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NOTE

Enumeration of Labelled (k, m)-Trees

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A k-graph is called a (k, m)-*tree* if it can be obtained from a single edge by consecutively adding edges so that every new edge contains $k-m$ new vertices while its remaining m vertices are covered by an already existing edge. We prove that there are

$$\frac{(e(k-m)+m)!(e\binom{k}{m}-e+1)^{e-2}}{e!m!((k-m)!)^e}$$

distinct vertex labelled (k, m)-trees with e edges.

The notion of a tree and its different extensions to k-*graphs*, that is, k-uniform set systems, play an important role in discrete mathematics and computer science. We will dwell upon the following, rather general, definition suggested independently by Dewdney [5] and Beineke and Pippert [2].

Let us agree that the vertex set is $[n]=\{1, ..., n\}$. Fix the *edge size* k and the *overlap size* m, $0 \leqslant m \leqslant k-1$. We refer to k-subsets and m-subsets of $[n]$ as *edges* and *laps* respectively. A non-empty k-graph without isolated vertices is called a (k, m)-tree if we can order its edges, say $E_1, ..., E_e$, so that for every i, $2 \leqslant i \leqslant e$, there is i', $1 \leqslant i' < i$, such that $|E_i \cap E_{i'}| = m$ and $(E_i \backslash E_{i'}) \cap (\bigcup_{j=1}^{i-1} E_j) = \varnothing$. In other words, we start with a single edge and can consecutively affix a new edge along an m-subset of an existing edge.

Thus, a (k, m)-*tree* with e edges has $n=e(k-m)+m$ vertices and its edges cover $f=e(\binom{k}{m}-1)+1$ laps. For example, a $(k, 0)$-tree consists of disjoint edges.

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The problem of counting $(m+1, m)$-*trees* which are known in the literature as m-*trees*, received great attention and was completely settled by Beineke and Pippert [1] and Moon [9]. This extends celebrated Cayley's theorem [3] as, clearly, 1-trees correspond to usual (Cayley) trees. Later, different bijective proofs for m-trees appeared as well, see [11, 7, 8, 6, 4].

In this paper we enumerate (k, m)-trees. It is not surprising that many of the above counting techniques are applicable here as is, for example, Foata's bijection [7] (details are available from the author [10]). Also, as observed by the referee, our formula can be obtained via Chen's method [4]. We decided to present here an inductive proof which is the shortest one, perhaps.

Let $T_{km}(e)$ denote the number of (k,m)-trees on $[n]$ with e edges, $n = e(k-m)+m$, and let $R_{km}(e)$ count the trees *rooted* at the lap $[m]$, that is, those trees for which $[m]$ is covered by some edge.

THEOREM 1. *Given integers k, m, e with $0 \leqslant m \leqslant k-1$ and $e \geqslant 1$, let $n = e(k-m)+m$, $l = \binom{k}{m}$ and $f = e(l-1)+1$. Then the number of different (k, m)-trees on $[n]$ equals*

$$T_{km}(e) = \frac{n! f^{e-2}}{e! m! ((k-m)!)^e}. \tag{1}$$

Proof. Like in Beineke and Pippert [1], to prove the theorem, we write down a recurrence relation for $T_{km}(e)$ and then verify that (1) does satisfy the relation. Let us agree that $T_{km}(0) = R_{km}(0) = 1$.

Counting in two different ways the number of pairs (H, L), where H is a (k, m)-tree on $[n]$ rooted at an m-subset L of $[n]$, we obtain

$$\binom{n}{m} R_{km}(e) = f \cdot T_{km}(e). \tag{2}$$

Next, consider the following method for constructing trees. Select an edge E, a k-subset of $[n]$, and label by $L_1, \ldots, L_l$ the laps of E. Represent $e-1$ as a sum of l non-negative integers, $e-1 = e_1 + \cdots + e_l$. Partition $[n] \backslash E$ into sets $X_1, \ldots, X_l$ of sizes $e_1(k-m), \ldots, e_l(k-m)$ respectively. On each $L_i \cup X_i$ build a (k, m)-tree H_i rooted at L_i, $i \in [l]$. Clearly, the union of all H_i's plus the edge E forms a (k, m)-tree with e edges and every such tree H is obtained exactly e times. Therefore, by (2) we obtain

$$\begin{aligned} eT_{km}(e) &= \binom{n}{k} \sum_{\mathbf{e}} \frac{(n-k)!}{(e_1(k-m))! \cdots (e_l(k-m))!} \prod_{i=1}^{l} R_{km}(e_i) \\ &= \frac{n!}{k!} \sum_{\mathbf{e}} \prod_{i=1}^{l} \frac{m!(e_i(l-1)+1)\, T_{km}(e_i)}{(e_i(k-m)+m)!}, \end{aligned} \tag{3}$$

where $\sum_{\mathbf{e}}$ denotes the summation over all representations $e-1=e_1+\cdots+e_l$ with non-negative integers summands.

Clearly, formula (1) gives correct values for $e=0$. Also, the substitution of (1) into the both sides of (3) gives (after routine cancellations)

$$l(e(l-1)+1)^{e-2}=\sum_{\mathbf{e}}\frac{(e-1)!}{e_1!\cdots e_l!}\prod_{i=1}^{l}(e_i(l-1)+1)^{e_i-1}. \tag{4}$$

The last identity (in slightly different notation) was established by Beineke and Pippert [1, Lemma 2], which proves our theorem by induction. ▮

Corollary 1. *The number of labelled m-trees on n vertices*, $n>m\geqslant 1$, *is*

$$T_{m+1,m}(n-m)=\binom{n}{m}(mn-m^2+1)^{n-m-2}.$$

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