The maximal length of a gap between r-graph Turán densities

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Abstract

The *Turán density* $\pi(\mathcal{F})$ of a family $\mathcal{F}$ of r-graphs is the limit as $n \to \infty$ of the maximum edge density of an $\mathcal{F}$-free r-graph on n vertices. Erdős [Israel J. Math **2** (1964):183–190] proved that no Turán density can lie in the open interval $(0, r!/r^r)$. Here we show that any other open subinterval of $[0, 1]$ avoiding Turán densities has strictly smaller length. In particular, this implies a conjecture of Grosu [arXiv:1403.4653, 2014].

1 Introduction

Let $\mathcal{F}$ be a (possibly infinite) family of *r-graphs* (that is, r-uniform set systems). We call elements of $\mathcal{F}$ *forbidden*. An r-graph G is *$\mathcal{F}$-free* if no member $F \in \mathcal{F}$ is a subgraph of G, that is, we cannot obtain F by deleting some vertices and edges from G. The *Turán function* $\mathrm{ex}(n, \mathcal{F})$ is the maximum number of edges that an $\mathcal{F}$-free r-graph on n vertices can have. This is one of the central questions of extremal combinatorics that goes back to the fundamental paper of Turán [16]. We refer the reader to the surveys of the Turán function by Füredi [8], Keevash [12], and Sidorenko [15].

As was observed by Katona, Nemetz, and Simonovits [11], the limit

$$\pi(\mathcal{F}) := \lim_{n\to\infty} \frac{\mathrm{ex}(n, \mathcal{F})}{\binom{n}{k}}$$

exists. It is called the *Turán density* of $\mathcal{F}$. Let $\Pi_{\infty}^{(r)}$ consist of all possible Turán densities of r-graph families and let $\Pi_{\mathrm{fin}}^{(r)}$ be the set of all possible Turán densities when *finitely*

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many r-graphs are forbidden. It is convenient to allow empty forbidden families, so that 1 is also a Turán density. Clearly, $\Pi^{(r)}_{\mathrm{fin}} \subseteq \Pi^{(r)}_{\infty}$. A result of Brown and Simonovits [3, Theorem 1] implies that the topological closure $\mathrm{cl}(\Pi^{(r)}_{\mathrm{fin}})$ of $\Pi^{(r)}_{\mathrm{fin}}$ contains $\Pi^{(r)}_{\infty}$ while the converse inclusion was established in [14, Proposition 1]; thus

$$\Pi^{(r)}_{\infty} = \mathrm{cl}(\Pi^{(r)}_{\mathrm{fin}}), \quad \text{for every integer } r \geqslant 2. \tag{1}$$

For $r = 2$, the celebrated Erdős-Stone-Simonovits Theorem [5, 6] determines the Turán density for every family $\mathcal{F}$. In particular, we have

$$\Pi^{(2)}_{\mathrm{fin}} = \Pi^{(2)}_{\infty} = \left\{ \frac{m-1}{m} : m = 1, 2, 3, \ldots, \infty \right\}. \tag{2}$$

Unfortunately, the Turán function for *hypergraphs* (that is, r-graphs with $r \geqslant 3$) is much more difficult to analyse and many problems (even rather basic ones) are wide open.

Fix some $r \geqslant 2$. A *gap* is an open interval $(a, b) \subseteq (0, 1)$ that is disjoint from $\Pi^{(r)}_{\infty}$ (which, by (1), is equivalent to being disjoint from $\Pi^{(r)}_{\mathrm{fin}}$). Here we consider g_r, the maximal possible length of a gap. In other words, g_r is the maximal g such that there is a real a with $(a, a+g) \subseteq (0,1) \setminus \Pi^{(r)}_{\infty}$. For example, (2) implies that $g_2 = 1/2$. Erdős [4] proved that $(0, r!/r^r)$ is a gap; in particular, $g_r \geqslant r!/r^r$. Here we show that this is equality and every other gap has strictly smaller length.

