Env_{l-1}(x_{s+1})$. Thus Spoiler can win in at most l further moves by Lemma 2, having made at most $s + l \le k$ moves in total. Hence, let us assume that $x_{s+1} \in V(G_s)$. In this case, let us swap the vertices y_s and y_{s+1} as well as x_s and x_{s+1}. It is clear that the new sequences $y_1, \ldots, y_{s+1}$ and $x_1, \ldots, x_{s+1}$ satisfy Properties 1–4. This completes the description of the case $s \le k - l$.

Case 2. Suppose that $s = k - l + 1$.

This means that H_s is a cocomponent of H_1 (and thus has at most c vertices). Spoiler selects all vertices in $V(H_s) \setminus \{y_s\}$. We claim that Duplicator has lost by now. Indeed, if Duplicator replies all the time inside G_s, then she has lost because $G_s \not\supset H_s$ by Assumption 3 and Property 3. Otherwise, her response to the whole set $V(H_s)$ cannot be complement-connected because it contains both a vertex outside of G_s and the vertex $x_s \in V(G_s)$. Thus Spoiler wins, having made at most $s - 1 + c - 1 \le k + c - 1$ further moves. ■

3.2. Finishing the proof

Lemma 4 (*Main Lemma*). *Let G be a connected uniform inclusion-free graph. Suppose that every cocomponent of G has at most c vertices. Then $D_0(G) \le rk\, G + c + 1$.*

Proof. Let $k = rk\, G$. Since the case of $k = 0$ is trivial (namely we have $D_0(G) \le v(G) + 1 \le c + 1$ by (2)), we assume that $k \ge 1$. This and the assumption that G is uniform inclusion-free imply that $c \ge 5$.

Fix a graph $H \ncong G$. We will design a strategy allowing Spoiler to win the 0-alternation Ehrenfeucht game on (G, H) in at most the required number of moves. There are a few cases to consider.

Case 1. H has a cocomponent C non-embeddable into any cocomponent of G.

If C has no more than c vertices, Spoiler selects all vertices of C. Otherwise he selects $c + 1$ vertices spanning a complement-connected subgraph in C which is possible by Lemma 1 (since $c \ge 5$). If Duplicator's response A is within a cocomponent of G, then $C \ncong A$ by the assumption. Otherwise A is not complement-connected and Duplicator loses anyway.

Case 2. There are an $l \in [0, k]$ and an $A \in Dec_l\, G$ properly embeddable into some $B \in Dec_l\, H$, and not Case 1.

Let H_0 be a copy of A in B. Fix an arbitrary vertex $y_0 \in V(B) \setminus V(H_0)$. Note that since we are not in Case 1, the connected graph B cannot be a cocomponent of H by Property 2 in the definition of an inclusion-free graph. Hence

$$V(Env_l(y_0; H)) \ne V(Env_{l+1}(y_0; H)). \tag{8}$$

Let $Z = V(B) \setminus V(H_0)$. We will need the following routine claim, whose proof uses the connectedness of H_0 and the fact that B is not a cocomponent of H.

Claim 2. *For any $m \ge 0$ and $y \in V(H_0)$ we have*

$$Env_m(y; H_0) = Env_{m+l}(y; H - Z).$$

Proof of Claim. It is enough to prove the case $m = 0$ only, because the remaining cases would follow by a straightforward induction on m. Since H_0 is connected, the claim for $m = 0$ amounts to proving that

$$H_0 = Env_l(y; H - Z). \tag{9}$$

We will suppose that H is connected because otherwise instead of H we can consider its component containing $V(B)$.

Let us introduce a syntactic notion capturing the depth l decomposition of a connected graph. We call a tree T with root r *decomposing* if it has the following two properties. First, all paths from the root to a leaf have the same length l. Second, if a vertex v of T has exactly one child (a neighbor on the way to a leaf) u, then u also has no more than one child. Let $b_1, \dots, b_p$ be all leaves of T and $B_1, \dots, B_p$ be vertex-disjoint graphs. If we assign each b_i the value $\nu(b_i) = B_i$, this determines evaluation of each vertex v of T as follows: If the children of v are $u_1, \dots, u_q$, then $\nu(v) = \overline{\nu(u_1) \sqcup \dots \sqcup \nu(u_q)}$. Denote $T(B_1, \dots, B_p) = \nu(r)$.

Let us make a simple but useful observation: If all $B_1, \dots, B_p$ are connected and, moreover, B_i is complement-connected whenever b_i is a single child of its parent, then $H = T(B_1, \dots, B_p)$ is connected and $Dec_l\, H = \{B_1, \dots, B_p\}$. Conversely, for every connected H with such depth l decomposition there is a decomposing tree T of height l such that $H = T(B_1, \dots, B_p)$.

Furthermore, assume that $Z \subset V(B_1)$. A simple induction on l shows that

$$T(B_1 - Z, B_2, \ldots, B_p) = H - Z$$

for any tuple of vertex-disjoint graphs $B_1, \ldots, B_p$.

We now apply these general considerations to infer (9). Let H, l, Z, and H_0 be as in (9). Suppose that $Dec_l\, H = \{B_1, \ldots, B_p\}$ and let T be a decomposing tree of height l such that $H = T(B_1, \ldots, B_p)$. Without loss of generality, suppose that $B_1 = B$. Since $H_0 = B_1 - Z$, we have $H - Z = T(H_0, B_2, \ldots, B_p)$. Recall that H_0 is connected. Since $B_1 = B$ is not a cocomponent of H, the leaf b_1 of T is not a single child of its parent. Thus, H_0 does not need to be complement-connected to participate in a proper decomposition. Hence $Dec_l\,(H - Z) = \{H_0, B_2, \ldots, B_p\}$, which immediately implies (9). ■

Spoiler plays in H. At the first move he selects y_0. Denote Duplicator's response in G by x_0 and set $G_0 = Env_l(x_0)$. There are two alternatives to consider.

