# The minimum size of 3-graphs without a 4-set spanning no or exactly three edges

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## ARTICLE INFO

## ABSTRACT

Let $G_i$ be the (unique) 3-graph with 4 vertices and $i$ edges. Razborov [A. Razborov, On 3-hypergraphs with forbidden 4-vertex configurations, SIAM J. Discrete Math. 24 (2010) 946–963] determined asymptotically the minimum size of a 3-graph on $n$ vertices having neither $G_0$ nor $G_3$ as an induced subgraph. Here we obtain the corresponding stability result, determine the extremal function exactly, and describe all extremal hypergraphs for $n \geq n_0$. It follows that any sequence of almost extremal hypergraphs converges, which answers in the affirmative a question posed by Razborov.

## 1. Introduction

For a set $X$ and an integer $k$, let $\binom{X}{k} = \{Y \subseteq X : |Y| = k\}$. A *k-graph G with vertex set V* is a subset of $\binom{V}{k}$, i.e., it is a collection of $k$-element subsets of $V$. Elements of $V$ and $G$ are called *vertices* and *edges* respectively. We will also call $G$ a *hypergraph*.

Let $\mathcal{G}$ be a family of $k$-graphs. A $k$-graph $F$ is $\mathcal{G}$-*free* if it contains no member of $\mathcal{G}$ as an induced subgraph. Let $t(n, \mathcal{G})$ be the minimum size of a $\mathcal{G}$-free $k$-graph on $n$ vertices. This function is related to the Turán problem; we refer the reader to surveys by Füredi [9], Sidorenko [24], and Keevash [12].

If $\mathcal{G} = \{G\}$ consists of one $k$-graph $G$, we may abbreviate $t(n, \{G\})$ to $t(n, G)$, etc. For $0 \leq i \leq 4$, let $G_i$ be the (unique) 3-graph with 4 vertices and $i$ edges.

One of the most famous open questions in extremal combinatorics is to determine $t(n, G_0)$. It goes back to the fundamental paper by Turán [26] who conjectured that

$$t(n, G_0) = t_n, \tag{1}$$

where $t_n$ is defined as follows.

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For pairwise disjoint sets $V_0$, $V_1$, and $V_2$, the *Turán pattern* $T_{V_0,V_1,V_2}$ is the 3-graph on $V = V_0 \cup V_1 \cup V_2$ whose edges are triples $\{x, y, z\}$ with $x, y \in V_i$ and $z \in V_i \cup V_{i+1}$ for some $i \in \mathbb{Z}_3$. (Here $\mathbb{Z}_m$ denotes the additive group of residues modulo $m$.) Let $t_{v_0,v_1,v_2}$ be the number of edges in $T_{V_0,V_1,V_2}$ where $|V_i| = v_i$. The *Turán 3-graph* $T_n$ is the (unique up to isomorphism) Turán pattern $T_{V_0,V_1,V_2}$ with $v_0 + v_1 + v_2 = n$ and $|v_i - v_j| \leq 1$ for all $i, j \in \mathbb{Z}_3$. It is not hard to show (see Lemma 4) that among all Turán patterns on $n$ vertices, the Turán 3-graph $T_n$ has the smallest size. Let

$$t_n = |T_n|.$$

We have $t_n = \left(\frac{4}{9} + o(1)\right) \binom{n}{3}$ as $n \to \infty$. Also, any Turán pattern is $G_0$-free; thus $t(n, G_0) \leq t_n$. The problem of obtaining a matching lower bound (even within a $(1 + o(1))$-factor) seems to be extremely difficult. Successively better lower bounds on $t(n, G_0)$ were proved by de Caen [5], Giraud (unpublished, see [4]), and Chung and Lu [4]. Razborov [20,22] presented a general framework for working with extremal problems of this kind. His solution of a certain semidefinite program with over 900 variables suggests that $t(n, G_0) \geq 0.43833 \binom{n}{3}$ for all sufficiently large $n$, see also [1]. One of many difficulties here is that, if Turán's conjecture (1) is correct, then there are many non-isomorphic extremal 3-graphs, see [16,3,8]. Also, we refer the reader to Razborov [21] for some related results.

