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Constructions of Turán systems that are tight up to a multiplicative constant

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ABSTRACT

For positive integers $n \geqslant s > r$, the *Turán function* $T(n, s, r)$ is the smallest size of an r-graph with n vertices such that every set of s vertices contains at least one edge. Also, define the *Turán density* $t(s, r)$ as the limit of $T(n, s, r)/\binom{n}{r}$ as $n \to \infty$. The question of estimating these parameters received a lot of attention after it was first raised by Turán in 1941. A trivial lower bound is $t(s, r) \geqslant 1/\binom{s}{s-r}$. In the 1990s, de Caen conjectured that $r \cdot t(r+1, r) \to \infty$ as $r \to \infty$ and offered 500 Canadian dollars for resolving this question.

We disprove this conjecture by showing more strongly that for every integer $R \geqslant 1$ there is μ_R (in fact, μ_R can be taken to grow as $(1 + o(1))\, R \ln R$) such that $t(r + R, r) \leqslant (\mu_R + o(1))/\binom{r+R}{R}$ as $r \to \infty$, that is, the trivial lower bound is tight for every R up to a multiplicative constant μ_R.

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1. Introduction

An *r-graph* G on a vertex set V is a collection of r-subsets of V, called *edges*. If the vertex set V is understood, then we may identify G with its edge set, that is, view G as a subset of $\binom{V}{r} := \{X \subseteq V : |X| = r\}$.

For positive integers $n \geqslant s > r$, a *Turán (n, s, r)-system* is an r-graph G on an n-set V such that every s-subset $X \subseteq V$ is *covered* by G, that is, there is $Y \in G$ with $Y \subseteq X$. Let the *Turán number* $T(n, s, r)$ be the smallest size of a Turán (n, s, r)-system. If we pass to the complements then $\binom{n}{r} - T(n, s, r)$ is $\mathrm{ex}(n, K_s^r)$, the maximum size of an n-vertex r-graph without K_s^r, the complete r-graph on s vertices. This is a key instance of the classical extremal *Turán problem* that goes back to Turán [29]. For the purposes of this paper, it is more convenient to work with the function T rather than with its complementary version ex. So we define the *Turán density*

$$t(s, r) := \lim_{n\to\infty} \frac{T(n, s, r)}{\binom{n}{r}}, \tag{1}$$

to be the asymptotically smallest edge density of a Turán (n, s, r)-system as $n \to \infty$. The limit in the right-hand size of (1) exists, since easy double-counting shows that $T(n, s, r)/\binom{n}{r}$ is non-decreasing in n.

We refer the reader to de Caen [6] and Sidorenko [24] for surveys of results and open problems on minimum Turán systems, and to Füredi [9] and Keevash [12] for surveys of the ex-function for general hypergraphs. Also, the table on Page 651 in Ruszinkó [23] lists some connections of Turán systems to various areas of combinatorial designs.

In the trivial case $r = 1$, it holds that $T(n, s, 1) = n-s+1$ for any $n \geqslant s > 1$. The case $r = 2$ was resolved in the fundamental paper of Turán [29] (with the special case when $(s, r) = (3, 2)$ previously done by Mantel [18]). In particular, it holds that $t(s, 2) = \frac{1}{s-1}$ with the upper bound coming from the disjoint union of $s - 1$ almost equal cliques. The problem of determining $T(n, s, r)$ for $r \geqslant 3$ was raised already in the above-mentioned paper of Turán [29] from 1941. Erdős [7, Section III.1] offered \$500 for determining $t(s, r)$ for at least one pair (s, r) with $s > r \geqslant 3$. This reward is still unclaimed despite decades of active research. It was conjectured by Turán and other researchers (see e.g. [4, Page 348]) that $t(s, 3) = 4/(s - 1)^2$ for each $r \geqslant 4$. Various constructions attaining this upper bound can be found in Sidorenko's survey [24, Section 7]. In the first open case $s = 4$, the computer-generated lower bound $t(4, 3) \geqslant 0.438...$ of Razborov [22] (improving on the earlier bounds in [11,5,1]) comes rather close to the conjectured value 4/9. As stated by Sidorenko [24, Section 8] (and this seems to be still true), the only pair (s, r) with $s > r \geqslant 4$ for which there is a plausible conjecture is $(5, 4)$, where a construction of Giraud [10] gives $t(5, 4) \leqslant \frac{5}{16} = 0.325$. (The best known lower bound is $t(5, 4) \geqslant \frac{627}{2380} = 0.2634...$ by Markström [19].)

