

Minimum H -decompositions of graphs

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Abstract

Given graphs G and H , an H -decomposition of G is a partition of the edge set of G such that each part is either a single edge or forms a graph isomorphic to H . Let $\phi_H(n)$ be the smallest number ϕ such that any graph G of order n admits an H -decomposition with at most ϕ parts.

Here we determine the asymptotic of $\phi_H(n)$ for any fixed graph H as n tends to infinity.

The exact computation of $\phi_H(n)$ for an arbitrary H is still an open problem. Bollobás [B. Bollobás, On complete subgraphs of different orders, Math. Proc. Cambridge Philos. Soc. 79 (1976) 19–24] accomplished this task for cliques. When H is bipartite, we determine $\phi_H(n)$ with a constant additive error and provide an algorithm returning the exact value with running time polynomial in $\log n$.

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1. Introduction

Given two graphs G and H , an H -decomposition of G is a partition of the edge set of G such that each part is either a single edge or forms an H -subgraph, i.e., a graph isomorphic to H . Let $\phi_H(G)$ be the smallest possible number of parts in an H -decomposition of G .

It is easy to see that, for non-empty H , $\phi_H(G) = e(G) - p_H(G)(e(H) - 1)$, where $p_H(G)$ is the maximum number of pairwise edge-disjoint H -subgraphs that can be packed into G and $e(G)$

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denotes the number of edges in G . Building upon a body of previous research, Dor and Tarsi [6] showed that if H has a component with at least 3 edges then the problem of checking whether an input graph G admits a partition into H -subgraphs is NP-complete. Hence, it is NP-hard to compute the function $\phi_H(G)$ for such H .

Here we study the function

$$\phi_H(n) = \max\{\phi_H(G) \mid v(G) = n\},$$

which is the smallest number such that any graph G of order n admits an H -decomposition with at most $\phi_H(n)$ parts. Motivated by the problem of representing graphs by set intersections, Erdős, Goodman and Pósa [8] proved that $\phi_{K_3}(n) = t_2(n)$, where K_r denotes the complete graph (clique) of order r , and $t_r(n)$ is the maximum size of an r -partite graph on n -vertices. This result was extended by Bollobás [4], who proved that

$$\phi_{K_r}(n) = t_{r-1}(n), \quad \text{for all } n \geq r \geq 3. \tag{1.1}$$

Here we determine the asymptotic of $\phi_H(n)$ for any fixed graph H as $n \rightarrow \infty$.

Theorem 1.1. *Let H be any fixed graph of chromatic number $r \geq 3$. Then,*

$$\phi_H(n) = t_{r-1}(n) + o(n^2).$$

The upper bound of Theorem 1.1 is proved in Section 2. The lower bound follows from the trivial inequalities $\phi_n(H) \geq \text{ex}(n, H) \geq t_{r-1}(n)$, where

$$\text{ex}(n, H) = \max\{e(G) \mid v(G) = n, H \not\subseteq G\}$$

is the Turán function. We make the following conjecture.

Conjecture 1.2. *For any graph H of chromatic number $r \geq 3$ there is $n_0 = n_0(H)$ such that $\phi_H(n) = \text{ex}(n, H)$ for all $n \geq n_0$.*

This conjecture is known to be true for cliques (Bollobás [4]), clique-extensions (Sousa [19]), the cycle of length 5 and some other graphs (Sousa [18]).

For a bipartite graph H it is easy to determine the asymptotic (see Sousa [18]):

Lemma 1.3. *For any non-empty graph H with m edges and any integer n , we have*

$$\phi_H(n) \leq \frac{1}{m} \binom{n}{2} + \frac{m-1}{m} \text{ex}(n, H). \tag{1.2}$$

In particular, if H is a fixed bipartite graph with m edges and $n \rightarrow \infty$, then

$$\phi_H(n) = \left(\frac{1}{m} + o(1)\right) \binom{n}{2}. \tag{1.3}$$

Proof. To prove (1.2) remove greedily one by one the edge-sets of H -subgraphs of a given graph G and then remove the remaining edges. The bound (1.2) follows as at most $\text{ex}(n, H)$ parts are single edges.

The upper bound in (1.3) follows from (1.2) and the inequality

$$\text{ex}(n, K_{t,t}) = O(n^{2-1/t}), \tag{1.4}$$

of Kóvari, Sós and Turán [13], where $K_{t,s}$ denotes the complete bipartite graph with parts of size t and s . The lower bound in (1.3) follows from $\phi_H(n) \geq \phi_H(K_n) \geq \frac{1}{m} \binom{n}{2}$. \square

We managed to determine $\phi_H(n)$ for any fixed bipartite graph H with an $O(1)$ additive error (see Theorem 1.4 below). Furthermore, our proof gives a procedure for computing the exact values of $\phi_H(n)$ for all large n , that runs in polylogarithmic time. Although it should be possible to write a closed formula for the exact value of $\phi_H(n)$ for H bipartite, it seems to be too cumbersome so we do not attempt this here.

For a non-empty graph H , let $\gcd(H)$ denote the greatest common divisor of the degrees of H . For example, $\gcd(K_{6,4}) = 2$ while for any tree T with at least 2 vertices we have $\gcd(T) = 1$. We will prove the following result in Section 3.

Theorem 1.4. *Let H be a bipartite graph with m edges and let $d = \gcd(H)$. Then there is $n_0 = n_0(H)$ such that for all $n \geq n_0$ the following statements hold.*

If $d = 1$, then if $\binom{n}{2} \equiv m - 1 \pmod{m}$,

$$\phi_H(n) = \phi_H(K_n) = \left\lfloor \frac{n(n-1)}{2m} \right\rfloor + m - 1, \tag{1.5}$$

otherwise,

$$\phi_H(n) = \phi_H(K_n^*) = \left\lfloor \frac{n(n-1)}{2m} \right\rfloor + m - 2 \tag{1.6}$$

where K_n^ denotes any graph obtained from K_n after deleting at most $m - 1$ edges in order to have $e(K_n^*) \equiv m - 1 \pmod{m}$. Furthermore, if G is extremal then G is either K_n or K_n^* .*

If $d \geq 2$, then

$$\phi_H(n) = \frac{nd}{2m} \left(\left\lfloor \frac{n}{d} \right\rfloor - 1 \right) + \frac{1}{2}n(d-1) + O(1). \tag{1.7}$$

Moreover, there is a procedure with running time polynomial in $\log n$ which determines $\phi_H(n)$ and describes a family \mathcal{D} of n -sequences such that a graph G of order n satisfies $\phi_H(G) = \phi_H(n)$ if and only if the degree sequence of G belongs to \mathcal{D} . (It will be the case that $|\mathcal{D}| = O(1)$ and each sequence in \mathcal{D} has $n - O(1)$ equal entries, so \mathcal{D} can be described using $O(\log n)$ bits.)