Note that $T_n$ is also $G_3$-free; thus $t(n, \{G_0, G_3\}) \leq t_n$. Applying his technique Razborov [22] proved the matching asymptotic lower bound. Thus

$$t(n, \{G_0, G_3\}) = \left(\frac{4}{9} + o(1)\right) \binom{n}{3}. \tag{2}$$

This result is interesting because there are very few non-trivial hypergraphs or hypergraph families for which the asymptotic of its Turán function is known. Also, it gives us a better understanding of the original conjecture of Turán. For example, if the conjecture is false, then any $G_0$-free 3-graph $G$ on $n$ vertices beating $t_n$ has to contain an induced copy of $G_3$. (In fact, if $|G| \leq (1 - \Omega(1))t_n$ as $n \to \infty$, then $G$ contains $\Omega(n^4)$ $G_3$-subgraphs by the super-saturation technique of Erdős and Simonovits [7].)

Here, we prove for all $n \geq n_0$ that $t(n, \{G_0, G_3\}) = t_n$ and the Turán hypergraph $T_n$ is the unique extremal 3-graph:

**Theorem 1** (*Exact Result*). *There is $n_0$ such that every $\{G_0, G_3\}$-free 3-graph $F$ on $n \geq n_0$ vertices has at least $t_n$ edges with equality if and only if $F \cong T_n$.*
*In particular, $t(n, \{G_0, G_3\}) = t_n$ for $n \geq n_0$.*

Theorem 1 is also interesting in the context of the rapidly developing theory of graph and hypergraph limits, see e.g. [17,2,6]. Although Razborov's proof of (2) is stated without any appeal to hypergraph limits, the flag algebras introduced by him provide a convenient and powerful language for manipulating limit objects. Also, any relations proved with the help of flag algebras or (hyper)graph limits hold only asymptotically as the order of the underlying (hyper)graph tends to infinity. So, at first sight, this technique can give asymptotic results only. However, the proof of Theorem 1 gives an example of how a solution of the "limiting" case may lead to an exact result for all sufficiently large $n$. The key ingredient here is the stability property which states, roughly speaking, that all almost extremal hypergraphs have essentially the same unique structure. Here is the precise formulation for the $\{G_0, G_3\}$-problem:

**Theorem 2** (*Stability Property*). *For every $\varepsilon > 0$ there is $c > 0$ such that the following holds. Let $G$ be a $\{G_0, G_3\}$-free 3-graph on $n > 1/c$ vertices with at most $t_n + cn^3$ edges. Then we can make $G$ isomorphic to $T_n$ by changing at most $\varepsilon n^3$ triples.*

Stability greatly helps in proving exact results (with one example being Theorem 1). This approach was pioneered by Simonovits [25] in the late 1960s and has led to exact solutions of numerous extremal problems since then. In recent years it has been actively used to prove exact results for the hypergraph Turán problem, see e.g. [13,11,14,15,18,10,19].

As an extra bonus, Theorem 2 also implies the following result, which answers in the affirmative a question posed by Razborov [22, Section 5]. For $F \subseteq \binom{V}{k}$ and $H \subseteq \binom{U}{k}$ let ind$(H, F)$ denote the *induced density* of $H$ in $F$, that is, the probability that a random injection $U \to V$ preserves all edges and non-edges of $H$.

**Theorem 3** (*Convergence*). *Let* $n_i \rightarrow \infty$. *Let* $F_i \subseteq \binom{[n_i]}{3}$ *be a* $\{G_0, G_3\}$-*free 3-graph with* $|F_i| = \left(\frac{4}{9} + o(1)\right) \binom{n_i}{3}$ *as* $i \rightarrow \infty$. *Then, for every fixed 3-graph H, the limit* $\lim_{i \rightarrow \infty} \mathrm{ind}(H, F_i)$ *exists (and is equal to* $\lim_{m \rightarrow \infty} \mathrm{ind}(H, T_m)$).

**Proof.** By Theorem 2 we can change $o(n_i^3)$ edges in $F_i$ and transform it into $T_{n_i}$. Relabel the vertices of $F_i$ so that $V(F_i) = V(T_{n_i})$ and the symmetric difference $F_i \triangle T_{n_i}$ has $o(n^3)$ triples, where $V(F)$ denotes the vertex set of a hypergraph $F$.