Subcase 2.1. $G_0 \ncong H_0$.

Suppose first that $l < k$. Since G_0 and H_0 are non-isomorphic copies of elements of $Dec_l\, G$ and G is inclusion-free, Spoiler is able to make his next choice y_1 in some $H_1 \in Dec\, \overline{H_0}$ with no isomorphic graph in $Dec\, \overline{G_0}$. Denote Duplicator's response by x_1.

If $x_1 \notin V(G_0)$, then Lemma 2 applies to $[x_0, x_1, y_0, y_1, G, H, l]$ in view of (8). Thus Spoiler can win by using at most $l + 3 \le k + 2$ moves in total. So, assume that $x_1 \in V(G_0)$. Lemma 3 applies to $[x_1, y_1, G, H - Z, l + 1]$ in view of Claim 2. (For example, Assumption 3 is satisfied because G is uniform inclusion-free and $Dec_l\, G$ contains both G_0 and an isomorphic copy of H_0.) Thus Spoiler can win in at most $2 + (k + c - 1)$ moves in total, as desired.

It remains to consider the case $l = k$. Spoiler selects all vertices of H_0. There are at most c of them because H_0 is isomorphic to a cocomponent of G. If Duplicator's replies lie in $V(G_0)$, she has already lost in view of $G_0 \not\supset H_0$ (which holds since G is inclusion-free). Otherwise, Duplicator's reply to $V(H_0)$ contains both a vertex outside G_0 and the vertex $x_0 \in V(G_0)$, so it cannot be complement-connected, and she loses. So, Spoiler wins having made at most c moves in total.

Subcase 2.2. $G_0 \cong H_0$.

Though the graphs are isomorphic, the crucial fact is that G_0, unlike H_0, contains a selected vertex. By the definition of an inclusion-free graph, every automorphism of $G_0 \cong H_0$ takes each cocomponent onto itself. Therefore all isomorphisms between G_0 and H_0 match cocomponents of these graphs in the same way. Let Y be the H_0-counterpart of the cocomponent $X = Env_{k-l}(x_0; G_0)$ with respect to this matching. In the second round Spoiler selects an arbitrary y_1 in Y. Denote Duplicator's answer by x_1.

Suppose first that $x_1 \in X$. Spoiler selects all vertices of $Y \setminus \{y_1\}$. At least one of Duplicator's replies lies outside $V(X)$ for otherwise she has already lost having chosen some vertex in X twice. But then Duplicator's reply to Y cannot be complement-connected. In any case Spoiler wins, having made at most $c + 1$ moves in total.

If $x_1 \in V(G_0) \setminus X$, then there is an $m \le k - l$ such that $Env_m(x_1; G_0)$ and $Env_m(y_1; H_0)$ are non-isomorphic. By Claim 2 Spoiler can apply the strategy of Lemma 3 to $[x_1, y_1, G, H - Z, l + m]$, winning in at most $2 + (k + c - 1)$ moves. If $x_1 \notin V(G_0)$, then Spoiler wins by Lemma 2 applied to $[x_0, x_1, y_0, y_1, G, H, l]$, having made at most $2 + l + 1 < k + c + 1$ moves in total.

Case 3. H has a component H_0 isomorphic to G, and not Cases 1–2.

Spoiler plays in H. In the first round he selects a vertex y_0 outside H_0 and further plays exactly as in Subcase 2.2 with $G_0 = G$.

Case 4. Neither of Cases 1–3.

Spoiler plays in $G_0 = G$. His first move x_0 is arbitrary. Denote Duplicator's response in H by y_0 and set $H_0 = Env_0(y_0)$. Since we are not in Cases 1–3, $G_0 \not\subset H_0$. As G_0 is inclusion-free, $\overline{G_0}$ has a connected component G_1 with no isomorphic component in $\overline{H_0}$.

If $x_0 \notin V(G_1)$, then Spoiler just selects any vertex $x_1 \in V(G_1)$. Let Duplicator respond with y_1. Assume that $y_1 \in V(H_0)$, for otherwise Duplicator has already lost: $\{y_0, y_1\} \notin E(H)$ while $\{x_0, x_1\} \in E(G)$.

If $x_0 \in V(G_1)$, then Spoiler selects any vertex $x_1 \in V(G_0) \setminus V(G_1)$. (The latter set is non-empty since $k \geq 1$.) Let Duplicator respond with y_1. As before we can assume that $y_1 \in V(H_0)$. Now, let us swap x_0 and x_1 as well as y_0 and y_1.

What we have achieved in both cases is that $G_1 \not\cong H_1$, where $H_1 = Env_1(y_1; H)$. Also, G_1 is a uniform inclusion-free graph of rank $k-1$. Lemma 3 applies to $[y_1, x_1, H, G, 1]$. (For example, Assumption 3 of the lemma holds because we are not in Cases 1–2.) This shows that Spoiler can win the 0-alternation game in at most $2 + (k + c - 1) = k + c + 1$ moves. This completes the proof of Lemma 4. ■

We will also need the following simple fact.

Lemma 5. *For some integer $c \leq 10$, there are $4c + 4$ pairwise non-embeddable into each other complement-connected graphs*

$$H_{i,j}, \qquad c \leq i \leq 2c, \quad 1 \leq j \leq 4,$$

such that $H_{i,j}$ has order i.