2. H -decompositions for a non-bipartite H

In this section we will prove the upper bound in Theorem 1.1. In outline, the proof is the following. First, we apply Szemerédi’s Regularity Lemma [20] to the graph G that we want to decompose. The regularity partition of G gives us a weighted graph K with large but bounded number k of vertices. By generalizing the method of Bollobás [4] we decompose K into weighted copies of K_r and K_2 with aggregate weight at most $t_{r-1}(k) + o(k^2)$. Then, we split G into subgraphs that correspond to the cliques from the above decomposition of K . Finally, each of the obtained r -partite subgraphs of G is almost perfectly decomposed into copies of H by using the theorem of Pippenger and Spencer [14]. The idea that a regularity partition allows us to relate combinatorial and fractional decompositions of graphs has already been used by various researchers, see Haxell and Rödl [11], Yuster [22] and others.

Before presenting the proof we need to introduce the tools.

Let $G = (V, E)$ be a graph and let A and B be two disjoint non-empty subsets of V . Let $e(A, B)$ denote the number of edges between A and B . The *density* of (A, B) is defined as

$$d(A, B) = \frac{e(A, B)}{|A||B|}.$$

For $\varepsilon > 0$ the pair (A, B) is said to be ε -regular if for every $X \subset A$ and $Y \subset B$ satisfying $|X| > \varepsilon|A|$ and $|Y| > \varepsilon|B|$ we have

$$|d(X, Y) - d(A, B)| < \varepsilon.$$

Theorem 2.1 (Regularity Lemma [20]). *For every $\varepsilon > 0$ and m there exist two integers $M(\varepsilon, m)$ and $N(\varepsilon, m)$ with the following property: for every graph $G = (V, E)$ with $n \geq N(\varepsilon, m)$ vertices there is a partition of the vertex set into $k + 1$ classes (clusters)*

$$V = V_0 \cup V_1 \cup \dots \cup V_k$$

such that

- (i) $m \leq k \leq M(\varepsilon, m)$,
- (ii) $|V_0| < \varepsilon n$,
- (iii) $|V_1| = |V_2| = \dots = |V_k|$,
- (iv) all but at most εk^2 of the pairs (V_i, V_j) , $1 \leq i < j \leq k$, are ε -regular.

Let \mathcal{H} be a t -uniform hypergraph, that is, every hyperedge of \mathcal{H} contains exactly t vertices. If v and w are vertices of \mathcal{H} , the *codegree* of v and w , denoted by $\text{codeg}(v, w)$, is the number of hyperedges in \mathcal{H} containing both v and w .

We will need the following theorem of Pippenger and Spencer [14], see also Rödl [15]. By $a \pm c$ we mean a real between $a - c$ and $a + c$.

Theorem 2.2. *For every integer t and real $c_2 > 0$, there are $c_3 = c_3(t, c_2) > 0$ and $d_0 = d_0(t, c_2)$ such that for any $n \geq D \geq d_0$ the following holds.*

Every t -uniform hypergraph \mathcal{H} on a set V of n vertices satisfying all of the following conditions

- (1) *for all vertices $x \in V$ but at most $c_3 n$ of them, $\text{deg}(x) = (1 \pm c_3)D$;*
- (2) *for all $x \in V$, $\text{deg}(x) \leq D/c_3$;*
- (3) *for any two distinct $x, y \in V$, $\text{codeg}(x, y) < c_3 D$;*

contains a matching consisting of at least $(1 - c_2)n/t$ hyperedges.

We will also need the following version of Turán’s Theorem, see, e.g., [4].

Theorem 2.3 (Turán’s Theorem, Min-Degree Version). *If in a graph with n vertices the degree of every vertex is greater than $\lfloor \frac{r-2}{r-1}n \rfloor$ then the graph contains a K_r .*

A *weighted graph* of order k is a graph K with k vertices together with a *weight function* ω that assigns to each edge of K a real number between 0 and 1. By assigning weight 0 to all non-edges, we may assume that K is a complete graph. A *weighted K_r -decomposition* of K is a collection A_1, \dots, A_t of subsets of $[k]$ and positive reals $\alpha_1, \dots, \alpha_t$, each A_i having 2 or r

vertices such that for any distinct $i, j \in [k]$ we have $\omega(ij) = \sum_{h: A_h \ni ij} \alpha_h$. The total weight of the decomposition is $\sum_{i=1}^l \alpha_i$. Thus we want to decompose our graph into weighted versions of K_r 's and K_2 's.

Lemma 2.4. *For any integer $r \geq 3$ and a positive real c_1 , there are $c_2 > 0$ and k_0 such that any weighted graph K on $k \geq k_0$ vertices admits a weighted K_r -decomposition of total weight at most $t_{r-1}(k) + 2c_1k^2$ in which every K_r has weight at least c_2 .*

Proof. Our proof is built upon the ideas from Bollobás [4]. Given r and c_1 choose, in this order, small $c_2 > 0$, large f and large C .

We will be iteratively updating our weighted graph K , decreasing the edge-weights by a corresponding amount after the removal of any clique in the obvious way, until all edge-weights are zero. Also, we agree that if at any stage the current graph K has an edge ij of weight $\omega(ij) < c_2$, then we immediately remove this edge (as a 2-clique). Since we do this at most $\binom{k}{2}$ times, the total weight of our decomposition will increase by at most $c_2\binom{k}{2}$.

Also, whenever we remove a K_r we take the maximal possible weight. Thus each K_r will have weight at least c_2 , and the second condition of the lemma is automatically satisfied.