Also, a lot of attention was paid to estimating $t(s, r)$ as $r \to \infty$. In the first interesting case $s = r + 1$, the trivial lower bound $t(r + 1, r) \geqslant \frac{1}{r+1}$ was improved to $1/r$

independently by de Caen [3], Sidorenko [27], and Tazawa and Shirakura [28]. Some further improvements (of order at most $O(1/r^2)$) for a growing sequence of r were made by Giraud (unpublished, see [1, Page 362]), Chung and Lu [1], and by Lu and Zhao [17].

In terms of the previously known upper bounds on $t(r+1,r)$ as $r \to \infty$ there was a sequence of better and better bounds: $O(1/\sqrt{r})$ by Sidorenko [26], $\frac{1+2\ln r}{r}$ by Kim and Roush [13], $\frac{\ln r + O(1)}{r}$ by Frankl and Rödl [8], and $(1+o(1))\frac{\ln r}{2r}$ by Sidorenko [25].

De Caen [6, Page 190] conjectured that $r \cdot t(r+1,r) \to \infty$ as $r \to \infty$ and offered 500 Canadian dollars for proving or disproving this. The question which of the bounds $\Omega(\frac{1}{r}) \leqslant t(r+1,r) \leqslant O(\frac{\ln r}{r})$ is closer to the truth was asked earlier by Kim and Roush [13, Page 243]. Also, Frankl and Rödl [8, Page 216] wrote that it is "conceivable" that $t(r+1,r) = O(1/r)$. Here we disprove de Caen's conjecture (and thus confirm the intuition of Frankl and Rödl), with the following explicit constants.

Theorem 1.1. *For all integers $n > r \geqslant 1$, it holds that $T(n,r+1,r) \leqslant \frac{6.239}{r+1}\binom{n}{r}$.*

Also, there is r_0 such that, for all integers $n > r \geqslant r_0$, it holds that $T(n,r+1,r) \leqslant \frac{4.911}{r+1}\binom{n}{r}$.

While the constant in the first part is worse than in the second part, we include both proofs as it may be useful to have a simple explicit bound valid for every pair (n,r).

In the general case, the trivial lower bound is $T(n,s,r) \geqslant \binom{n}{r}/\binom{s}{s-r}$: indeed, if $G \subseteq \binom{[n]}{r}$ has fewer edges than the stated bound then the expected number of edges inside a random s-set is $\binom{s}{r}\cdot|G|/\binom{n}{r} < 1$ so some s-set is not covered at all. Thus $t(s,r) \geqslant 1/\binom{s}{s-r}$. This was improved to

$$t(s,r) \geqslant \frac{1}{\binom{s-1}{s-r}}, \tag{2}$$

by de Caen [4], and this is still the best known general lower bound. In terms of upper bounds, Frankl and Rödl [8] proved that, for any integer $R \geqslant 1$, we have

$$t(r+R,r) \leqslant (1+o(1))\frac{R(R+4)\ln r}{\binom{r+R}{R}}, \quad \text{as } r \to \infty. \tag{3}$$

We can also remove the factor $\ln r$ in the above result of Frankl and Rödl:

Theorem 1.2. *For every integer $R \geqslant 1$, it holds that*

$$t(r+R,r) \leqslant (\mu + o(1))\,\frac{1}{\binom{r+R}{R}}, \quad \textit{as } r \to \infty, \tag{4}$$

where $\mu := (c_0+1)^{R+1}/c_0^R$ with $c_0 = c_0(R)$ being the largest real root of the equation $\mathrm{e}^c = (c+1)^{R+1}$.