Proof. The existence of such graphs can be easily deduced by choosing each $H_{i,j}$ uniformly at random from all graphs of order i, independently from the other graphs. Indeed, for any $(i, j) \neq (f, g)$ with $c \leq i \leq f \leq 2c$ the probability that H_{ij} is embeddable into H_{fg} is at most

$$\frac{f!}{(f-i)!}\, 2^{-\binom{i}{2}}$$

while the probability of $H_{i,j}$ not being complement-connected is at most

$$\frac{1}{2}\sum_{h=1}^{i-1}\binom{i}{h} 2^{-h(i-h)+1},$$

where the factor $\frac{1}{2}$ accounts for the fact that each vertex partition is counted twice.

Hence, by looking at the expected number of 'bad' events, we conclude that if

$$16 \sum_{c\leq i\leq f\leq 2c} \frac{f!}{(f-i)!}\, 2^{-\binom{i}{2}} + 4 \times \frac{1}{2}\sum_{i=c}^{2c}\sum_{h=1}^{i-1}\binom{i}{h} 2^{-h(i-h)+1} < 1, \tag{10}$$

then the required graphs exist. The exact-arithmetic calculation with *Mathematica* shows that $c = 10$ works in (10). ■

Proof of the upper bound in Theorem 1. Fix an integer $c \leq 10$ and graphs $H_{\ell,j}$, where $c \leq \ell \leq 2c$, $1 \leq j \leq 4$, as in Lemma 5.

We define, inductively on i, a family R_i of graphs, starting with

$$R_0 = \{H_{c,1}, H_{c,2}, H_{c,3}, H_{c,4}\}.$$

Assume that R_{i-1} is already specified. Given a non-empty subset $S \subset R_{i-1}$, we define the graph

$$G_{i,S} = \overline{\bigsqcup_{G \in S} G},$$

or, in words, $G_{i,S}$ is the complement of the vertex-disjoint union of the graphs in S. We let

$$R_i = \left\{G_{i,S} : |S| = |R_{i-1}|/2\right\},$$

where we view R_i as the set of isomorphism types of graphs. It is proved in Claim 3 below that the graphs $G_{i,S}$ are pairwise non-isomorphic. (In particular, this implies by induction on i that $|R_i|$ is even because $\binom{2m}{m}$ is even for any integer $m \geq 1$.) Let $r_i = |R_i|$.

Let us list some properties of these graphs.

Claim 3. 1. *For any $S \subset R_{i-1}$ with $|S| \geq 2$, $G_{i,S}$ is a connected inclusion-free uniform graph of rank i.*
2. *For any $S, T \subset R_{i-1}$ with $S \not\subset T$, the graph $G_{i,S}$ is not embeddable into $G_{i,T}$.*
3. $r_i = \binom{r_{i-1}}{r_{i-1}/2}$.

Proof of Claim. We prove all claims by induction on i, the case $i = 1$ directly following from the definition of R_0. Let $i \geq 2$.

First, we verify Property 1, assuming that Properties 1–3 hold for all smaller values of i. Since $|S| \geq 2$, $G_{i,S}$ is connected. The components of $\overline{G_{i,S}}$ belong to R_{i-1}, each being isomorphic to $G_{i-1,S'}$ for some $S' \subset R_{i-2}$. From Property 3 and the initial value $r_0 = 4$, it is easy to deduce that $|S'| = r_{i-2}/2 \geq 2$. By the inductive Property 1, all components of $\overline{G_{i,S}}$ are uniform of rank $i - 1$, so $G_{i,S}$ is uniform of rank i.

Next, let us verify that $G_{i,S}$ is an inclusion-free graph. For any $j \in [i]$ all elements of $Dec_j\, G_{i,S}$ belong to R_{i-j}; by induction, each is inclusion-free. Let us show that none of these graphs is properly embeddable into another. Assume that $j < i$ for otherwise the claim follows from the definition of R_0. Take any two non-isomorphic $G_{i-j,S'}, G_{i-j,S''} \in R_{i-j}$. We have $S' \not\subset S''$ because $S' \neq S''$ and $|S'| = |S''| = r_{i-j-1}/2$. By induction (Property 2), we conclude that $G_{i-j,S'} \not\subset G_{i-j,S''}$, giving the stated. Since $G_{i,S}$ is connected, it remains to observe that $\overline{G_{i,S}}$ has no two isomorphic components, which follows from Property 2 again. Thus $G_{i,S}$ is indeed inclusion-free. We have completely finished the inductive step for Property 1.

Let us turn to Property 2. All components of $\overline{G_{i,S}}$ and $\overline{G_{i,T}}$ belong to R_{i-1}. Take any $H \in S \setminus T$. The graph $H \in R_{i-1}$ appears as a component in $\overline{G_{i,S}}$. By induction (Property 2) and the definition of R_{i-1}, H cannot be embedded into any component of $\overline{G_{i,T}}$. Thus $G_{i,S} \not\subset G_{i,T}$, as required. Property 3 follows from Property 2 which implies that the graphs $G_{i,S}$, for $S \subset R_{i-1}$, are pairwise non-isomorphic. ■

All graphs in R_i have the same order which we denote by n_i. Thus $v(G_{i,S}) = |S|\, n_{i-1}$. We have $n_0 = c$ and, for $i \geq 1$,

$$n_i = n_{i-1} r_{i-1}/2.$$

If we denote $m_i = r_i/2$, then we have $m_0 = 2$ and $m_1 = 3$. Thus for $i \geq 1$ we have

$$m_{i+1} = \frac{1}{2} r_{i+1} = \frac{1}{2} \binom{r_i}{r_i/2} = \frac{1}{2} \binom{2m_i}{m_i} \geq 2^{m_i}.$$