We use induction on k to prove the bound

$$t_{r-1}(k) + c_1k^2 + C, \tag{2.1}$$

on the total weight of our decomposition. If $k \leq f$, then the required bound follows from the C term alone since $\binom{k}{2} \leq C$. So assume that $k > f$. Let the weighted degree of a vertex x be $\omega(x) = \sum_{y \in \Gamma(x)} \omega(xy)$, where $\Gamma(x)$ denotes the neighborhood of x . Let x have the smallest weighted degree, call it γ . We want to decompose all edges incident to x .

If $\gamma \leq t_{r-1}(k) - t_{r-1}(k-1) + c_1(2k-1)$, then we just remove all single edges at x and decompose the remaining graph of order $k-1$ by induction, obtaining (2.1) as required. So suppose that

$$\gamma > t_{r-1}(k) - t_{r-1}(k-1) + c_1(2k-1). \tag{2.2}$$

Let A_x consist of all y such that $\omega(xy) > 0$. Let $\alpha = |A_x|$. As each edge-weight is at most 1, $\alpha \geq \gamma$. Let us greedily remove maximum weight K_r 's through x . Suppose that the removed K_r 's have total weight h . Let $B \subset A_x$ consist of those $y \in A_x$ for which we still have $\omega(xy) > 0$. The weighted graph induced by B contains no K_{r-1} . Thus, by the min-degree version of Turán's Theorem, Theorem 2.3, and since each edge-weight is at most 1, for some $y \in B$ we must have $\omega_B(y) \leq \frac{r-3}{r-2}\beta$, where $\beta = |B|$ and

$$\omega_B(y) = \sum_{z \in \Gamma(y) \cap B} \omega(yz).$$

We have

$$\beta \geq \gamma - (r-1)h - c_2k, \tag{2.3}$$

since

$$\gamma = \omega(x) \leq \sum_{z \in B} \omega(xz) + (r-1)h + c_2k \leq \beta + (r-1)h + c_2k$$

and each edge-weight is at most 1. Moreover, those of the removed K_r 's that contain y have total weight at most 1, again because each edge-weight is at most 1.

Since initially we had $\omega(y) \geq \gamma$ and $\omega(y) = \omega_B(y) + \sum_{z \notin B} \omega(yz) + (r - 1)\theta$, where θ denotes the weight of the removed K_r 's that contain y , we conclude that

$$\gamma \leq \omega(y) \leq \frac{r-3}{r-2}\beta + k - \beta + r - 1.$$

Using (2.3) we obtain

$$\gamma \leq k + r - 1 - \frac{\gamma - (r - 1)h - c_2k}{r - 2}.$$

Thus,

$$h \geq \gamma - \frac{r-2}{r-1}k - r + 2 - \frac{c_2k}{r-1},$$

and the total weight removed through x is at most

$$h + \gamma - (r - 1)h = \gamma - (r - 2)h \leq \gamma - (r - 2)\left(\gamma - \frac{r-2}{r-1}k - r + 2 - \frac{c_2k}{r-1}\right).$$

The right-hand side is a non-increasing function of γ (recall that $r \geq 3$), so it is maximized when γ attains equality in (2.2), giving at most

$$t_{r-1}(k) - t_{r-1}(k - 1) + c_1(2k - 1),$$

since $\gamma - \frac{r-2}{r-1}k - r + 2 - \frac{c_2k}{r-1} \geq 0$ in view of $2c_1 < \frac{c_2}{r-1}$ and $k > f$ being large.

This proves the bound (2.1) by induction. The lemma clearly follows from (2.1). \square

Let us return to Theorem 1.1.

Proof of the upper bound in Theorem 1.1. Let $c_0 > 0$ be arbitrary. We choose, in this order, sufficiently small $c_1 \gg \dots \gg c_5 > 0$ and then let n_0 be sufficiently large. Let G be any graph of order $n \geq n_0$. We will show that $\phi_H(G) \leq t_{r-1}(n) + c_0n^2$.

Apply the Regularity Lemma to G to find a $c_4/2$ -regular partition $V(G) = V_0 \cup V_1 \cup \dots \cup V_k$ with $1/c_3 \leq k < 1/c_5$. Remove all edges inside parts, in non-regular pairs and in regular pairs of density less than c_1 —these will be removed as single edges. We removed at most $c_1n^2 \ll c_0n^2$ edges.

Let K be the weighted complete graph on $[k]$ where the weight $\omega(ij)$ is the density of $G[V_i, V_j]$ (after the removals), where $G[V_i, V_j]$ denotes the bipartite graph on $V_i \cup V_j$ consisting of all edges of G between V_i and V_j . As $k \geq 1/c_3$ is large, by Lemma 2.4 we can find a weighted K_r -decomposition of K with total weight at most $t_{r-1}(k) + 2c_1k^2$, where each K_r has weight at least c_2 . Let A_1, \dots, A_t be all the K_r 's with weights $\alpha_1, \dots, \alpha_t$ respectively. Note that

$$t \leq \frac{\binom{k}{2}}{c_2 \binom{r}{2}}. \tag{2.4}$$

Perform the following procedure for each pair ij with $\omega(ij) > 0$. For $l \in [t]$, let $p_{ij,l} = \alpha_l/\omega(ij)$ if $i, j \in A_l$ and let $p_{ij,l} = 0$ otherwise. Let $p_{ij,0} = 1 - \sum_{l=1}^t p_{ij,l} \geq 0$. Partition $G[V_i, V_j]$ into bipartite subgraphs $B_{ij,0}, \dots, B_{ij,t}$ with vertex sets $V_i \cup V_j$, where each edge of $G[V_i, V_j]$ is included into $B_{ij,l}$ with probability $p_{ij,l}$, independently of the other edges. For $1 \leq l \leq t$, the expected density of $B_{ij,l}$ is α_l if $ij \in A_l$ and 0 otherwise.

Let us call a bipartite graph $G[A, B]$ (c, ε) -regular if for every $X \subset A$ and $Y \subset B$ satisfying $|X| > \varepsilon|A|$ and $|Y| > \varepsilon|B|$ we have $|d(X, Y) - c| < \varepsilon$. For example, if a bipartite graph is (c, ε) -regular, then it is 2ε -regular (as defined in Section 2).