While for a given integer R it is possible to numerically approximate the above constant μ (in particular, to see that $\mu < 4.911$ for $R = 1$), it is also interesting to see how μ grows with R. This is done in the following corollary to Theorem 1.2:

Corollary 1.3. *There is R_0 such that for every integer $R \geqslant R_0$, it holds that*

$$\limsup_{r\to\infty} t(r+R,r)\cdot\binom{r+R}{R} \leqslant R\ln R + 3R\ln\ln R. \tag{5}$$

Let us discuss the known upper bounds in the case when $r \to \infty$ while $R = R(r)$ is a function of r that also tends to the infinity. By analysing the construction of Frankl and Rödl [8], Sidorenko [25, Theorem 2] proved that $\mu(r+R,r) \leqslant (1+o(1))R\ln\binom{r+R}{r}$ provided $R \geqslant r/\log_2 r$. Liu and Pikhurko [16] observed that the restriction on R in Sidorenko's bound can be weakened to just $R \to \infty$. Also, Liu and Pikhurko [16] calculated that the recursive construction presented in this paper yields $\mu(r+R,r) \leqslant (1+o(1))R\ln R$ for any function $R = o(\sqrt{r})$.

2. Proofs

Our construction is motivated by the recursive constructions of covering codes in [2,14,15], of which Theorems 6 and 9 in Lenz, Rashtchian, Siegel and Yaakobi [15] are probably closest to the presented results. This connection was previously used by Verbitsky and Zhukovskii (personal communication) to prove new results on insertion covering codes using some methods developed for Turán systems; this project later developed into a joint paper [21]. Here, we exploit this connection by using the recursion in [15] as guiding intuition for our construction.

It will be convenient to extend the definition of $T(n,s,r)$ to allow all triples (n,s,r) of integers with $s > r \geqslant 0$ and $n \geqslant 0$. We agree (and this formally matches the general definition) that $T(n,s,r) = 0$ for $n < s$ (in particular, for $n \leqslant r$), while $T(n,s,0) = 1$ for $n \geqslant s \geqslant 1$. These degenerate cases will be used in our inductive proofs.

We need some definitions first. For integers $0 \leqslant m \leqslant n$, we denote $[n] := \{1,2,\dots,n\}$ and $[m,n] := \{m,m+1,\dots,n\}$. For an integer $m \geqslant 0$ and an ℓ-graph $H \subseteq \binom{[n]}{\ell}$, let $H \otimes_n K_*^m$ denote the $(\ell+m)$-graph on $[n]$ that consists of those $X \in \binom{[n]}{\ell+m}$ for which there is $Y \in H$ with Y being the initial ℓ-segment of X, that is, if we order the elements of X as $x_1 < \cdots < x_{\ell+m}$ under the natural ordering of $[n]$ then $\{x_1,\dots,x_\ell\} \in H$. Informally speaking, $H \otimes_n K_*^m$ is obtained from H by extending its edges into $(\ell+m)$-subsets of $[n]$ in all possible ways to the right. Also, we define

$$\mathcal{B}(H) := \left\{ B \in \binom{[n]}{\ell+1} : \binom{B}{\ell} \cap H \neq \emptyset \right\},$$

to consist of all $(\ell+1)$-subsets of $[n]$ covered by the ℓ-graph H.

The first part of Theorem 1.1 will be derived from the following lemma.

Lemma 2.1. *Let reals $\beta \in (0,1)$ and $c, \mu > 1$ be fixed such that $\lfloor \beta\mu \rfloor \geqslant c$ and*

$$\frac{c}{\beta\mu - 1} + \frac{\mathrm{e}^{-c}}{1-\beta} \leqslant 1. \tag{6}$$

Then, for all integers $n, r \geqslant 0$, there is a Turán $(n, r+1, r)$-system G_n^r with $|G_n^r| \leqslant \frac{\mu}{r+1}\binom{n}{r}$.

Proof. We construct G_n^r using induction on r and then on n. Let $r_0 := \lfloor \mu \rfloor - 1$.

For $r \leqslant r_0$ and any $n \geqslant 0$, we can let $G_n^r := \binom{[n]}{r}$ be the complete r-graph on $[n]$. Note that $\frac{\mu}{r+1} \geqslant 1$ so the desired upper bound $|G_n^r| \leqslant \frac{\mu}{r+1}\binom{n}{r}$ trivially holds.

Let $r > r_0$. Given r, we construct G_n^r inductively on n. For $n \in [0, r]$, we let $G_n^r := \emptyset$ be the empty r-graph on $[n]$, which trivially satisfies the lemma.