For every fixed 3-graph $H$ we have $|\mathrm{ind}(H, F_i) - \mathrm{ind}(H, T_{n_i})| = o(1)$ because the probability that a random injection $V(H) \rightarrow V(F_i)$ hits one of the triples where $F_i$ and $T_{n_i}$ differ is $o(1)$. Also, $\mathrm{ind}(H, T_m)$ tends to an (explicitly computable) limit $\lambda_H$ as $m \rightarrow \infty$. Thus $\mathrm{ind}(H, F_i) \rightarrow \lambda_H$, as required. $\square$

**Remark.** A simple application of the Principle of Inclusion–Exclusion shows that the conclusion of Theorem 3 is equivalent to the statement that the sequence $(F_i)$ of 3-graphs converges, as defined by Elek and Szegedy [6, Definition 2.5].

## 2. Some notation

We denote $[n] = \{1, \ldots, n\}$. For brevity, we often omit punctuation signs when writing sets; for example, $abc$ is a shorthand for $\{a, b, c\}$.

Let $G \subseteq \binom{V}{k}$ be a $k$-graph on $V$. For $A \subseteq V$, $G[A] = \{D \in G : D \subseteq A\}$ denotes the subgraph of $G$ induced by $A$. For disjoint subsets $V_1, \ldots, V_k \subseteq V$, let

$$G[V_1, \ldots, V_k] = \{D \in G : \forall i \in [k] \ |D \cap V_k| = 1\}$$

denote the $k$-partite subgraph of $G$ induced by the sets $V_i$. For $A \subseteq V$ with $a \leq k - 1$ elements, the *link* $(k - a)$-*graph* of $A$ is

$$G_A = \{D : D \subseteq V \setminus A, \ D \cup A \in G\}.$$

When $a = k - 1$, we view $G_A$ as a set of vertices rather than a set of 1-element sets. The *maximum degree* of $G$ is $\Delta(G) = \max\{|G_x| : x \in V\}$.

Let $G$ and $H$ be two $k$-graphs with the same number of vertices. They are *isomorphic* (written as $G \cong H$) if there is a bijection $f : V(G) \rightarrow V(H)$ such $A \in G$ if and only if $f(A) \in H$ for every $A \in \binom{V(G)}{k}$. The *edit distance* $\delta_1(G, H)$ is the minimum of $|\sigma(G) \triangle H|$ over all bijections $\sigma : V(G) \rightarrow V(H)$. In other words, $\delta_1(G, H)$ is the smallest number of $k$-tuples whose inclusion into $G$ one has to change in order to make $G$ isomorphic to $H$.

## 3. Auxiliary results

Here we list a few lemmas needed later. Their proofs are fairly straightforward and are included here for the sake of completeness.

**Lemma 4.** *Let* $n \geq 3$. *For every Turán pattern* $T_{X,Y,Z}$ *on* $[n]$ *we have* $|T_{X,Y,Z}| \geq t_n$ *and, if we have equality, then* $T_{X,Y,Z} \cong T_n$.

**Proof.** Let $x, y, z$ be the cardinalities of $X, Y, Z$ respectively. The claim is trivial for $n = 3$, so let us assume that $n \geq 4$.

It is enough to show that no two of $x, y, z$ differ by more than by 1. Suppose on the contrary that this is false. We will give an example of a triple strictly better than $(x, y, z)$, thus proving the lemma. Up to a symmetry, there are two cases.

*Case* 1. $x \geq y \geq z$ and $x \geq z + 2$.

Routine simplifications show that

$$\partial := t_{x,y,z} - t_{x-1,y,z+1} = \frac{x^2}{2} + xy - xz - \frac{y^2}{2} - \frac{3x}{2} - \frac{y}{2} + z + 1.$$

It is enough to show that this expression is strictly positive. This a linear function of $z$ with the coefficient $1 - x < 0$, so it suffices to show that $\partial > 0$ under the additional assumption that $z = \min(x - 2, y)$.

If $z = x - 2$, then $y$ can be one of $x$, $x - 1$, and $x - 2$ and $\partial$ is $x - 1$, $x - 1$, and $x - 2$ respectively. Since $n \geq 4$, we have $x \geq 2$. Also, if $x = 2$, then $z = 0$, $n = 4$, and $y = 2$. In all cases, $\partial$ is strictly positive, as desired.