Claim 1. *With high probability for every i, j, l with $\omega(ij) > 0$ and $ij \in A_l$ the graph $B_{ij,l}$ is (α_l, c_4) -regular.*

Proof. Recall that $a \pm c$ means a real between $a - c$ and $a + c$. Let $v = |V_i| = |V_j| \geq (1 - c_4/2)n/k$.

Fix any $U_i \subset V_i$ and $U_j \subset V_j$, each of size at least c_4v . By the $c_4/2$ -regularity of $G[V_i, V_j]$, the pair U_i, U_j spans $(\omega(ij) \pm c_4/2)|U_i||U_j|$ edges in G . The number of edges in $B_{ij,l}[U_i, U_j]$ has binomial distribution with parameters $(e(G[U_i, U_j]), p_{ij,l})$.

Using Chernoff’s bound [5] we can bound the probability that the pair U_i, U_j violates the (α_l, c_4) -regularity by $e^{-\lambda v^2}$, where λ can be chosen to depend on c_4 only. (Recall that $\alpha_l \geq c_2$.) Hence, for fixed i, j, l , the expected number of pairs U_i, U_j violating the (α_l, c_4) -regularity is at most

$$(2^v)^2 e^{-\lambda v^2} = o(k^{-4}).$$

Since the total number of choices for i, j and l is at most $k^2t = O(k^4)$ by (2.4), it follows that the expected number of pairs U_i, U_j violating the (α_l, c_4) -regularity is $o(1)$. Markov’s inequality implies the claim. \square

Fix any choice of $B_{ij,l}$ satisfying the conclusions of Claim 1.

Claim 2. *Let $r \geq 3$ and $\chi(H) = r$. Let $c_2 \gg c_3 \gg c_4 \gg 1/v$. Let $\lambda > c_2$ and G' be an r -partite graph on $V_1 \cup \dots \cup V_r$ with each $|V_i| = v$ such that each $G'[V_i, V_j]$ is (λ, c_4) -regular. Then G' minus at most $c_2e(G')$ edges can be perfectly decomposed into edge disjoint copies of H .*

Proof. Fix a coloring $h : V(H) \rightarrow [r]$ of H . Let H have m edges and s vertices. We can assume that the number of edges between any two color classes of the coloring h is the same, namely $q = m/\binom{s}{2}$. Indeed, otherwise, instead of H , we consider $r!$ vertex disjoint copies of H and extend h onto the new graph by taking all possible permutations of colors on copies of H .

We will apply Theorem 2.2 to the hypergraph \mathcal{H} whose vertex set consists of all edges of G' and whose hyperedges are the edge-sets of (not necessarily induced) H -subgraphs of G' such that $x \in V(H)$ is embedded into $V_{h(x)}$. Thus $v(\mathcal{H}) = e(G') = (\lambda \pm c_4)v^2\binom{r}{2}$. Let

$$D = v^{s-2}\lambda^{m-1}q.$$

First, let us briefly recall the standard argument for counting vertex-labeled H -subgraphs, see, e.g., Simonovits and Sós [17, Theorem 5]. It is slightly modified to better suit our purpose. Arbitrarily order the vertices of H as x_1, \dots, x_s . For $i \in [s]$ let $U_{i,1} = V_{h(x_i)}$. We will be constructing the embedding $f : V(H) \rightarrow V(G')$ one by one as follows. Suppose we have already embedded x_1, \dots, x_{j-1} and have the current potential sets $U_{1,j}, \dots, U_{s,j}$ where $U_{i,j} = \{f(x_i)\}$ for $i = 1, \dots, j - 1$. We are about to embed x_j . For $i > j$ with $x_jx_i \in E(H)$ let the *bad* set $B_{j,i}$ consist of all vertices $x \in U_{j,j}$ such that $|\Gamma(x) \cap U_{i,j}| \neq (\lambda \pm c_4)|U_{i,j}|$. (For all other i ’s, we let $B_{j,i} = \emptyset$ for convenience.)

If we assume that

$$|U_{i,j}| \geq c_4v, \tag{2.5}$$

then $|B_{j,i}| \leq 2c_4v$. Indeed, let X (resp. Y) consist of those $x \in U_{j,j}$ that have more than $(\lambda + c_4)|U_{i,j}|$ (resp. less than $(\lambda - c_4)|U_{i,j}|$) neighbors in $U_{i,j}$. The (λ, c_4) -regularity of $G'[V_{h(x_i)}, V_{h(x_j)}]$ implies that $|X| \leq c_4v$ and $|Y| \leq c_4v$. Since $B_{j,i} = X \cup Y$, the claim follows.

Hence, in total there are at most $2c_4sv$ bad vertices in $U_{j,j}$. For $f(x_j)$ choose any vertex of $U_{j,j}$ that is not bad. Update:

$$U_{i,j+1} = \begin{cases} \{f(x_i)\}, & i \leq j, \\ U_{i,j} \setminus \{f(x_j)\}, & i > j \text{ and } x_jx_i \notin E(H), \\ (U_{i,j} \setminus \{f(x_j)\}) \cap \Gamma(f(x_j)), & i > j \text{ and } x_jx_i \in E(H). \end{cases}$$

For any $i > j$ we have $|U_{i,j+1}| \geq (\lambda - c_4)^m v - s \geq c_4v$, so (2.5) and all above estimates are valid by induction on j .

Recall that $c_4 \ll c_3 \ll \lambda$. Rather crudely, it follows that the number of the above embeddings is

$$(\lambda \pm c_4 \pm 2c_4s)^m (v \pm 2c_4sv)^s = (1 \pm c_3)v^s \lambda^m.$$

In all other embeddings that preserve the coloring h , we have to use a *bad* vertex (that is, a vertex in a bad set given the fixed ordering x_1, \dots, x_s) at least once. Hence, the number of the remaining embeddings is at most

$$2c_4s^2v^s \ll (1 \pm c_3)v^s \lambda^m.$$

Now call an edge xy , with say $x \in V_i$ and $y \in V_j$, of G' *good* if

- x has $(\lambda \pm c_4)(v - 1)$ neighbors in $V_j \setminus \{y\}$,
- y has $(\lambda \pm c_4)(v - 1)$ neighbors in $V_i \setminus \{x\}$,
- for any $g \in [r] \setminus \{i, j\}$, each of x, y has $(\lambda \pm c_4)v$ neighbors in V_g while their common neighborhood in V_g has size $(\lambda \pm c_4)^2v$.