Let $n \geqslant r+1$. Define $k := \lfloor \beta(r+1) \rfloor$. Note that $k \geqslant 1$ since $r + 1 \geqslant r_0 + 2 > \mu$ while $\beta\mu \geqslant c > 1$ by our assumptions. Also, $k \leqslant r$ since $\beta < 1$. We will consider a random r-graph G_n^r (which will be a Turán $(n, r+1, r)$-system deterministically) and fix an outcome whose size is at most the expected value.

Let $S \subseteq \binom{[n]}{k-1}$ be a $\frac{c}{k}$-*binomial* random $(k-1)$-graph on $[n]$, that is, we include each $(k-1)$-subset of $[n]$ into S with probability c/k, with all choices being mutually independent. (Note that $k \geqslant \lfloor \beta(r_0 + 2) \rfloor \geqslant \lfloor \beta\mu \rfloor$ which is at least c by one of the assumptions, so $c/k \leqslant 1$.) The expected size of the r-graph $S^* := S \otimes_n K_*^{r-k+1}$ is exactly $\frac{c}{k}\binom{n}{r}$ because every r-set $Y \in \binom{[n]}{r}$ is included into it with probability c/k: indeed, $Y \in S^*$ if and only if the initial $(k-1)$-segment of Y is in S, which happens with probability c/k.

Let $T := \binom{[n]}{k} \setminus \mathcal{B}(S)$, that is, T consists of those k-subsets Y of $[n]$ such that no $(k-1)$-subset of Y belongs to S. Note that every $Y \in \binom{[n]}{k}$ is included into T with probability exactly $\left(1 - \frac{c}{k}\right)^k$: each of its k subsets of size $k-1$ has to be omitted from S. Let $T^* := T \otimes_n G_*^{r-k}$ be the r-graph on $[n]$ obtained as follows: for every edge $Y \in T$, let $y := \max Y$, take a copy G_Y of G_{n-y}^{r-k} on $[y+1, n]$ and add to T^* all sets $Y \cup Z$ with $Z \in G_Y$. (Note that if $y \geqslant n - r + k$ then no edges are added to T^* for this Y.) By the inductive assumptions (since $k \geqslant 1$) and by $1 - x \leqslant \mathrm{e}^{-x}$, the expected size of T^* can be upper bounded as follows, where y plays the role of $\max Y$ for $Y \in T$:

$$\begin{aligned}
\mathbb{E}|T^*| &= \sum_{y=k}^{n} \left(1 - \frac{c}{k}\right)^k \binom{y-1}{k-1} \cdot |G_{n-y}^{r-k}| \\
&\leqslant \sum_{y=k}^{n} \mathrm{e}^{-c} \binom{y-1}{k-1} \cdot \frac{\mu}{r-k+1}\binom{n-y}{r-k} \\
&= \frac{\mathrm{e}^{-c}\,\mu}{r-k+1}\binom{n}{r}.
\end{aligned}$$

Note that we may have $k = r$ or $y = n$ in the above expressions; this is why we found is convenient to allow any $n, r \geqslant 0$ when defining Turán (n, s, r)-systems.

Fix S such that $|S^* \cup T^*|$ is at most its expected value, and let $G_n^r := S^* \cup T^*$. We have by above that

$$|G_n^r| \leqslant \mathbb{E}|S^* \cup T^*| \leqslant \mathbb{E}|S^*| + \mathbb{E}|T^*| \leqslant \left(\frac{c}{k} + \frac{\mathrm{e}^{-c}\,\mu}{r-k+1}\right)\binom{n}{r}. \tag{7}$$

Since $r \geqslant r_0 + 1 \geqslant \lfloor \mu \rfloor$ and thus $r + 1 \geqslant \mu$, we can lower bound k as

$$k \geqslant \beta(r+1) - 1 = (r+1)\left(\beta - \frac{1}{r+1}\right) \geqslant (r+1)\left(\beta - \frac{1}{\mu}\right) = \frac{(r+1)(\beta\mu - 1)}{\mu}.$$

Using this and the trivial bound $r - k + 1 \geqslant (r+1)(1-\beta)$, we obtain from (7) that

$$|G_n^r| \leqslant \left(\frac{c\mu}{(r+1)(\beta\mu-1)} + \frac{\mathrm{e}^{-c}\,\mu}{(r+1)(1-\beta)}\right)\binom{n}{r},$$

which is at most the claimed bound $\frac{\mu}{r+1}\binom{n}{r}$ by (6).