We conclude that $m_i > \text{Tower}(i)$ for all $i \geq 0$ and thus

$$n_i \geq m_{i-1} > \text{Tower}(i-1). \tag{11}$$

At this point we are able to prove the required upper bound on $q_0(n)$ for an infinite sequence of n, namely,

$$\dots, n_{i-1}, 2n_{i-1}, 3n_{i-1}, \dots, m_{i-1}n_{i-1} = n_i, 2n_i, \dots. \tag{12}$$

Indeed, by Lemma 4 for every $2 \leq s \leq m_{i-1}$ and an s-set $S \subset R_{i-1}$, we have

$$q_0(sn_{i-1}) \leq D_0(G_{i,S}) \leq i + c + 1.$$

Also, we have $i \leq \log^* n_{i-1} + 1$ by (11). Thus

$$q_0(sn_{i-1}) \leq \log^*(n_{i-1}) + c + 2 \leq \log^*(sn_{i-1}) + 12.$$

It now remains to fill in the gaps in (12). We need some auxiliary notions and claims first. We define the operation of a *cocomponent replacement* as follows. Suppose that A is a cocomponent of a graph G and B is a complement-connected graph. The result of the *replacement of A with B in G* is the graph G' with $V(G') = (V(G) \setminus V(A)) \cup V(B)$ such that $G'[V(B)] = B$, $G' - B = G - A$, and every vertex in B is adjacent to a vertex v outside B in G' if and only if every vertex in A is adjacent to v in G. (Here, we assume that $V(G) \cap V(B) = \emptyset$, and we use the fact that any two vertices x and y inside a cocomponent have the same adjacency pattern to the rest of the graph, i.e., for every z outside the cocomponent, x and y are equally adjacent or not adjacent to z.)

Claim 4. *Let G be a uniform inclusion-free graph of rank i with all cocomponents being isomorphic to one of $H_{c,l}$ with $1 \leq l \leq 4$. Let G' be obtained from G by replacing each cocomponent $A \cong H_{c,l}$ with some $H_{j,l}$, where $j \in [c, 2c]$ may depend on A. Then G' is a uniform inclusion-free graph of rank i.*

Proof of Claim. The partitions $P_0, \dots, P_i$ defined in (6) are completely determined by the vertex sets of the cocomponents and the adjacencies between then. This shows that G' is uniform of rank i. Let us check that G' is inclusion-free.

Let $0 \leq j \leq i$, $K' \in Dec_j\, G'$, and C'_1, C'_2 be some distinct components of the complement of K'. Suppose on the contrary that a bijection $f' : V(C'_1) \to V(C'_2)$ establishes an isomorphism between C'_1 and C'_2. The isomorphism f' induces a correspondence g' between the cocomponents of C'_1 and C'_2.

The description of the component replacement we made to obtain G' from G allows us to point the corresponding $K \in Dec_j\, G$, $C_1, G_2 \in Dec\, \overline{K}$, and g. Since $C_1 \not\cong C_2$, there is a cocomponent X_1 of C_1 such that the cocomponent $X_2 = g(X_1)$ is not isomorphic to X_1. It means that, if $X_1 \cong H_{c,l_1}$ and $X_2 \cong H_{c,l_2}$, then $l_1 \neq l_2$. But in G' these are replaced by $X'_1 \cong H_{j_1,l_1}$ and $X'_2 \cong H_{j_2,l_2}$, which are still non-isomorphic since $l_1 \neq l_2$. This contradicts the assumption that f' is an isomorphism. Thus G' satisfies Property 1 of the definition of an inclusion-free graph. The other property in the definition can be checked similarly. ■

If $n \leq 2c \leq 20$, then the upper bound (3) follows from the trivial inequality $q_0(n) \leq n+1$. So assume that $n > 2c = 2n_0$. Choose the integer i satisfying $2n_i \leq n < 2n_{i+1}$. Since $n_{i+1} = n_i m_i$, let $s \in [2, 2m_i - 1]$ satisfy $sn_i \leq n < (s+1)n_i$. Pick any s-set $S \subset R_i$ and let $G = G_{i+1,S}$. We have $v(G) = sn_i \leq n$ and, by Claim 3, the graph G is inclusion-free and uniform of rank $i+1$.

Let $f : Dec_{i+1}\, G \to [c, 2c]$ be some function. We construct a new graph G_f by replacing every cocomponent A of G by a copy of $H_{f(A),j}$, where j is defined by $A \cong H_{c,j}$. If f is the constant function assuming the value $2c$, then $v(G_f) = 2v(G) > n$. Hence there is some choice of f such that $v(G_f) = n$. By Claims 3 and 4, the graph G_f is a uniform inclusion-free graph of rank $i+1$. By Lemma 4, we have $D_0(G_f) \le i+2c+2$. On the other hand, $n > n_i >$ Tower$(i-1)$, that is, $\log^* n \ge i$. It means that

$$q_0(n) \le \log^* n + 2c + 2 \le \log^* n + 22.$$

This finishes the proof of the upper bound in (3). ■

4. Lower bound: Proof of Theorem 2

From now on we will be dealing with digraphs.

Given a first order formula Φ in which the negation sign occurs only in front of atomic subformulas, let the *alternation number* of Φ, denoted by $alt(\Phi)$, be the maximum number of quantifier alternations, i.e. the occurrences of ∃∀ and ∀∃, in a sequence of nested quantifiers of Φ. For a non-negative integer a, we denote

$$\Lambda_a = \{\Phi : alt(\Phi) \le a\}.$$

We also define $\Lambda_{1/2}$ to be the class of formulas Φ with $alt(\Phi) \le 1$ such that any sequence of nested quantifiers of Φ starts with ∃ or has no quantifier alternation. We introduce the latter class for the sake of generality, despite it is not used to prove Theorem 2. Note that $\Lambda_0 \subset \Lambda_{1/2} \subset \Lambda_1 \subset \Lambda_2 \subset \ldots$.