The above argument gives that all but at most

$$\binom{r}{2} (2c_4v(r - 1) \times v + v \times 2c_4(2r - 3)) < c_3e(G')$$

edges of G' are good and that any good edge belongs to $(1 \pm c_3)v^{s-2}\lambda^{m-1}q = (1 \pm c_3)D$ vertex-labelled copies of H . This shows that \mathcal{H} satisfies Condition (1) of Theorem 2.2.

For any edge, there are at most $v^{s-2} < D/c_3$ H -subgraphs containing it. For any two edges, there are at most $v^{s-3} < c_3D$ H -subgraphs containing both of them. Hence, all assumptions of Theorem 2.2 are satisfied.

Therefore \mathcal{H} contains a matching consisting of at least $(1 - c_2)v(\mathcal{H})/m$ hyperedges, that is, our graph G' contains at least $(1 - c_2)e(G')/m$ edge disjoint copies of H . We are left with at most $c_2e(G')$ edges of G' not decomposed. So Claim 2 holds. \square

This shows that for each $l \in [t]$, we can find at least

$$(1 - c_2)\alpha_l \binom{r}{2} / m \times ((1 - c_4/2)n/k)^2 \geq (1 - 2c_2) \frac{\alpha_l}{m} \binom{r}{2} (n/k)^2$$

pairwise edge disjoint H -subgraphs in B_l , where B_l is the union of bipartite graphs $B_{ij,l}$, $ij \in \binom{[k]}{2}$. All the remaining edges of our graph G are removed one by one as single edges.

Let $\alpha = \sum_{i=1}^t \alpha_i$ and $\omega(K) = \sum_{ij \in E(K)} \omega(ij)$. We have $m \geq \binom{r}{2}$ and one can easily prove that $e(G) \leq \omega(K)n^2/k^2 + c_1n^2$. Furthermore, the total weight of the decomposition of the weighted graph K is $\alpha + \omega(K) - \binom{r}{2}\alpha$ which is at most $t_{r-1}(k) + 2c_1k^2$ by Lemma 2.4. Therefore, the total number of parts in our decomposition of G is at most

$$\begin{aligned}
 & \alpha(1 - 2c_2) \binom{r}{2} \frac{n^2}{mk^2} + e(G) - m\alpha(1 - 2c_2) \binom{r}{2} \frac{n^2}{mk^2} \\
 &= \left(\frac{1 - 2c_2}{m} - (1 - 2c_2) \right) \alpha \binom{r}{2} \frac{n^2}{k^2} + e(G) \\
 &\leq \left(\alpha - \binom{r}{2} \alpha + \omega(K) + (m - 1)2c_2\alpha \right) \frac{n^2}{k^2} + c_1n^2 \\
 &\leq (t_{r-1}(k) + 2c_1k^2) \frac{n^2}{k^2} + 2c_1n^2 \\
 &\leq t_{r-1}(n) + c_0n^2
 \end{aligned}$$

as required. This finishes the proof of Theorem 1.1. \square

Our proof can be converted to a randomized algorithm that for given H , $\varepsilon > 0$ and G produces an H -decomposition of G with at most $t_{r-1}(n) + \varepsilon n^2$ parts, where $r = \chi(H)$, $n = v(G)$, and n is sufficiently large. We have to use the algorithmic version of the Regularity Lemma by Alon, Duke, Lefmann, Rödl and Yuster [2] while the proofs of Theorem 2.2 and Claim 1 of Section 2 naturally give randomized algorithms. (Since it is co-NP-complete to decide if a bipartite graph is ε -regular, see [2], we do not verify the regularity of each output graph $B_{i,j,l}$ of Claim 1 but check whether each hypergraph \mathcal{H} of Claim 2 satisfies the assumptions of Theorem 2.2.) The running time of our algorithm can be bounded by a polynomial P in n whose degree depends only on H . Unfortunately, the coefficients of P will grow very fast with ε since the required number of parts in an ε -regularity partition can grow as tower-like function of $1/\varepsilon$, see Gowers [9].

3. H -decompositions for a bipartite H

In this section we will prove Theorem 1.4. Before we start with the proof, we provide some auxiliary results.

Lemma 3.1. *For any bipartite graph H with bipartition (V_1, V_2) and any $A \subset V_1$ with $a \geq 1$ elements, there are integers C and n_0 such that the following holds. In any graph G of order $n \geq n_0$ with minimum degree $\delta(G) \geq \frac{2}{3}n$ there is a family of edge disjoint copies of H such that the vertex subsets corresponding to $A \subset V(H)$ are disjoint and cover all but at most C vertices of G . One can additionally ensure that each vertex of G belongs to at most $3(v(H))^2$ copies of H .*

Proof. Let $|V_1| = h_1$, $|V_2| = h_2$ and let $t = 2\lceil h_1/a \rceil h_2 a$. Let K be the complete 3-partite graph with t vertices in each color class. Let n_0 be sufficiently large. Let G be a graph with $n \geq n_0$ vertices and minimum degree at least $\frac{2}{3}n$.

A theorem of Shokoufandeh and Zhao [16] (see also Alon and Yuster [3] and Komlós, Sárközy, and Szemerédi [12]) implies that, in G , we can find vertex disjoint K -subgraphs covering all but at most C vertices, where C is a constant. Therefore, it suffices to prove that K contains $3t/a$ edge disjoint copies of H having vertex disjoint sets corresponding to A .

Claim. *The complete bipartite graph $K_{t,t}$ contains t/a edge disjoint copies of H with vertex disjoint sets A in one part.*

Proof of claim. Let (X, Y) be a bipartition of $K_{t,t}$. For $1 \leq i \leq t/a$ define $X_i = \{(i - 1)a + 1, \dots, (i - 1)a + h_1\}$ and $A_i = \{(i - 1)a + 1, \dots, ia\}$ where the elements are taken modulo t .