Let us check that, regardless of the choice of S, the obtained r-graph G_n^r is a Turán $(n, r+1, r)$-system. Take any $(r+1)$-subset X of $[n]$. Let its elements be $x_1 < \dots < x_{r+1}$ and let $Y := \{x_1, \dots, x_k\}$. If Y is in T, then the Turán $(n - x_k, r-k+1, r-k)$-system G_Y on $[x_k + 1, n]$ contains an edge Z which is a subset of $\{x_{k+1}, \dots, x_{r+1}\} \in \binom{[x_k+1,n]}{r-k+1}$; thus G_n^r contains $Y \cup Z$ which is a subset of X, as desired. So suppose that Y is not in T. By definition, this means that there is $i \in [k]$ such that $Z := Y \setminus \{x_i\}$ is in S. But then $X \setminus \{x_i\}$ has Z as its initial $(k-1)$-segment and thus belongs to $S^* \subseteq G_n^r$, as desired. This finishes the proof of the lemma. □

It is easy to find a triple (β, c, μ) satisfying Lemma 2.1: for example, take $\beta = 1/2$, $c = 2$ and sufficiently large μ. The constant in the first part of Theorem 1.1 is the optimal μ coming from Lemma 2.1, rounded up in the third decimal digit.

Proof of the first part of Theorem 1.1. The assignment $\beta := 0.784$, $c := 2.89$ and $\mu := 6.239$ can be checked to satisfy Lemma 2.1. □

Remark 2.2. The best constant $\mu = 6.2387...$ for Lemma 2.1 comes from some small values of r. It can be improved in various ways, even just by using $T(n,3,2) \leqslant \frac{1}{2}\binom{n}{2}$ in the base case of the induction in the proof.

Next, we turn to general R (with the result also including the case $R = 1$, which will be used to derive the second part of Theorem 1.1).

Lemma 2.3. *If an integer $R \geqslant 1$, and reals $\beta \in (0,1)$ and $c, \mu > 0$ satisfy*

$$\frac{c}{\beta^R} + \frac{\mathrm{e}^{-c}\mu}{(1-\beta)^R} \leqslant \mu \tag{8}$$

(in particular, it holds that $\mathrm{e}^{-c} < (1-\beta)^R$*), then there is a constant* D *such that, for all integers* $n, r \geqslant 0$*, there is a Turán* $(n, r+R, r)$*-system* H_n^r *with*

$$|H_n^r| \leqslant \left(\mu + \frac{D}{\ln(r+3)}\right)\binom{r+R}{R}^{-1}\binom{n}{r}. \tag{9}$$

Proof. Given R, β, c and μ, fix constants C, r_0 and D in this order, with each being sufficiently large depending on the previous constants (and with r_0 being an integer). We construct $H_n^r \subseteq \binom{[n]}{r}$ by induction on r and, for each r, by induction on n. For $r \in [0, r_0]$ and any n, we can take the complete r-graph on $[n]$ for H_n^r. Note that (9) holds since we can assume that $\binom{r+R}{R} \leqslant D/\ln(r+3)$ for every such r. (We use that $\ln(r+3) \geqslant 1 > 0$ for every $r \geqslant 0$, which was the reason why 3 was added to the argument of ln.)

So let $r > r_0$. Define $k := \lfloor \beta r \rfloor$. We have that $k \geqslant \lfloor \beta r_0 \rfloor$ is at least R because r_0 is sufficiently large. For $n \in [0, r]$, we let H_n^r be the empty r-graph on $[n]$, which trivially satisfies the lemma. So let $n \geqslant r+1$.