Theorem 1. *For every $r \geqslant 3$, we have that $g_r = r!/r^r$ and, furthermore, $(0, r!/r^r)$ is the only gap of length $r!/r^r$ for r-graphs.*

In particular we obtain the following result that was conjectured by Grosu [9, Conjecture 10].

Corollary 2. *The union of r-graph Turán densities over all $r \geqslant 2$ is dense in $[0, 1]$, that is, $\mathrm{cl}(\cup_{r=2}^{\infty} \Pi^{(r)}_{\infty}) = [0, 1]$.* □

The question whether the set $\Pi^{(r)}_{\infty}$ is a well-ordered subset of $([0, 1], \leqslant)$ for $r \geqslant 3$ was a famous \$1000 problem of Erdős that was answered in the negative by Frankl and Rödl [7]. Despite a number of results that followed [7], very little is known about other gaps in $\Pi^{(r)}_{\infty}$ for $r \geqslant 3$. For example, let g'_r be the second largest gap length, that is, the maximum $g \geqslant 0$ such that $(a, a+g) \subseteq (r!/r^r, 1) \setminus \Pi^{(r)}_{\infty}$ for some a. The computer-generated proof of Baber and Talbot [2] implies that $g'_3 \geqslant 0.0017$. Klas Markström and Fei Song [13] conjectured that $(2/7, 8/27)$ is the (unique) second largest gap for 3-graphs (and, in particular, $g'_3 = 2/189$). However, not for a single $r \geqslant 4$ is it known, for example, whether g'_r is zero (i.e. whether $\Pi^{(r)}_{\infty}$ is dense in $[r!/r^r, 1]$).

This paper is organised as follows. In Section 2 we give some definitions and auxiliary results. Theorem 1 is proved in Section 3. We give another proof of Corollary 2 in Section 4. Although the latter proof is not strong enough to prove Theorem 1, its advantage is that it produces explicit elements of $\Pi^{(r)}_{\mathrm{fin}}$ (as opposed to the implicit values of certain maximisation problems returned by the proof in Section 3). So we include both proofs here, even though the second one is longer.

2 Preliminaries

For $n \in \mathbb{N}$, define $[n] := \{1, \ldots, n\}$. For reals $a \leqslant b$, let (a, b) and $[a, b]$ be respectively the open and closed interval of reals with endpoints a and b. The *standard* $(m-1)$*-dimensional simplex* is

$$\mathbb{S}_m := \{\boldsymbol{x} \in \mathbb{R}^m : x_1 + \cdots + x_m = 1,\ \forall\, i \in [m]\ x_i \geqslant 0\}.$$

An *r-pattern* is a collection P of r-multisets on $[m]$, for some $m \in \mathbb{N}$. (By an *r-multiset* we mean an unordered collection of r elements with repetitions allowed.) Let $V_1, \ldots, V_m$ be disjoint sets and let $V = V_1 \cup \cdots \cup V_m$. The *profile* of an r-set $X \subseteq V$ (with respect to $V_1, \ldots, V_m$) is the r-multiset on $[m]$ that contains $i \in [m]$ with multiplicity $|X \cap V_i|$. For an r-multiset Y on $[m]$, let $Y(\!(V_1, \ldots, V_m)\!)$ consist of all r-subsets of V whose profile is Y. We call this r-graph the *blow-up of* Y (with respect to $V_1, \ldots, V_m$) and the r-graph

$$P(\!(V_1, \ldots, V_m)\!) := \bigcup_{Y \in P} Y(\!(V_1, \ldots, V_m)\!)$$

is called the *blow-up of* P. Let the *Lagrange polynomial* of P be

$$\lambda_P(x_1, \ldots, x_m) := r! \sum_{D \in P} \prod_{i=1}^{m} \frac{x_i^{D(i)}}{D(i)!} \in \mathbb{R}[x_1, \ldots, x_m],$$

where $D(i)$ denotes the multiplicity of i in D. This definition is motivated by the fact that, for every partition $[n] = V_1 \cup \cdots \cup V_m$, we have that

$$|P(\!(V_1, \ldots, V_m)\!)| = \lambda_P\left(\frac{|V_1|}{n}, \ldots, \frac{|V_m|}{n}\right) \times \binom{n}{r} + O(n^{r-1}), \qquad \text{as } n \to \infty.$$