Consider the graph \mathcal{G} with vertex set $X_1, \dots, X_{t/a}$ and $\{X_i, X_j\}$ is an edge if and only if $X_i \cap X_j \neq \emptyset$. For $i = 1, \dots, t/a$, $\deg X_i$ is at most the number of other sets, not equal to X_i , that contain an endpoint of the interval X_i . Thus, $\Delta(\mathcal{G}) \leq 2(\lceil h_1/a \rceil - 1)$. Properly color the vertices of \mathcal{G} using at most $\Delta(\mathcal{G}) + 1$ colors.

Let $I_1, \dots, I_{t/h_2}$ be disjoint subsets of Y of size h_2 . We pair all color- k vertices of \mathcal{G} with I_k . All X_i get paired since the number of colors is at most t/h_2 . Observe that a pair X_i and I_j induces a copy of K_{h_1, h_2} . Inside this graph choose an arbitrary H -subgraph so that $A_i \subset X_i$ corresponds to $A \subset V_1$. Since I_j is paired with pairwise disjoint subsets of X , the obtained copies of H are edge disjoint. This completes the proof of the claim. \square

Returning to the proof of the lemma, let (X, Y, Z) be a 3-partition of K . Apply the claim to the complete bipartite graphs with bipartitions (X, Y) , (Y, Z) and (Z, X) . To complete the proof observe that each vertex of K appears in at most

$$2 \left\lceil \frac{h_1}{a} \right\rceil + \frac{t}{a} \leq 2 \left\lceil \frac{h_1}{a} \right\rceil + 2h_2 \left\lceil \frac{h_1}{a} \right\rceil \leq 2v(H) + 2(v(H))^2 \leq 3(v(H))^2$$

copies of H . \square

The following results appearing in Alon, Caro and Yuster [1, Theorem 1.1, Corollary 3.4, Lemma 3.5] which follow with some extra work from the powerful decomposition theorem of Gustavsson [10], are crucial to the proof of Theorem 1.4.

Recall that for a non-empty graph H , $\gcd(H)$ denotes the greatest common divisor of the degrees of H .

Lemma 3.2. *For any non-empty graph H with m edges, there are $\gamma > 0$ and N_0 such that the following holds. Let $d = \gcd(H)$. Let G be a graph of order $n \geq N_0$ and of minimum degree $\delta(G) \geq (1 - \gamma)n$.*

If $d = 1$, then

$$p_H(G) = \left\lfloor \frac{e(G)}{m} \right\rfloor. \tag{3.1}$$

If $d \geq 2$, let $\alpha_u = d \lfloor \frac{\deg(u)}{d} \rfloor$ for $u \in V(G)$ and let X consist of all vertices whose degree is not divisible by d . If $|X| \geq \frac{n}{10d^3}$, then

$$p_H(G) = \left\lfloor \frac{1}{2m} \sum_{u \in V(G)} \alpha_u \right\rfloor. \tag{3.2}$$

If $|X| < \frac{n}{10d^3}$, then

$$p_H(G) \geq \frac{1}{m} \left(e(G) - \frac{n}{5d^2} \right). \tag{3.3}$$

Proof of Theorem 1.4. Given H , let $\gamma(H)$ and N_0 be given by Lemma 3.2. Assume that $\gamma \leq \gamma(H)$ is sufficiently small and that $n_0 \geq N_0$ is sufficiently large to satisfy all the inequalities we will encounter. Let $n \geq n_0$ and let G be any graph of order n with $\phi_H(G) = \phi_H(n)$.

Let $G_n = G$ and $i = n$. Repeat the following at most $\lfloor n/\log n \rfloor$ times. (Here the function $\lfloor n/\log n \rfloor$ was chosen to suit our needs and it is not meant to be the best one.)

If the current graph G_i has a vertex x_i of degree at most $(1 - \gamma/2)i$, let $G_{i-1} = G_i - x_i$ and decrease i by 1.

Suppose we stopped after s repetitions. Then, either $\delta(G_{n-s}) \geq (1 - \gamma/2)(n - s)$ or $s = \lfloor n/\log n \rfloor$. Let us show that the latter cannot happen. Otherwise, we have

$$e(G) \leq \binom{n-s}{2} + \left(1 - \frac{\gamma}{2}\right) \sum_{i=n-s+1}^n i < \binom{n}{2} - \frac{\gamma n^2}{4 \log n}. \tag{3.4}$$

Let t satisfy $K_{t,t} \supset H$. Using (1.2), (1.4), and (3.4) we obtain

$$\phi_H(G) < \frac{1}{m} \left(\binom{n}{2} - \frac{\gamma}{4} \frac{n^2}{\log n} \right) + \frac{m-1}{m} cn^{2-1/t} < \frac{1}{m} \binom{n}{2} \leq \phi_H(K_n),$$

which contradicts our assumption on G . Therefore, $s < \lfloor n/\log n \rfloor$ and we have $\delta(G_{n-s}) \geq (1 - \gamma/2)(n - s)$.

Let $\alpha = 2\gamma$. We will have another pass over the vertices x_n, \dots, x_{n-s+1} , each time decomposing the edges incident to x_i by H -subgraphs and single edges. It will be the case that each time we remove the edges incident to the current vertex x_i , the degree of any other vertex drops by at most $3h^4$, where $h = v(H)$. Here is a formal description. Initially, let $G'_n = G$ and $i = n$. If in the current graph G'_i we have $\deg_{G'_i}(x_i) \leq \alpha n$, then we remove all G'_i -edges incident to x_i as single edges and let $G'_{i-1} = G'_i - x_i$.

Suppose that $\deg_{G'_i}(x_i) > \alpha n$. Then, the set

$$X_i = \{y \in V(G_{n-s}): x_i y \in E(G'_i)\},$$

has at least $\alpha n - s + 1$ vertices. The minimum degree of $G[X_i]$ is

$$\delta(G[X_i]) \geq |X_i| - s - \frac{\gamma n}{2} - s \times 3h^4 \geq \frac{2}{3}|X_i|.$$

Let $y \in V(H)$, $A = \Gamma_H(y)$ and $a = |A|$. By Lemma 3.1 there is a constant C such that all but at most C vertices of $G[X_i]$ can be covered by edge disjoint copies of $H - y$ each of them having vertex disjoint sets A . Therefore, all but at most C edges between x_i and X_i can be decomposed into copies of H . All other edges incident to x_i are removed as single edges. Let G'_{i-1} consist of the remaining edges of $G'_i - x_i$ (that is, those edges that do not belong to an H -subgraph of the above x_i -decomposition). This finishes the description of the case $\deg_{G'_i}(x_i) > \alpha n$.