Let S be a random subset of $\binom{[n]}{k-R}$ where each $(k-R)$-subset of $[n]$ is included with probability $c/\binom{k}{R}$ independently of the others. Note that $c/\binom{k}{R} \leqslant 1$ since $r \geqslant r_0$ is sufficiently large depending on β, c and R. Let $S^* := S \otimes_n K_*^{r-k+R}$. Recall that this is the r-graph obtained by extending the edges of S to the right into all possible r-subsets of $[n]$. Also, let $T := \binom{[n]}{k} \setminus \mathcal{B}_R(S)$, where

$$\mathcal{B}_R(S) := \left\{X \in \binom{[n]}{k} : \binom{X}{k-R} \cap S \neq \emptyset\right\}$$

consists of all k-subsets of $[n]$ covered by S. Let $T^* := T \otimes_n H_*^{r-k}$ be the r-graph on $[n]$ obtained as follows. For every edge Y in T, let $y := \max Y$, take a copy H_Y of H_{n-y}^{r-k} on $[y+1, n]$ and, for every $Z \in H_Y$, add $Y \cup Z$ to T^*. Take S such that the size of $H_n^r := S^* \cup T^*$ is at most its expected value.

Let us show that H_n^r is a Turán $(n, r+R, r)$-system, regardless of the choice of S. Take any $(r+R)$-subset $X \subseteq [n]$ with elements $x_1 < \cdots < x_{r+R}$. Let $Y := \{x_1, \ldots, x_k\}$. If Y is in T then some edge Z in the Turán $(n-x_k, r-k+R, r-k)$-system H_Y satisfies $Z \subseteq X \setminus Y$ and thus $Y \cup Z \in T^*$ is a subset of X. Otherwise there is an R-subset $Z \subseteq Y$ such that $Y \setminus Z \in S$; then $X \setminus Z$ is in S^* and is a subset of X. Thus every $(r+R)$-subset of $[n]$ is covered by H_n^r, as desired.

It remains to show that H_n^r satisfies (9). Similarly as in Lemma 2.1, we have by induction that

$$\begin{aligned}
|H_n^r| &\leqslant \mathbb{E}|S^*| + \mathbb{E}|T^*| \\
&\leqslant \frac{c}{\binom{k}{R}}\binom{n}{r} + \sum_{y=k}^{n}\left(1 - c\binom{k}{R}^{-1}\right)^{\binom{k}{R}}\binom{y-1}{k-1}\cdot|H_{n-y}^{r-k}| \\
&\leqslant \left(\frac{c}{\binom{k}{R}} + \frac{\mathrm{e}^{-c}}{\binom{r-k+R}{R}}\left(\mu + \frac{D}{\ln(r-k+3)}\right)\right)\binom{n}{r}.
\end{aligned}$$

Thus we have

$$|H_n^r|\,\frac{\binom{r+R}{R}}{\binom{n}{r}} \leqslant \frac{c}{\beta^R}+\frac{\mathrm{e}^{-c}\mu}{(1-\beta)^R}+\frac{C}{r}+\left(\left(\frac{\mathrm{e}^{-c}}{(1-\beta)^R}+\frac{C}{r}\right)\frac{D}{\ln(r-k+3)}\right)$$
$$\leqslant \mu+\frac{D}{\ln(r+3)}+\left(\left(\frac{\mathrm{e}^{-c}}{(1-\beta)^R}-1+\frac{1}{C}\right)\frac{D}{\ln(r-k+3)}\right)\ \leqslant\ \mu+\frac{D}{\ln(r+3)},$$

as desired. Here, the first inequality uses the fact that the ratio $\binom{r+R}{R}/\binom{k}{R}$ (resp. $\binom{r+R}{R}/\binom{r-k+R}{R}$) deviates from the "main" term β^{-R} (resp. $(1-\beta)^{-R}$) by $O(1/r)$, and the resulting error can be absorbed by C/r. The second inequality uses (8) and absorbs all error terms by the (much larger) term $D/(C\ln(r-k+3))$. The last inequality uses that $\mathrm{e}^{-c}<(1-\beta)^R$.

This finishes the proof of the lemma. □

Proof of Theorem 1.2. It is enough to show that if we take $c := c_0$ as in the theorem (that is, the largest root of $\mathrm{e}^c=(c+1)^{R+1}$), $\beta := \frac{c}{1+c}$ and $\mu := (c+1)^{R+1}/c^R$ then all assumptions of Lemma 2.3 are satisfied. Note that $\mathrm{e}^c-(c+1)^{R+1}$ is negative (resp. positive) for sufficiently small (resp. large) $c>0$ so c_0 is well-defined and positive.