If $z = y$, then $\partial = \frac{x^2}{2} - \frac{3x}{2} - \frac{y^2}{2} + \frac{y}{2} + 1$, which is an increasing function of $x \geq 2$. So it follows from the case $x = z + 2$ which we have just done.

*Case* 2. $x \geq z \geq y$ and $x \geq y + 2$.

Routine simplifications show that

$$\partial := t_{x,y,z} - t_{x-1,y+1,z} = -\frac{y^2}{2} + xy - yz + \frac{z^2}{2} - \frac{y}{2} - \frac{z}{2}.$$

This is a non-decreasing function of $x$, so it is enough to consider the case $x = \max(y+2, z)$. If $y = x-2$, then $z$ is one of $x$, $x - 1$, $x - 2$ with $\partial$ being $x - 1$, $x - 2$ and $x - 2$ respectively. The assumption $n \geq 4$ implies that $\partial > 0$ in each case. If $z = x$, then $\partial = \frac{x^2}{2} - \frac{x}{2} - \frac{y^2}{2} - \frac{y}{2}$, which is increasing in $x \geq 2$, so it is enough to assume that $x = z = y + 2$; we have $\partial = x - 1 > 0$ in this case.   □

**Lemma 5.** *For every* $\varepsilon > 0$ *there is* $c > 0$ *such that for every* $n > 1/c$ *and for every non-negative integer* $v_0, v_1, v_2$ *with* $v_0 + v_1 + v_2 = n$ *and* $t_{v_0,v_1,v_2} \leq \left(\frac{4}{9} + c\right)\binom{n}{3}$ *we have* $|v_i - n/3| \leq \varepsilon n$ *for every* $i \in \mathbb{Z}_3$.

**Proof.** Since we are not interested in an explicit dependence of $c$ on $\varepsilon$, we present a "non-constructive" but short proof. Suppose that the lemma is false, that is, there is $\varepsilon > 0$ such that for every integer $m$ we have a counterexample $(v_0, v_1, v_2)$ for $c = 1/m$. By choosing a subsequence of $m$, we can assume that $v_i/n$ converges for each $i \in \mathbb{Z}_3$; let $x_i$ be the limit of $v_i/n$. By Lemma 4, we have $t_{v_0,v_1,v_2} = \left(\frac{4}{9} + o(1)\right)\binom{n}{3}$. Thus

$$P(x_0, x_1, x_2) = \frac{x_0^3 + x_1^3 + x_2^3}{6} + \frac{x_0^2 x_1 + x_1^2 x_2 + x_2^2 x_0}{2}$$

assumes the value $\frac{4}{9} \times \frac{1}{3!} = \frac{2}{27}$.

Let us minimize $P(x, y, z)$ over non-negative reals $x, y, z$ with $x + y + z = 1$. If, for example, $x \geq y \geq z$ with $x > z$, then the following difference of partial derivatives

$$\frac{\partial}{\partial z}P(x, y, z) - \frac{\partial}{\partial x}P(x, y, z) = (y - x)\frac{x + y}{2} + (z - y)x$$

is strictly negative (because at least one of $y - x \leq 0$ and $z - y \leq 0$ is strictly negative while $x \geq \frac{1}{3}$). Thus $P(x - \delta, y, z + \delta) < P(x, y, z)$ for all small $\delta > 0$. Likewise, if $x \geq z \geq y$ with $x > y$, then

$$\frac{\partial}{\partial y}P(x, y, z) - \frac{\partial}{\partial x}P(x, y, z) = (z - x)y + (y - z)\frac{y + z}{2} \leq 0.$$

Moreover, if we have equality here, then $y = z = 0$, $x = 1$ and $P$ assumes value $\frac{1}{6} > P\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{2}{27}$. In any case, $P(x, y, z)$ is not minimum. This implies that the only extremal point is $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ and the minimum value of $P$ is $\frac{2}{27}$.