For example, if $r = 3$, $m = 3$, and P consists of multisets $\{1, 1, 2\}$ and $\{1, 2, 3\}$, then $P(\!(V_1, \ldots, V_m)\!)$ contains all triples that have two vertices in V_1 and one vertex in V_2 plus all triples with exactly one vertex in each part; here $\lambda_P(x_1, x_2, x_3) = 3x_1^2 x_2 + 6x_1 x_2 x_3$.

Let the *Lagrangian of* P be $\varLambda_P := \max\{\lambda_P(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{S}_m\}$, the maximum value of the polynomial λ_P on the compact set $\mathbb{S}_m$. One obvious connection of this parameter to r-graph Turán densities is that, if each blow-up of P is $\mathcal{F}$-free, then $\pi(\mathcal{F}) \geqslant \varLambda_P$. Also, it is not hard to show that $\varLambda_P = \pi(\mathcal{F})$, where $\mathcal{F}$ consists of all r-graphs F such that every blow-up of P is F-free; thus $\varLambda_P \in \varPi_\infty^{(r)}$. As shown in [14, Theorem 3], we have in fact that

$$\varLambda_P \in \varPi_{\text{fin}}^{(r)}, \quad \text{for every } r\text{-pattern } P. \tag{3}$$

We will use the special case of Muirhead's inequality (see e.g. [10, Theorem 45]) which states that, for any $0 \leqslant i < j \leqslant k$, we have

$$x^{k+i}y^{k-i} + x^{k-i}y^{k+i} \leqslant x^{k+j}y^{k-j} + x^{k-j}y^{k+j}, \quad \text{for } x, y \geqslant 0. \tag{4}$$

3 Proof of Theorem 1

Let $r \geqslant 3$. Fix a sufficiently large integer $m = m(r)$ so that $r!\binom{m}{r}/m^r > 1 - r!/r^r$. Consider r-graphs $G_0, \ldots, G_{\binom{m}{r}}$ on $[m]$ such that G_0 has no edges and, for $i = 1, \ldots, \binom{m}{r}$, the r-graph G_i is obtained from G_{i-1} by adding a new edge. In other words, we enumerate all r-subsets of $[m]$ as $R_1, \ldots, R_{\binom{m}{r}}$ and let $G_i := \{R_1, \ldots, R_i\}$. Let

$$\lambda_i(\boldsymbol{x}) := \lambda_{G_i}(\boldsymbol{x}) = r! \sum_{D \in G_i} \prod_{j \in D} x_j,$$

be the Lagrange polynomial of G_i and $\Lambda_i := \Lambda_{G_i}$ be its Lagrangian, where we view G_i as an r-pattern. Since $G_{i-1} \subseteq G_i$, we have that $\Lambda_{i-1} \leqslant \Lambda_i$.

We claim that for every $i \in [\,\binom{m}{r}\,]$

$$\Lambda_i - \Lambda_{i-1} \leqslant r!/r^r. \tag{5}$$

Indeed, pick $\boldsymbol{x} \in \mathbb{S}_m$ with $\Lambda_i = \lambda_i(\boldsymbol{x})$. Let $R_i = \{u_1, \ldots, u_r\}$. When we remove the term $r!\, x_{u_1} \ldots x_{u_r}$ from $\lambda_i(\boldsymbol{x})$, we get the evaluation of λ_{i-1} on $\boldsymbol{x} \in \mathbb{S}_m$. By definition, $\Lambda_{i-1} \geqslant \lambda_{i-1}(\boldsymbol{x})$. Also, since $x_{u_1} + \cdots + x_{u_r} \leqslant 1$, we have $x_{u_1} \ldots x_{u_r} \leqslant r^{-r}$ by the Geometric-Arithmetic Mean Inequality. Thus we obtain the stated bound:

$$\Lambda_i = \lambda_i(\boldsymbol{x}) = \lambda_{i-1}(\boldsymbol{x}) + r!\, x_{u_1} \ldots x_{u_r} \leqslant \Lambda_{i-1} + r!/r^r.$$