Now we somewhat extend our notation. Let F be some class of first order formulas. If a digraph G has a defining sentence in F, let $D_F(G)$ (resp. $L_F(G)$) denote the minimum quantifier rank (resp. length) of such a sentence; otherwise, we let $D_F(G) = L_F(G) = \infty$. The *succinctness function* is defined as

$$q_F(n) = \min\{D_F(G) : v(G) = n\}.$$

Whenever the index F is omitted, it is supposed that F is the class of all first order formulas. We also simplify notation by $D_a(G) = D_{\Lambda_a}(G)$ and similarly with $L_a(G)$ and $q_a(n)$. Clearly,

$$q(n) \le \cdots \le q_2(n) \le q_1(n) \le q_{1/2}(n) \le q_0(n).$$

Lemma 6. *For every $a \in \{0, 1/2, 1, 2, 3, \ldots\}$ and any digraph G we have*

$$L_a(G) < \text{Tower}(D_a(G) + \log^* D_a(G) + 2).$$

An analog of this lemma for $L(G)$ and $D(G)$ appears in [11, Theorem 10.1]. However, the proof of Lemma 6 we give below is not just an adaptation of the proof in [11] because the restrictions on the class of formulas do not allow us to run the same argument directly. Moreover, if $a = 1/2$, there appears another obstacle—the class of formulas $\Lambda_{1/2}$ is not closed with respect to negation.

Lemma 6 is proved in the next section in a stronger form since the argument is presentable more naturally in a more general situation. Here, let us show how Lemma 6 implies Theorem 2.

Given n, denote $k = q_0(n)$ and fix a digraph G on n vertices such that $D_0(G) = k$. By Lemma 6, G is definable by a 0-alternating sentence Φ of length less than Tower$(k + \log^* k + 2)$. We convert Φ to an equivalent *prenex* $\exists^*\forall^*$-*sentence* Ψ, i.e. of form (13). This can be easily done as follows. By renaming variables, ensure that each variable is quantified exactly once. Let the existential (resp. universal) quantifiers appear with variables $x_1, \ldots, x_l$ (resp. $y_1, \ldots, y_m$) in this order as we scan Φ from left to right. To obtain the required sentence Ψ simply 'pull' all quantifiers at front:

$$\Psi = \exists x_1 \cdots \exists x_l \forall y_1 \cdots \forall y_m \text{ (quantifier-free part)} \tag{13}$$

The obtained sentence Ψ is equivalent to Φ, which can be shown by induction on $L(\Phi)$ using the fact that Φ does not contain an $\exists$-quantifier in the range of a $\forall$-quantifier. Also, this reduction does not increase the total number of quantifiers. Therefore, as a rather crude estimate, we have $D(\Psi) \leq L(\Phi)$.

It is well known and easy to see that, if a sentence of the form (13) is true on some structure H, then it is true on some structure of order at most $l \leq D(\Psi)$. (Indeed, fix any satisfying assignment for $x_1, \ldots, x_l$ and take the substructure of H induced by the corresponding elements.) Since the defining sentence Ψ is true only on G, we have

$$n \leq D(\Psi) \leq L(\Phi) < \text{Tower}(k + \log^* k + 2).$$

This implies that

$$\log^* n \leq k + \log^* k + 2. \tag{14}$$

Suppose on the contrary to Theorem 2 that $k \leq \log^* n - \log^* \log^* n - 3$. Then $\log^* k \leq \log^* \log^* n$ and (14) implies that

$$\log^* n \leq (\log^* n - \log^* \log^* n - 3) + \log^* \log^* n + 2,$$

which is a contradiction, proving Theorem 2.

Note that identically the same argument works for $a = 1/2$ as well, strengthening Theorem 2 to the bound

$$q_{1/2}(n) \geq \log^* n - \log^* \log^* n - 2$$

for all n.

5. Length versus depth for restricted classes of defining sentences

Writing $A(x_1, \ldots, x_s)$, we mean that $x_1, \ldots, x_s$ are all free variables of A. We allow $s = 0$ which means that A is a sentence. A formula of type $x_i = x_j$ or $x_i \mapsto x_j$ is called *atomic*. A formula $A(x_1, \ldots, x_s)$ of quantifier rank $k - s$ is *normal* if

- all negations occurring in A stay only in front of atomic subformulas,
- A has occurrences of variables $x_1, \ldots, x_k$ only,
- every sequence of nested quantifiers of A has length $k - s$ and quantifies the variables $x_{s+1}, \ldots, x_k$ exactly in this order (here and later we consider only *maximal* sequences of nested quantifiers).

A simple inductive syntactic argument shows that any $A(x_1, \ldots, x_s)$ has an equivalent normal formula $A'(x_1, \ldots, x_s)$ of the same quantifier rank. Such a formula A' will be called a *normal form* of A.

Below F will always denote a non-empty class of first order formulas. The class of sentences (i.e. formulas without free variables) in F of quantifier rank k will be denoted by F^k. We call F *regular* if

- F contains all atomic formulas and Boolean combinations thereof,
- F is closed under subformulas and renaming of bound variables,
- with each $A(x_1, \ldots, x_s)$ in F, the class F contains a normal form of A,
- for any $k \geq 1$, F^k has *pattern set* $P^k \subseteq \{\forall, \exists\}^k$ such that a normal sentence A belongs to F^k iff every sequence of nested quantifiers of A belongs to P^k. (By the normality of A all quantifier sequences have the same length.)

Theorem 3. *Suppose that F is regular and G is definable in F. Then*

$$L_F(G) < \text{Tower}\left(D_F(G) + \log^* D_F(G) + 2\right).$$

Note that Theorem 3 generalizes Lemma 6 because the classes Λ_a are regular.