It follows that $x_0 = x_1 = x_2 = \frac{1}{3}$, which contradicts the fact some two of the ratios $v_0/n$, $v_1/n$, and $v_2/n$ differ by at least $\varepsilon$ for every $m$.   □

## 4. Stability for the $\{G_0, G_3\}$-problem

In this section we will prove Theorem 2. Suppose on the contrary that it is false. Thus there is $\varepsilon > 0$ and a sequence $(F_i)$ with $|F_i| \leq \left(\frac{4}{9} + o(1)\right)\binom{n_i}{3}$ as $i \to \infty$, where $F_i$ is a $\{G_0, G_3\}$-free 3-graph on $n_i > i$ vertices that is $\varepsilon n_i^2$-far in the edit distance from $T_{n_i}$.

Fix any such sequence $(F_i)$. We will split the whole proof resulting in a final contradiction into a sequence of claims.

Let us call a 3-graph $H$ *singular* if $H$ is $\{G_0, G_3\}$-free but for every $n$ the Turán graph $T_n$ does not contain $H$ as an induced subgraph. Clearly, it is enough to check this inclusion for $n = 3|V(H)|$ only. There are exactly 26 non-isomorphic singular 3-graphs on 6 vertices, denoted by $H_9, \ldots, H_{34}$ in [22].

*Claim* 1. For every singular 3-graph $H$ on 6 vertices we have $\mathrm{ind}(H, F_i) \to 0$ as $i \to \infty$.

Proof of Claim. Although this claim is stated in [22, Section 5], let us sketch its proof very briefly. Let $n \to \infty$ and let $F$ be an arbitrary $\{G_0, G_3\}$-free 3-graph on $n$ vertices. Let $\rho = |F|/\binom{n}{3}$ be the edge density of $F$. Razborov [22, Section 3] derives the following identity:

$$\frac{5}{9}(\rho - 4/9) = [\![ (e - 4/9)^2 ]\!]_1 + [\![ Q_1(f_1, \ldots, f_4) ]\!]_{\tau_1} + [\![ Q_2(g_0, \ldots, g_5) ]\!]_{\tau_2} + R + o(1). \qquad (3)$$

Rather than formally defining all terms appearing here, we state only those properties that we need in order to prove Claim 1, referring the reader to [22] for all details.

- Each term involving the brackets $[\![ \ldots ]\!]$ is non-negative by Inequality (6) in [22].
- The last term, which we denoted by $R$, is of the form $\sum_H \alpha_H \mathrm{ind}(H, F)$, where the following applies.
  - The sum runs over $\{G_0, G_3\}$-free 3-graphs $H$ with 6 vertices.
  - $\alpha_H$ are explicit non-negative reals that are listed in Table 3 in [22].
  - $\alpha_H \geq 1/360$ for every singular $H$.

It follows that if $|F| \leq \left(\frac{4}{9} + \varepsilon\right)\binom{n}{3}$, then $\mathrm{ind}(H, F) \leq 360 \times \frac{5\varepsilon}{9} + o(1)$ for every singular 6-vertex hypergraph $H$. The claim follows. □

Now, we can apply the Strong Hypergraph Removal Lemma of Rödl and Schacht [23] to each 3-graph $F_i$ with respect to induced singular subgraphs $H_9, \ldots, H_{34}$. The lemma shows that we can change $o(n_i^3)$ edges in $F_i$ as $i \to \infty$ and ensure that it contains no induced singular subgraph on 6 vertices. Hence, by making $\varepsilon$ slightly smaller, it is enough to derive a contradiction under the additional assumption that $F_i$ has no induced singular subgraph on 6 vertices.

Fix large $i$, and let $n = n_i$, $F = F_i$, and $V = V(F)$ for the remainder of this section. Let $T$ be the logical predicate that takes three disjoint sets $U_0, U_1, U_2 \subseteq V$ as input and is true if and only if the induced subgraph $F[U_0 \cup U_1 \cup U_2]$ follows the Turán pattern, that is, its edges are precisely triples $xyz$ with $xy \in U_j$ and $z \in U_j \cup U_{j+1}$ for some $j \in \mathbb{Z}_3$. Thus we have the following claim.