Also, we have $\Lambda_{\binom{m}{r}} \geqslant \lambda_{\binom{m}{r}}(\frac{1}{m}, \ldots, \frac{1}{m}) = r!\binom{m}{r}/m^r > 1 - r!/r^r$. This and (3) imply that $g_r \leqslant r!/r^r$ (while the above-mentioned result of Erdős [4] gives the converse inequality). Also, if we have equality in (5), then necessarily $x_{u_1} = \cdots = x_{u_r} = 1/r$, each other x_j is zero, and $\Lambda_{i-1} = \lambda_{i-1}(\boldsymbol{x}) = 0$, implying the uniqueness part of Theorem 1.

4 Alternative proof of Corollary 2

For integers $r, s \geqslant 2$, let $\mathcal{P}_{r,s}$ consist of ordered s-tuples $(r_1, \ldots, r_s)$ of non-negative integers such that $r_1 \geqslant \ldots \geqslant r_s$ and $r_1 + \cdots + r_s = r$. This set admits a partial order, where $\boldsymbol{x} \succcurlyeq \boldsymbol{y}$ if $\sum_{i=1}^k x_i \geqslant \sum_{i=1}^k y_i$ for every $k \in [s-1]$. For example, the (unique) maximal element is $(r, 0, \ldots, 0)$ and the (unique) minimal element is $(\lceil r/s \rceil, \ldots, \lfloor r/s \rfloor)$.

Let $A \subseteq \mathcal{P}_{r,s}$. The set A is called *down-closed* if $\boldsymbol{y} \in A$ whenever $\boldsymbol{x} \in A$ and $\boldsymbol{x} \succcurlyeq \boldsymbol{y}$. Let G_A consist of all r-multisets X on $[s]$ such that the multiplicities of X satisfy $\langle X(1), \ldots, X(s) \rangle \in A$, where $\langle \boldsymbol{x} \rangle$ denotes the non-increasing ordering of a vector $\boldsymbol{x}$. Also, we use the shorthand $\lambda_A := \lambda_{G_A}$ and $\Lambda_A := \Lambda_{G_A}$.

Lemma 3. *Let $r, s \geqslant 2$. If $A \subseteq \mathcal{P}_{r,s}$ is down-closed, then $\Lambda_A = \lambda_A(\frac{1}{s}, \ldots, \frac{1}{s})$.*

Proof. We use induction on s.

First, we prove the base case $s = 2$. Let $k := r/2$. For $h \geqslant 0$, let I_h consist of all integer translates of k whose absolute value is at most h, that is, $I_h := (\mathbb{Z} + k) \cap [-h, h]$.

Also, let $I_h^+ := I_h \cap [0,h]$. (These definitions will allow us to deal with the cases of even and odd r uniformly.) For example, $\mathcal{P}_{r,2} = \{(k+i, k-i) : i \in I_k^+\}$.

Take a down-closed set $A \subseteq \mathcal{P}_{r,2}$. It consists of pairs $(k+i, k-i)$ with $i \in I_h^+$ for some $h \leqslant k$. Then G_A consists of all $2k$-multisets on $\{1,2\}$ that contain 1 with multiplicity $k+i$ for $i \in I_h$. By the homogeneity of the polynomials involved, the required inequality can be rewritten as

$$\sum_{i\in I_h} \binom{2k}{k+i} \left(\frac{x+y}{2}\right)^{2k} - \sum_{i\in I_h} \binom{2k}{k+i} x^{k+i} y^{k-i} \geqslant 0, \quad \text{for } x, y \geqslant 0. \tag{6}$$

We will apply the so-called *bunching method* where we try to write the desired inequality as a positive linear combination of Muirhead's inequalities (4). If $j \in I_h$, then the coefficient in front of $x^{k+j}y^{k-j}$ in (6) is

$$2^{-2k}\binom{2k}{k+j}\sum_{i\in I_h}\binom{2k}{k+i} - \binom{2k}{k+j} \leqslant 0.$$