When we write $\bar{z}$, we will mean an s-tuple $(z_1, \ldots, z_s)$. If $\bar{u} \in V(G)^s$, we write $G, \bar{u} \models A(\bar{x})$ if $A(\bar{x})$ is true on G with each x_i assigned the respective u_i. Notation $\models A(\bar{x})$ will mean that $A(\bar{x})$ is true on all digraphs with s designated vertices.

A formula $A(\bar{x})$ in F is called an *F-description* of $(G, \bar{u})$ if

- $G, \bar{u} \models A(\bar{x})$, and
- for every $B(\bar{x}) \in F$ such that $G, \bar{u} \models B(\bar{x})$, we have $\models A(\bar{x}) \Rightarrow B(\bar{x})$, where $X \Rightarrow Y$ is a shorthand for $(\neg X) \vee Y$.

Lemma 7. *Suppose that G is definable in F. Let A be a sentence in F. Then A defines G iff A is an F-description of G.*

Proof. Suppose that A defines G. Then $G \models A$. Let $B \in F$ satisfy $G \models B$. We have to show that $H \models A \Rightarrow B$ for any H. If $H \not\models A$, we are done immediately. If $H \models A$, then $H \cong G$ and $H \models B$, as required.

For the other direction, suppose that A is an F-description of G. We have to show that $H \not\models A$ for any $H \not\cong G$. Fix a sentence $B \in F$ defining G. Since $H \not\models B$ and $\models A \Rightarrow B$, we conclude that $H \not\models A$, as required. ■

Let G and H be digraphs, $\bar{u} \in V(G)^s$, and $\bar{v} \in V(H)^s$. We write $G, \bar{u} \equiv H, \bar{v} \pmod{F}$ if, for any $A(\bar{x})$ in F, we have $G, \bar{u} \models A(\bar{x})$ exactly when $H, \bar{v} \models A(\bar{x})$.

Lemma 8. *Suppose that $G, \bar{u} \equiv H, \bar{v} \pmod{F}$ and let $A(\bar{x}) \in F$. Then A is an F-description of $(G, \bar{u})$ iff it is an F-description of $(H, \bar{v})$.*

Proof. As A is in F, we have $G, \bar{u} \models A(\bar{x})$ iff $H, \bar{v} \models A(\bar{x})$. Let $B(\bar{x}) \in F$. Again $G, \bar{u} \models B(\bar{x})$ iff $H, \bar{v} \models B(\bar{x})$. It follows that $G, \bar{u} \models B(\bar{x})$ implies $\models A(\bar{x}) \Rightarrow B(\bar{x})$ iff $H, \bar{v} \models B(\bar{x})$ implies $\models A(\bar{x}) \Rightarrow B(\bar{x})$. ■

Furthermore, we define

$$(G, \bar{u}) \bmod F = \{(H, \bar{v}) : G, \bar{u} \equiv H, \bar{v} \pmod{F}\}.$$

Let $0 \leq s \leq k$. The class of formulas in F with s free variables and quantifier rank $k - s$ is denoted by $F^{k,s}$. In particular, $F^{k,0} = F^k$. Given K, a non-empty subset of $F^{k,s}$, we define

$$E(K) = \left\{(G, \bar{u}) \bmod K : G \text{ is a digraph, } \bar{u} \in V(G)^s\right\}.$$

We will also use the following notation. Given $P^k \subseteq \{\forall, \exists\}^k$ and $\sigma \in \{\forall, \exists\}^s$, let

$$P^{k,s}_\sigma = \left\{\rho \in \{\forall, \exists\}^{k-s} : \sigma\rho \in P^k\right\}.$$

Note that, if $\sigma \in P^k$, then $P_\sigma^{k,k}$ consists of the empty word only, while if $\sigma \notin P^k$, then $P_\sigma^{k,k}$ is empty. Furthermore, given a regular F with pattern set P^k, let $F_\sigma^{k,s}$ consist of the normal formulas in $F^{k,s}$ whose sequences of nested quantifiers are in $P_\sigma^{k,s}$. We say that a formula $A(\bar{x}) \in F_\sigma^{k,s}$ *describes* a class $\alpha \in E(F_\sigma^{k,s})$ if $A(\bar{x})$ is an $F_\sigma^{k,s}$-description of some $(G, \bar{u}) \in \alpha$. By Lemma 8, this definition does not depend on the particular choice of a representative $(G, \bar{u})$ of α, and the word *some* in the definition can be replaced with *every*.

Proof of Theorem 3. For each $\alpha \in E(F_\sigma^{k,s})$ we will construct a formula $A_\alpha(\bar{x}) \in F_\sigma^{k,s}$ describing α. We will use induction on $k - s$. Afterwards we will estimate the length of the obtained A_α and show how this implies Theorem 3.

We start with $s = k$. Let $\sigma \in \{\forall, \exists\}^k$. Assume that $\sigma \in P^k$, for otherwise $F_\sigma^{k,k}$ is empty and there is nothing to do. For any such σ, $F_\sigma^{k,k} = F^{k,k}$ is exactly the class of all quantifier-free formulas in F over the set of variables $\{x_1, \ldots, x_k\}$. Clearly, $(G, \bar{u}) \equiv (H, \bar{v}) (\text{mod } F_\sigma^{k,k})$ iff the componentwise correspondence between $\bar{u}$ and $\bar{v}$ gives a partial isomorphism. So, any given class $\alpha \in E(F_\sigma^{k,s})$ can be described as follows. Pick any representative $(G, u_1, \ldots, u_k)$ of α and let $A_\alpha(x_1, \ldots, x_k)$ be the conjunction of all atomic formulas $x_i \mapsto x_j$ for (u_i, u_j) in G, all negations $\neg(x_i \mapsto x_j)$ for (u_i, u_j) not in G, all $x_i = x_j$ for identical u_i, u_j, and all $\neg(x_i = x_j)$ for distinct u_i, u_j. Clearly, $H, \bar{v} \models A_\alpha(\bar{x})$ iff $(H, \bar{v}) \in \alpha$. It follows that A_α indeed describes α. Note that $L(A_\alpha) \le 18k^2$.