*Claim* 2. For any set $U \subseteq V$ with $|U| \leq 6$, there is a partition $U = U_0 \cup U_1 \cup U_2$ such that $T(U_0, U_1, U_2)$ holds. □

Let the logical predicate $S(ab, cd)$ state that the vertices $a, b, c, d \in V$ are pairwise distinct, $abc, abd \in F$, and $acd, bcd \notin F$. Also, for $a, b \in V$ let us write $a \sim b$ if $a = b$ or there are $c, d \in V$ satisfying $S(ab, cd)$. In the latter case, we call the pair $cd$ a *witness* of $a \sim b$. Clearly, the binary relation $\sim$ is symmetric. The following claim can be checked by a trivial case analysis.

*Claim* 3. If $T(U_0, U_1, U_2)$ holds and $a, b, c, d \in U_0 \cup U_1 \cup U_2$ satisfy $S(ab, cd)$, then for some $j \in \mathbb{Z}_3$ we have $a, b \in U_j$ and $c, d \in U_{j+1}$. □

*Claim* 4. The relation $\sim$ is transitive.

Proof of Claim. Suppose that $a \sim b$ and $b \sim f$, which is witnessed by $S(ab, cd)$ and $S(bf, gh)$ respectively. Let $U = \{a, b, c, d, f, g, h\}$.

If $|U| \leq 6$, then take a partition $U = U_0 \cup U_1 \cup U_2$ given by Claim 2. By Claim 3 and symmetry, we can assume that e.g. $ab \in U_0$ and $cd \in U_1$. Since $S(bf, gh)$ holds, the vertices $b$ and $f$ are in the same part $U_j$ by Claim 3. Thus $f \in U_0$. Then $S(af, cd)$ holds, giving $f \sim a$ as required.

So suppose that $|U| = 7$ (i.e. all involved vertices are pairwise distinct). Consider $U' = U \setminus \{g\}$. It has 6 elements, so by Claim 2 there is a partition $U' = U'_0 \cup U'_1 \cup U'_2$ satisfying $T(U'_0, U'_1, U'_2)$. Assume by Claim 3 that $a, b \in U'_0$ and $c, d \in U'_1$.

If $f \in U'_0$, then $S(af, cd)$ holds and thus $a \sim f$, as required. So suppose that $f \notin U'_0$.

If $f$ or $h$ is in $U'_1$, say $h \in U'_1$, then we have $S(ab, dh)$. We can replace $c$ by $h$ in $U$, reducing its size to 6 and conclude as above that $a \sim f$. So suppose that $f, h \notin U'_1$.

Since $bfh \in F$, it must be the case that $f, h \in U'_2$. Thus $T(ab, cd, fh)$ holds, in particular, $afh \in F$. By the symmetry, we can swap $f$ with $a$ and $cd$ with $gh$ in the above analysis and conclude that either

we are done or that $T(fb, gh, ac)$ holds. But the latter relation implies that $afh \notin F$. This contradiction proves Claim 4.  $\square$

Thus all vertices of $F$ are partitioned into $\sim$-equivalence classes, say

$$V = V_1 \cup \cdots \cup V_k.$$

Let us call two vertices $a, b \in V$ *twins* if swapping $a$ and $b$ we get an automorphism of $F$.

*Claim* 5. If $a \sim b$, then $a$ and $b$ are twins.

Proof of Claim. Pick $c$ and $d$ such that $S(ab, cd)$ holds. Let $e, f \in V \setminus \{a, b\}$ be arbitrary. Let $U = \{a, b, c, d, e, f\}$. Apply Claim 3 to $U$ to obtain a Turán partition $U = U_0 \cup U_1 \cup U_2$. Without loss of generality assume that $a, b \in U_0$. Wherever the vertices $e, f$ fall, we have $aef \in F$ if and only if $bef \in F$ by the symmetry of $U_0$ with respect to $F[U]$, as required.  $\square$

Hence, we can define the *skeleton* 3-graph $F'$ on $[k]$: namely, a triple $fgh \in \binom{[k]}{3}$ is an edge of $F'$ if and only if the induced 3-partite 3-graph $F[V_f, V_g, V_h]$ is complete (or, equivalently by Claim 5 non-empty).

*Claim* 6. $F'$ does not contain an induced copy of $G_0$, $G_2$, nor $G_3$.

Proof of Claim. If some 4-set $uwxy \in \binom{[k]}{4}$ spans exactly two triples in $F'$, say $uwx, uwy \in F'$, then $S(ab, cd)$ holds for arbitrary representatives $a \in V_u, b \in V_w, c \in V_x$, and $d \in V_y$. Thus $a \sim b$, a contradiction to $u \neq w$.