If $j \in I_k \setminus I_h$, then the coefficient is $2^{-2k}\binom{2k}{k+j}\sum_{i\in I_h}\binom{2k}{k+i} \geqslant 0$. Thus, if we group the left-hand side of (6) into terms $x^{k+j}y^{k-j} + x^{k-j}y^{k+j}$, then we get non-positive coefficients for $0 \leqslant j \leqslant h$ followed by non-negative coefficients for $j > h$. Also, the total sum of coefficients is zero because (6) becomes equality for $x = y = 1$. Thus we can "bunch" I_h-terms with $(I_k \setminus I_h)$-terms and use (4) to derive the desired inequality (6). This proves the case $s = 2$.

Now, let $s \geqslant 3$ and suppose that we have proved the lemma for $s-1$ (and all r). The function λ_A is a continuous function on the compact set $\mathbb{S}_s$. Let it attain its maximum on some $\boldsymbol{x} \in \mathbb{S}_s$. If there is more than one choice, then choose $\boldsymbol{x}$ so that $\Delta := \sum_{i\neq j}|x_i - x_j|$ is minimised. Suppose that $\Delta \neq 0$, say $x_1 \neq x_2$. Note that λ_A is a homogeneous polynomial of degree r, and the coefficient at $x_1^{r_1}\dots x_s^{r_s}$ is $\binom{r}{r_1,\dots,r_s}$ if the ordering $\langle \boldsymbol{r}\rangle$ of $\boldsymbol{r}$ is in A and 0 otherwise.

Fix $j \in \{0,\dots,r\}$. If we collect all terms in front of x_s^j, we get

$$\sum_{\substack{\langle \boldsymbol{r},j\rangle \in A\\ r_1+\dots+r_{s-1}=r-j}} \binom{r}{r_1,\dots,r_{s-1},j}\prod_{i=1}^{s-1} x_i^{r_i} = \binom{r}{j}\lambda_{A\setminus j}(x_1,\dots,x_{s-1}),$$

where $\langle \boldsymbol{y}, j\rangle$ is obtained from $\boldsymbol{y}$ by appending j and ordering the obtained sequence, while $A \setminus j$ consists of those $\boldsymbol{y} \in \mathcal{P}_{r-j,s-1}$ such that $\langle \boldsymbol{y}, j\rangle \in A$.

Let us show that $A \setminus j \subseteq \mathcal{P}_{r-j,s-1}$ is down-closed. Take arbitrary $\boldsymbol{z} \in A\setminus j$ and $\boldsymbol{y} \preccurlyeq \boldsymbol{z}$. We have to show that $\boldsymbol{y} \in A \setminus j$. Since $A \ni \langle \boldsymbol{z}, j\rangle$ is down-closed, it is enough to show that $\langle \boldsymbol{z}, j\rangle \succcurlyeq \langle \boldsymbol{y}, j\rangle$. We have to compare the sums of the first i terms of $\langle \boldsymbol{z}, j\rangle$ and of $\langle \boldsymbol{y}, j\rangle$ for each $i \in [s-1]$. A problem could arise only if the new entry j was included into these terms for $\langle \boldsymbol{y}, j\rangle$, say as the term number $h \leqslant i$, but not for $\langle \boldsymbol{z}, j\rangle$. Since $\boldsymbol{z} \succcurlyeq \boldsymbol{y}$, we have that $\sum_{f=1}^{h-1} z_f \geqslant \sum_{f=1}^{h-1} y_f$ (and these are also the initial sums for $\langle \boldsymbol{z}, j\rangle$ and $\langle \boldsymbol{y}, j\rangle$).

Furthermore, each of the subsequent $i-(h-1)$ entries is at least j for $\langle \boldsymbol{z}, j\rangle$ and at most j for $\langle \boldsymbol{y}, j\rangle$. It follows that $\langle \boldsymbol{z}, j\rangle \succcurlyeq \langle \boldsymbol{y}, j\rangle$. Thus $A\setminus j$ is down-closed, as claimed.