Assume now that $0 \le s < k$ and that for any $\tau \in \{\forall, \exists\}^{s+1}$ with $F_\tau^{k,s+1} \neq \emptyset$ and $\beta \in E(F_\tau^{k,s+1})$ we have a formula $A_\beta(\bar{x}, x_{s+1}) \in F_\tau^{k,s+1}$ describing β. Given a digraph G, an s-tuple of vertices $\bar{u} \in V(G)^s$, and a non-empty class of formulas K, we set

$$S(G, \bar{u}; K) = \{(G, \bar{u}, u) \text{ mod } K : u \in V(G)\}.$$

We also set $S(G, \bar{u}; \emptyset) = \emptyset$. Let $\sigma \in \{\forall, \exists\}^s$ and $\alpha \in E(F_\sigma^{k,s})$ (thus, $F_\sigma^{k,s} \neq \emptyset$ and hence $F_{\sigma *}^{k,s+1} \neq \emptyset$ for at least one $* \in \{\exists, \forall\}$). To construct $A_\alpha(\bar{x})$, we fix $(G, \bar{u})$ being an arbitrary representative of α and put[1]

$$A_\alpha(\bar{x}) = \bigwedge_{\beta \in S(G, \bar{u}; F_{\sigma\exists}^{k,s+1})} \exists x_{s+1}\, A_\beta(\bar{x}, x_{s+1}) \;\wedge\; \forall x_{s+1} \bigvee_{\beta \in S(G, \bar{u}; F_{\sigma\forall}^{k,s+1})} A_\beta(\bar{x}, x_{s+1}).$$

Claim 5. $A_\alpha(\bar{x}) \in F_\sigma^{k,s}$.

Proof of Claim. This follows from the assumption that $A_\beta(\bar{x}, x_{s+1}) \in F_{\sigma *}^{k,s+1}$ for $\beta \in S(G, \bar{u}; F_{\sigma *}^{k,s+1})$. ■

Claim 6. $G, \bar{u} \models A_\alpha(\bar{x})$.

Proof of Claim. Let us show first that all conjunctive members over $\beta \in S(G, \bar{u}; F_{\sigma\exists}^{k,s+1})$ are satisfied. Each such β is of the form $(G, \bar{u}, u_\beta) \text{ mod } F_{\sigma\exists}^{k,s+1}$ for some $u_\beta \in V(G)$. By assumption, $G, \bar{u}, u_\beta \models A_\beta(\bar{x}, x_{s+1})$ and hence $G, \bar{u} \models \exists x_{s+1} A_\beta(\bar{x}, x_{s+1})$.

It remains to show that the universal member of the conjunction is also satisfied. Consider an arbitrary $u \in V(G)$. Let $\beta_u = (G, \bar{u}, u) \text{ mod } F_{\sigma\forall}^{k,s+1}$. By assumption, $G, \bar{u}, u \models A_{\beta_u}(\bar{x}, x_{s+1})$ and hence the disjunction is always true. ■

[1] Here A_α has the same form as the *Hintikka formula* in [5, page 18]. Curiously, in a similar context in [11, Lemma 3.4] we use another generic defining formula borrowed from [15, Theorem 2.3.2], which is not usable now because F may be not closed with respect to negation.

Claim 7. *We have* $\models A_\alpha(\bar{x}) \Rightarrow B(\bar{x})$ *for any* $B(\bar{x}) \in F_\sigma^{k,s}$ *such that*

$$G, \bar{u} \models B(\bar{x}). \quad (15)$$

Proof of Claim. Given a class of formulas K, let $K|_\exists$ (resp. $K|_\forall$) denote the class of those formulas in K having form $\exists x(\ldots)$ (resp. $\forall x(\ldots)$). First, we settle two special cases of the claim.

Case 1. $B \in F_\sigma^{k,s}|_\exists$

Let $B = \exists x_{s+1} C(\bar{x}, x_{s+1})$. Note that $C(\bar{x}, x_{s+1}) \in F_{\sigma\exists}^{k,s+1}$. Assume that $H, \bar{v} \models A_\alpha(\bar{x})$. We have to verify that $H, \bar{v} \models B(\bar{x})$. By (15) we can choose a vertex $u \in V(G)$ such that $G, \bar{u}, u \models C(\bar{x}, x_{s+1})$. Let

$$\beta = (G, \bar{u}, u) \bmod F_{\sigma\exists}^{k,s+1}.$$

We have $H, \bar{v} \models \exists x_{s+1} A_\beta(\bar{x}, x_{s+1})$ and hence $H, \bar{v}, v \models A_\beta(\bar{x}, x_{s+1})$ for some $v \in V(H)$. Since we have assumed that $A_\beta(\bar{x}, x_{s+1})$ is an $F_{\sigma\exists}^{k,s+1}$-description of β, we have $H, \bar{v}, v \models C(\bar{x}, x_{s+1})$ and hence $H, \bar{v} \models B(\bar{x})$ as needed.