The $\{G_0, G_3\}$-freeness of $F'$ follows from the $\{G_0, G_3\}$-freeness of $F$ and Claim 5.  $\square$

*Claim* 7. Each $V_j$ spans a complete 3-graph.

Proof of Claim. Let $a, b, c \in V_j$ be distinct. Choose a witness $fg$ to $a \sim b$. By Claim 5, the pair $fg$ also witnesses $b \sim c$ and $a \sim c$. Thus $abf, acf, bcf \in F$. Since we do not have $G_3$, $abc \in F$.  $\square$

*Claim* 8. If $fgh \in F'$, then $V_f \cup V_g \cup V_h$ spans the complete 3-graph in $F$.

Proof of Claim. Take $c \in V_f$, $d \in V_g$ and distinct $a, b \in V_h$. We have $acd, bcd \in F$. Since $c \not\sim d$, at least one of $abc$ or $abd$ is an edge of $F$. But $F$ does not contain $G_3$. Hence both $abc$ and $abd$ are edges of $F$. The result follows from Claim 7.  $\square$

*Claim* 9. If two edges $D, E \in F'$ intersect in two vertices, then $D \cup E$ induces a complete subgraph in $F'$.

Proof of Claim. This follows from Claim 6 (the $\{G_2, G_3\}$-freeness of $F'$).  $\square$

*Claim* 10. If two edges $D, E \in F'$ intersect in one vertex, then $D \cup E$ induces a complete subgraph in $F'$.

Proof of Claim. Let $D = abc$ and $E = cde$. The 4-set $abde$ spans at least one edge (since $F'$ is $G_0$-free), say $abd \in F'$. By Claim 9 applied to $abc, abd \in F'$, the quadruple $abcd$ induces $G_4$. Since $cde \in F'$ intersects each of $acd, bcd \in F'$ in two vertices, we have $G[acde] \cong G[bcde] \cong G_4$ by Claim 9. This implies that every triple of $abcde$ is in $F'$ except perhaps $abe$. But Claim 9 applied to $abc, bce \in F'$ shows that $abe \in F'$, as required.  $\square$

By the above claims, $F'$ is a vertex-disjoint union of complete subgraphs on sets $W_1, \ldots, W_l$ respectively. Let us agree that each isolated vertex of $F'$, if there are any, forms a separate part $W_j$. The sets $W_1, \ldots, W_l$ partition $[k] = V(F')$. Since $F'$ is $G_0$-free, we have $l \leq 3$ (otherwise pick one vertex from some four parts $W_j$ to obtain a $G_0$-subgraph in $F'$). Moreover, every triple of $F$ intersects at most two of the parts $W_j$. For $j \in [l]$, define $U_j = \cup_{h \in W_j} V_h$ and $u_j = |U_j|$. The sets $U_1, \ldots, U_l$ partition $V = V(F)$.

*Claim* 11. For each $j \in [l]$, the set $U_j \subseteq V$ spans a complete subgraph in $F$.

Proof of Claim. If $W_j = \{h\}$ has only one element, then $U_j = V_h$ and the result follows from Claim 7. If $|W_j| \geq 3$, then $W_j$ spans a (non-trivial) complete subgraph in $F'$ and the result follows from Claim 8. Finally, it is impossible to have $|W_j| = 2$ for otherwise the two vertices of $W_j$ would be isolated in $F'$ and would form a separate part $W_h$ each.  $\square$

Suppose first that $l = 3$. Then the following holds.

*Claim* 12. For every $h \in [3]$ there is $j \neq h$ such that $abc \in F$ for every $ab \in \binom{U_h}{2}$ and $c \in U_j$.

Proof of Claim. Without loss of generality assume $h = 3$. For $j = 1, 2$, let $H_j$ be the 2-graph that consists of all pairs $ab \in \binom{U_3}{2}$ such that $abc \in F$ for every $c \in U_j$.