By the induction assumption (and since $\lambda_{A\setminus j}$ is a homogeneous polynomial), we have that $\lambda_{A\setminus j}(x_1,\ldots,x_{s-1}) \leqslant \lambda_{A\setminus j}(\frac{1-x_s}{s-1},\ldots,\frac{1-x_s}{s-1})$. Thus

$$\Lambda_A = \lambda_A(\boldsymbol{x}) = \sum_{j=0}^{r} \binom{r}{j} \lambda_{A\setminus j}(x_1,\ldots,x_{s-1})\, x_s^j \leqslant \lambda_A\left(\frac{1-x_s}{s-1},\ldots,\frac{1-x_s}{s-1},x_s\right).$$

Clearly, the sum $\sum_{i=1}^{s-1}|x_s-x_i|$ does not increase if we replace each of $x_1,\ldots,x_{s-1}$ by their arithmetic mean $(1-x_s)/(s-1)$. Since $x_1\neq x_2$, we have found another optimal element of $\mathbb{S}_s$ with strictly smaller Δ, a contradiction. The lemma is proved. $\square$

Fix some enumeration $\mathcal{P}_{r,r}=\{R_1,\ldots,R_t\}$ such that if $R_i \succcurlyeq R_j$ then $i\geqslant j$. For $j\in\{0,\ldots,t\}$, let $A_j := \{R_i : i\in[j]\}$. Thus, for example, $A_0=\varnothing$ and $A_t=\mathcal{P}_{r,r}$. By (3), $\Pi_{\mathrm{fin}}^{(r)}$ contains all of the following numbers:

$$0=\Lambda_{A_0}\leqslant \Lambda_{A_1}\leqslant\cdots\leqslant\Lambda_{A_t}=1.$$

Let us show that $\max\{\Lambda_{A_i}-\Lambda_{A_{i-1}} : i\in[t]\}=o(1)$ as $r\to\infty$. By definition, each $A_j\subseteq\mathcal{P}_{r,r}$ is down-closed. Thus, by Lemma 3 the difference $\Lambda_{A_i}-\Lambda_{A_{i-1}}$ is the probability that, when r balls are uniformly and independently distributed into r cells, the ordered ball distribution is given by R_i. Expose the first $r-m$ balls, where, for example, $m:=\lfloor \log r\rfloor$. Let k be the number of empty cells. Its expected value is $r(1-1/r)^{r-m}=(\mathrm{e}^{-1}+o(1))\,r$. By Azuma's inequality (see e.g. [1, Theorem 7.2.1]), we have *whp* (i.e. with probability $1-o(1)$ as $r\to\infty$) that k is in $I:=[r/4,3r/4]$. Assume that $k\in I$ and expose the remaining m balls. Let J be the number of balls that land inside the k cells that were empty after the first round. The probability that $J=j$ for any particular integer $j\in[m/8,7m/8]$ is

$$\begin{aligned}\binom{m}{j}\left(\frac{k}{r}\right)^{j}\left(\frac{r-k}{r}\right)^{m-j} &= (1+o(1))\sqrt{\frac{m}{2\pi j(m-j)}}\left(\frac{mk}{jr}\right)^{j}\left(\frac{m(r-k)}{(m-j)r}\right)^{m-j}\\ &\leqslant (1+o(1))\sqrt{\frac{m}{2\pi j(m-j)}} = o(1),\end{aligned}$$

where we used Stirling's formula and the Arithmetic-Geometric Mean Inequality. On the other hand, we have whp that $m/8\leqslant J\leqslant 7m/8$ (by Azuma's inequality and our assumption $k\in I$) and that the last m balls all go into different cells (since $m^2=o(r)$). Once the first $r-m$ balls are exposed and we condition on the event that the last m balls all land into distinct cells, there is at most one value of J for which the final ball distribution is R_i. Thus the probability of getting R_i is $o(1)$ uniformly in i, as desired. This finishes the second proof of Corollary 2.

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