Case 2. $B \in F_\sigma^{k,s}|_\forall$

Let $B = \forall x_{s+1} C(\bar{x}, x_{s+1})$. Note that $C(\bar{x}, x_{s+1}) \in F_{\sigma\forall}^{k,s+1}$. Assume that $H, \bar{v} \models A_\alpha(\bar{x})$. It follows that for every $v \in V(H)$ there is a $\beta_v \in S(G, \bar{u}; F_{\sigma\forall}^{k,s+1})$ such that $H, \bar{v}, v \models A_{\beta_v}(\bar{x}, x_{s+1})$. By (15) we have $G, \bar{u}, u \models C(\bar{x}, x_{s+1})$ for all $u \in V(G)$. Let u_v be such that

$$\beta_v = (G, \bar{u}, u_v) \bmod F_{\sigma\forall}^{k,s+1}.$$

We have $G, \bar{u}, u_v \models C(\bar{x}, x_{s+1})$, and, by our assumption that A_{β_v} describes β_v, we have $H, \bar{v}, v \models C(\bar{x}, x_{s+1})$. Since v is arbitrary, we conclude that $H, \bar{v} \models B(\bar{x})$, finishing the proof of Case 2.

Finally, take an arbitrary $B(\bar{x}) \in F_\sigma^{k,s}$. Since B is normal (and $s < k$), it is equivalent to a monotone DNF formula $\vee_i(\wedge_j B_{i,j})$ with all $B_{i,j}$ belonging to $F_\sigma^{k,s}|_\exists \cup F_\sigma^{k,s}|_\forall$. This can be routinely shown by induction on $L(B)$. For example, if $B = B_1 \wedge B_2$, where, by induction, B_h is equivalent to $\vee_{i_h}(\wedge_{j_h} B_{i_h,j_h})$, $h = 1, 2$, then we can take $\vee_{i_1,i_2}((\wedge_{j_1} B_{i_1,j_1}) \wedge (\wedge_{j_2} B_{i_2,j_2}))$ for B.

Since $G, \bar{u} \models B(\bar{x})$, we have $G, \bar{u} \models B_{i_0,j}(\bar{x})$ for some i_0 and all j. From Cases 1–2 it follows that $H, \bar{v} \models B_{i_0,j}(\bar{x})$ for all j whenever $H, \bar{v} \models A_\alpha(\bar{x})$. This means that $H, \bar{v} \models B(\bar{x})$ whenever $H, \bar{v} \models A_\alpha(\bar{x})$, as required. ■

Let us now estimate the length of the constructed formulas. The estimates are similar to those in [11, Theorem 10.1]. Our bound will depend on k and s only, so we define

$$l(k, s) = \max_{\tau \in \{\forall,\exists\}^s} \max \left\{ L(A_\beta) : \beta \in E(F_\tau^{k,s}) \right\}.$$

Let $f(k, s) = |Ehrv(k, s)|$, where $Ehrv(k, s) = E(FO^{k,s})$ with FO being the class of all first order formulas. (According to [15], the elements of $Ehrv(k, s)$ are called *digraph Ehrenfeucht values*.) The function $f(k, s)$ is an upper bound on $|E(F_\tau^{k,s})|$ for any $\tau \in \{\forall, \exists\}^s$. The number of Ehrenfeucht values for (undirected) graphs was estimated in [15, Theorem 2.2.1]. The obvious modifications of the proofs from [15] give the following bounds for digraphs:

$$f(k, k) \le 4^{k^2},$$
$$f(k, s) \le 2^{f(k,s+1)}.$$

We already know that $l(k,k) \le 18k^2$. Our construction of $A_\alpha(\bar{x})$ shows for $0 \le s < k$ that

$$l(k,s) \le 2f(k,s+1)(l(k,s+1)+9). \tag{16}$$

Let $k \ge 2$. Set $g(x) = 2 \cdot 2^x(x+9)$. A simple inductive argument shows that

$$f(k,s) \le 2^{g^{(k-s)}(18k^2)} \quad \text{and} \quad l(k,s) \le g^{(k-s)}(18k^2),$$

where $g^{(i)}(x) = g(g(\cdots g(x)\cdots))$ is obtained by iteratively applying g i times. Define the two-parameter function $\text{Tower}(i,x)$ inductively on i by $\text{Tower}(0,x) = x$ and $\text{Tower}(i+1,x) = 2^{\text{Tower}(i,x)}$ for $i \ge 1$. This is a generalization of the old function: $\text{Tower}(i,1) = \text{Tower}(i)$. One can prove by induction on i that for any $x \ge 5$ and $i \ge 1$ we have

$$g^{(i)}(x) < \text{Tower}(i+1,x)/2. \tag{17}$$

Indeed, it is easy to check the validity of (17) for $i = 1$, while for $i \ge 2$ we have

$$g^{(i)}(x) < g(\text{Tower}(i,x)/2) < 2^{\text{Tower}(i,x)-1} = \text{Tower}(i+1,x)/2. \tag{18}$$

If $k \ge 12$, then $18k^2 < 2^k$ and by (17) we have

$$l(k,0) \le g^{(k)}(18k^2) < \text{Tower}(k+1,18k^2)/2 < \text{Tower}(k+\log^* k+2). \tag{19}$$

Also, $18 \cdot 11^2 < \text{Tower}(4)/2$ and, similarly to (18), we have $g^{(k)}(18k^2) < \text{Tower}(k+4)/2$ for $k \le 11$. Thus (19) holds for $k \in [3,11]$ too. For $k = 2$ one can still prove (19) using (16) and the sharper initial estimates $f(2,2) = 18$ and $l(2,2) \le 48$.

To finish the proof of Theorem 3, let $k = D_F(G) \ge D(G) \ge 2$ and $\alpha = G \bmod F^{k,0}$. Since G is definable in $F^{k,0}$, the sentence A_α defines G by Lemma 7. By (19),

$$L_F(G) \le L(A_\alpha) \le l(k,0) < \text{Tower}(k+\log^* k+2),$$

completing the proof. ■

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