Consider the sets $S = \{x_n, \dots, x_{n-s+1}\}$, $S_1 = \{x_i \in S: \deg_{G'_i}(x_i) \leq \alpha n\}$, and $S_2 = S \setminus S_1$. Let their sizes be s , s_1 , and s_2 respectively, so $s = s_1 + s_2$.

Let F be the graph with vertex set $V(G_{n-s}) \cup S_2$, consisting of the edges coming from the removed H -subgraphs when we processed the vertices in S_2 . We have

$$\phi_H(G) \leq \phi_H(G'_{n-s}) + \frac{e(F)}{m} + s_1 \alpha n + s_2 C + \binom{s}{2}. \tag{3.5}$$

We know that $\phi_H(G'_{n-s}) = e(G'_{n-s}) - p_H(G'_{n-s})(m - 1)$. The last statement of Lemma 3.1 guarantees that $\delta(G'_{n-s}) \geq (1 - \gamma)(n - s)$. Thus, $p_H(G'_{n-s})$ can be estimated using Lemma 3.2.

Consider first the case $d = 1$. Using the inequalities $\alpha \leq (2 - \gamma)/2m$ and $e(G'_{n-s}) + e(F) \leq \binom{n-s}{2} + (1 - \gamma/2)ns_2$, we obtain

$$\begin{aligned} \phi_H(G) &\leq e(G'_{n-s}) - \left\lfloor \frac{e(G'_{n-s})}{m} \right\rfloor (m-1) + \frac{e(F)}{m} + s_1\alpha n + s_2C + \binom{s}{2} \\ &\leq \left(\frac{1}{m} \binom{n-s}{2} + m-1 \right) + \frac{2-\gamma}{2m} s_2n + s_1\alpha n + s_2C + \binom{s}{2} \\ &\leq \frac{1}{m} \binom{n}{2} - \frac{(n-1)s}{m} + \frac{s(s-1)}{2m} + \frac{2-\gamma}{2m} sn + \binom{s}{2} + s_2C + m-1. \end{aligned}$$

If $S \neq \emptyset$ then in order to prove that $\phi_H(G) < \frac{1}{m} \binom{n}{2} \leq \phi_H(K_n)$ and hence a contradiction to our assumption on G , it suffices to show that

$$\frac{s}{m} + \frac{s(s-1)}{2m} + \binom{s}{2} + s_2C + m-1 < \left(\frac{1}{m} - \frac{2-\gamma}{2m} \right) ns = \frac{\gamma}{2m} ns.$$

But this last inequality holds since we have $s < \frac{n}{\log n}$ and n is sufficiently large. Thus, $S = \emptyset$ and

$$\phi_H(G) = e(G) - (m-1) \left\lfloor \frac{e(G)}{m} \right\rfloor, \tag{3.6}$$

is a function of $e(G)$ alone. By the optimality of G we cannot increase the right-hand side of (3.6) by increasing $e(G)$ by 1 or by m . Thus $e(G)$ is $\binom{n}{2}$ or the largest integer below $\binom{n}{2}$ congruent to $m-1$ modulo m . (In fact, the optimal value for $e(G)$ is unique unless $m=2$ and $\binom{n}{2}$ is even when both of the above values give the maximum.) This proves the theorem for the case $d=1$.

Consider the case $d \geq 2$. To prove the lower bound in (1.7) we consider a graph L of order $n \geq n_0$, which is r -regular (except at most one vertex of degree $r-1$) where $r \in [n-d, n-1]$ has residue $d-1$ modulo d . (Such a graph L exists, which can be seen either directly or from Erdős and Gallai’s result [7].)

Let $r = qd + d - 1$. Then $p_H(L) \leq \frac{ndq}{2m}$ and

$$\phi_H(L) = e(L) - p_H(L)(m-1) \geq \frac{1}{2}n(qd + d - 1) - \frac{1}{2} - \frac{ndq}{2m}(m-1),$$

giving the required lower bound in view of $q = \lfloor n/d \rfloor - 1$.

We will now prove the upper bound in (1.7).

Assume first that (3.3) holds. Then, by (3.5)

$$\begin{aligned} \phi_H(G) &\leq e(G'_{n-s}) - \frac{1}{m} \left(e(G'_{n-s}) - \frac{n-s}{5d^2} \right) (m-1) + \frac{e(F)}{m} + s_1\alpha n + s_2C + \binom{s}{2} \\ &\leq \frac{1}{m} \binom{n-s}{2} + \frac{m-1}{m} \frac{n-s}{5d^2} + \frac{2-\gamma}{2m} s_2n + s_1\alpha n + s_2C + \binom{s}{2} \\ &\leq \frac{1}{m} \binom{n}{2} - \frac{(n-1)s}{m} + \frac{s(s-1)}{2m} + \frac{m-1}{m} \frac{n-s}{5d^2} + \frac{2-\gamma}{2m} sn + s_2C + \binom{s}{2}. \end{aligned}$$

For $s > \frac{2(m-1)}{5\gamma d^2}$ we have $\frac{\gamma}{2m} - \frac{m-1}{5md^2s} > 0$. Thus, for n sufficiently large

$$\frac{s}{m} + \frac{s(s-1)}{2m} - \frac{m-1}{m} \frac{s}{5d^2} + \binom{s}{2} + s_2C < \left(\frac{1}{m} - \frac{2-\gamma}{2m} - \frac{m-1}{5md^2s} \right) ns.$$

That is, $\phi_H(G) < \frac{1}{m} \binom{n}{2} \leq \phi_H(K_n)$ which contradicts the optimality of G . Otherwise, s is bounded by a constant independent of n , and the terms of order n^2 and n alone give us the contradiction $\phi_H(G) < \phi_H(L)$, where L is the (almost) r -regular graph from the lower bound

on $\phi_H(n)$. In fact, the coefficient of sn is $-\frac{1}{m} + \frac{2-\gamma}{2m} < 0$, so to get a contradiction it is enough to show

$$\frac{1}{m} \binom{n}{2} + \frac{n}{5d^2} \leq \frac{nd}{2m} \left(\frac{n}{d} - 2 \right) + \frac{1}{2}n(d-1),$$

that is,

$$\frac{n}{5d^2} \leq \frac{1-2d}{2m}n + \frac{1}{2}n(d-1).$$

The worst case is when $m = 4$ (note $m \geq 4$ since $d \geq 2$). Therefore, it suffices to show that

$$\frac{8n}{5d^2} \leq (2d-3)n,$$

which holds as $d \geq 2$.