In fact, this assignment was obtained as follows: first, we solved (8) as equality for μ, obtaining that $\mu = h(\beta, c)$, where

$$h(\beta,c) := \frac{c}{\beta^R-\mathrm{e}^{-c}(\frac{\beta}{1-\beta})^R}, \tag{10}$$

and then took a point at which the partial derivatives $\frac{\partial h}{\partial c}$ and $\frac{\partial h}{\partial \beta}$ vanish. (It seems that this choice of (β,c) minimises μ over the feasible region; however, this is not needed in our proof.)

Let us check that all assumptions of Lemma 2.3 are satisfied. Clearly, $\beta\in(0,1)$. For h as in (10) (and $c=c_0$), we have

$$h\left(\frac{c}{1+c},c\right)=\frac{c}{(\frac{c}{1+c})^R-\mathrm{e}^{-c}c^R}=\frac{c}{(\frac{c}{1+c})^R-(\frac{1}{c+1})^{R+1}c^R}=\frac{(c+1)^{R+1}}{c^R}=\mu,$$

that is, (8) holds (with equality). □

Proof of the second part of Theorem 1.1. The constant μ given by Theorem 1.2 for $R=1$ can be seen to be 4.9108..., which is less than the constant in the stated upper bound. Alternatively, it is enough just to give some feasible value of (β,c) such that (8) holds for $\mu := 4.911$; one can check that a pair $(0.715, 2.51)$ satisfies this. □

Proof of Corollary 1.3. Let R be sufficiently large and let c_0 be as defined in Theorem 1.2, that is, c_0 is the largest root of the equation $\mathrm{e}^c=(c+1)^{R+1}$.

Let us show that

$$R\ln R + R\ln\ln R < c_0 < R\ln R + 2R\ln\ln R. \tag{11}$$

For the upper bound, we have to show that, for any $c \geqslant R\ln R + 2R\ln\ln R$, it holds that $\mathrm{e}^c > (c+1)^{R+1}$, or by taking the logarithms that $c > (R+1)\ln(c+1)$. If, say, $c+1 \leqslant \mathrm{e}R\ln R$ then

$$\begin{aligned} c - (R+1)\ln(c+1) &\geqslant R\ln R + 2R\ln\ln R - (R+1)(\ln R + \ln\ln R + 1) \\ &= R\ln\ln R - O(R) > 0. \end{aligned}$$

Otherwise we have e.g. $R+1 \leqslant c/(2\ln(c+1))$ and thus $c-(R+1)\ln(c+1) \geqslant c/2 > 0$, as desired. On the other hand, for $c = R\ln R + R\ln\ln R$, we have very crudely that, say, $c+1 \leqslant \mathrm{e}R\ln R$ and thus

$$c-(R+1)\ln(c+1) \leqslant R\ln R + R\ln\ln R - (R+1)(\ln R + \ln\ln R + 1) = (-1+o(1))R < 0.$$

Thus the largest root c_0 of $\mathrm{e}^c = (c+1)^{R+1}$ indeed falls in the interval specified in (11).

We conclude that

$$\begin{aligned} \mu = (c_0+1)\left(1+\frac{1}{c_0}\right)^R &\leqslant (R\ln R + 2R\ln\ln R + 1)\left(1+\frac{1}{R\ln R + R\ln\ln R}\right)^R \\ &\leqslant (R\ln R + 2R\ln\ln R + 1)\left(1+\frac{2}{\ln R}\right) < R\ln R + 3R\ln\ln R. \end{aligned}$$

Now, Corollary 1.3 follows from Theorem 1.2. □

3. Concluding remarks

One can re-write the proof of Lemma 2.3 to also contain the conclusion of Lemma 2.1, with some minor changes for correctly handling the cases when $k < R$. However, the author feels that having a separate proof for the case $s = r+1$ is a good way to introduce the main ideas.

One can make the new lower bounds constructive, that is, for any fixed R, there is an algorithm that on input (n,r) outputs a Turán $(n, r+R, r)$-system in time polynomial in n^r. One has to replace a random subset $S \subseteq \binom{[n]}{k-R}$ by one constructed by the standard "conditioning method" (see e.g. [20]), in a very similar way as described in [14, Section IV].

Now that we know that $t(r+1,r) = O(1/r)$, the most intriguing remaining open question is whether $t(r+1,r) = (1+o(1))/r$ as $r \to \infty$ or not.

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