Finally, assume that (3.2) holds. It follows that $p_H(G)$ and thus $\phi_H(G)$, depends only on the degree sequence d_1, \dots, d_n of G . Namely, the packing number $\ell = p_H(G)$ equals $\lfloor \frac{1}{2m} \sum_{i=1}^n r_i \rfloor$, where $r_i = d \lfloor d_i/d \rfloor$ is the largest multiple of d not exceeding d_i .

Thus, is enough for us to prove the upper bound in (1.7) on ϕ_{\max} , the maximum of

$$\phi(d_1, \dots, d_n) = \frac{1}{2} \sum_{i=1}^n d_i - (m-1) \left\lfloor \frac{1}{2m} \sum_{i=1}^n \left\lfloor \frac{d_i}{d} \right\rfloor d \right\rfloor, \tag{3.7}$$

over all (not necessarily graphical) sequences d_1, \dots, d_n of integers with $0 \leq d_i \leq n-1$.

Let d_1, \dots, d_n be an optimal sequence attaining the value ϕ_{\max} . For $i = 1, \dots, n$ let $d_i = q_i d + r_i$ with $0 \leq r_i \leq d-1$. Then, $\ell = \lfloor \frac{(q_1 + \dots + q_n)d}{2m} \rfloor$.

Let $n = qd + r$ with $0 \leq r \leq d-1$ and $q = \lfloor n/d \rfloor$. Define $R = qd - 1$ to be the maximum integer which is at most $n-1$ and is congruent to $d-1$ modulo d . Let $C_1 = \{i \in [n]: r_i = d-1 \text{ and } d_i < R\}$ and $C_2 = \{i \in [n]: d_i = n-1\}$ if $n-1 \neq R$ and $C_2 = \emptyset$ otherwise.

Since d_1, \dots, d_n is an optimal sequence, we have that if $r_i \neq d-1$ then $d_i = n-1$ for all $i \in [n]$. Also, $|C_1| \leq \frac{2m}{d} - 1$ and $|C_2| \leq 2m - 1$. We have

$$\begin{aligned} \frac{1}{2} \sum_{i=1}^n d_i &= \frac{1}{2}(n - |C_1 \cup C_2|)R + \frac{1}{2} \sum_{i \in C_1} d_i + \frac{1}{2}|C_2|(n-1) \\ &\leq \frac{1}{2}nd(q-1) + \frac{1}{2}n(d-1) - \frac{d}{2} \sum_{i \in C_1} (q-1-q_i) + O(1), \\ \ell &\geq \left(\frac{1}{2m} \sum_{i=1}^n \left\lfloor \frac{d_i}{d} \right\rfloor d \right) - 1 \\ &\geq \frac{1}{2m}nd(q-1) - \frac{d}{2m} \sum_{i \in C_1} (q-1-q_i) + O(1). \end{aligned}$$

These estimates give us the required bound:

$$\phi_{\max} = \frac{1}{2} \sum_{i=1}^n d_i - (m-1)\ell \leq \frac{1}{2m}nd(q-1) + \frac{1}{2}n(d-1) + O(1). \tag{3.8}$$

If we want to compute the function $\phi_H(n)$ exactly we proceed as follows. From the obtained lower and upper bounds it follows that $\delta(G) \geq n - O(1)$ and $|C_1 \cup C_2| = O(1)$. Our algorithm

generates all such sequences, representing each one by listing the number n and then all degrees that are not equal to R . (Recall that R is the element of $[n - d, n - 1]$ congruent to $d - 1$ modulo d .) Each representation has only $O(1)$ terms, so it can be represented (and manipulated) in time polylogarithmic in n . Next, we eliminate all sequences that are not graphical. As it was shown by Tripathi and Vijay [21] it is enough to check as many inequalities in the Erdős and Gallai [7] criterion as there are distinct degrees, so we can do this in time $O(\log n)$. Finally, we compute $\phi(d_1, \dots, d_n)$ using (3.7) for each remaining sequence.

To finish the proof it remains to obtain a contradiction if $S \neq \emptyset$ holds. Let $\bar{d}_1, \dots, \bar{d}_n$ be the degree sequence of the graph with vertex set $V(G)$ and edge set $E(G'_{n-s}) \cup E(F)$. Consider the new sequence of integers

$$d'_i = \begin{cases} \bar{d}_i, & \text{if } x_i \notin S, \\ \bar{d}_i + \lceil \frac{(1-3\gamma)}{m} n \rceil m, & \text{if } x_i \in S_1, \\ \bar{d}_i + \lceil \frac{\gamma}{4m} n \rceil m, & \text{if } x_i \in S_2. \end{cases}$$

Each d'_i lies between 0 and $n - 1$, so $\phi(d'_1, \dots, d'_n) \leq \phi_{\max}$. We obtain

$$\begin{aligned} \phi_H(G) &\leq \phi(\bar{d}_1, \dots, \bar{d}_n) + s_1 \alpha n + s_2 C + \binom{s}{2} \\ &< \phi(d'_1, \dots, d'_n) - \frac{1-3\gamma}{2m} s_1 n - \frac{\gamma}{8m} s_2 n + s_1 \alpha n + s_2 C + \binom{s}{2} \\ &\leq \phi_{\max} - \frac{\gamma}{10m} s n, \end{aligned}$$

which contradicts the already established facts that the right-hand side of (1.7) is at most $\phi_H(G)$ by the optimality of G and is at least ϕ_{\max} by (3.8). \square

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