Let us show that the union of $H_1$ and $H_2$ is $\binom{U_3}{2}$. If, on the contrary, some pair $ab \in \binom{U_3}{2}$ is not in $H_1 \cup H_2$, then pick $c_j \in U_j$ with $abc_j \notin F$ for $j = 1, 2$ and observe that $abc_1c_2$ spans $G_0$ in $F$, a contradiction.

Also, for $j = 1, 2$, the 2-graph $H_j$ contains no induced path of length 2. Indeed, if $ad, bd \in H_j$ but $ab \notin H_j$, then pick $c \in U_j$ with $abc \notin F$ and observe that $abcd$ spans $G_3$ (note that $abd \in F$ by Claim 11), a contradiction.

Hence, each of $H_1$ and $H_2$ is a union of vertex-disjoint cliques. Since $H_1 \cup H_2 = \binom{U_3}{2}$, it easily follows that $H_1$ or $H_2$ is equal to $\binom{U_3}{2}$, proving the claim. $\quad\square$

**Claim** 13. For every distinct $j, h \in [3]$, $F$ contains at least

$$\frac{u_j^2 u_h + u_j u_h^2}{2} - \frac{u_j u_h \sqrt{u_j^2 + u_h^2}}{2} + O(n^2) \tag{4}$$

triples within $U_j \cup U_h$ that intersect both $U_j$ and $U_h$.

Proof of Claim. Suppose without loss of generality that $j = 1$ and $h = 2$.

Let $i = 1$ or $2$. Let $m_i$ be the number of triples that do not belong to $F$ and have exactly two vertices in $U_i$ and one vertex in $U_{3-i}$. For $ab \in \binom{U_i}{2}$, let $m_{ab}$ be the number of $c \in U_{3-i}$ such that $abc \notin F$. The sum

$$s_i = \sum_{ab \in \binom{U_i}{2}} \binom{m_{ab}}{2}$$

counts 4-tuples $abcd$ such that $ab \in \binom{U_i}{2}$, $cd \in \binom{U_{3-i}}{2}$, and $abc, abd \notin F$. Since $F$ is $G_0$-free, no 4-tuple is counted twice (i.e. for both $i = 1$ and $i = 2$). Thus $s_1 + s_2 \leq \binom{u_1}{2}\binom{u_2}{2}$. On the other hand, the convexity of the function $\binom{x}{2}$ and the identity $m_i = \sum_{ab \in \binom{U_i}{2}} m_{ab}$ imply that

$$s_i \geq \binom{u_i}{2}\binom{m_i / \binom{u_i}{2}}{2} = \frac{m_i^2}{u_i^2} + O(n^3).$$

We conclude that

$$\frac{u_1^2 u_2^2}{4} \geq \frac{m_1^2}{u_1^2} + \frac{m_2^2}{u_2^2} + O(n^3). \tag{5}$$

If we ignore the error term and maximize $m = m_1 + m_2$ over non-negative reals $m_1, m_2$ satisfying (5), then any optimal assignment makes (5) equality. Using this to eliminate $m_2$, we obtain that $m = m_1 + u_2(u_1^2 u_2^2/4 - m_1^2/u_1^2)^{1/2}$. This function of $m_1$ has the unique maximum $\frac{1}{2}u_1 u_2(u_1^2 + u_2^2)^{1/2}$ attained at the unique positive root $m_1 = u_1^3 u_2(u_1 + u_2)^{1/2}/2$ of its derivative. This gives an upper bound on the number of triples between $U_2$ and $U_3$ that are missing from $F$, proving the claim. $\quad\square$

By Claim 12, fix $j(h)$ for each $h \in [3]$. Up to a symmetry we have two cases.

*Case* 1. $(j(1), j(2), j(3)) = (2, 1, 1)$.

Here $U_1 \cup U_2$ spans a complete subgraph in $F$. By Claim 13, the number of edges in $F$ is at least $P(u_1, u_2, u_3) + O(n^2)$, where

$$P(u_1, u_2, u_3) = \frac{(u_1 + u_2)^3 + u_3^3}{6} + \frac{u_1 u_3^2}{2} + \frac{u_2 u_3^2 + u_2^2 u_3}{2} - \frac{u_2 u_3 \sqrt{u_2^2 + u_3^2}}{2}.$$

**Claim** 14. The minimum value of $P(x, y, z)$ over non-negative $x, y, z$ with $x + y + z = 1$ is strictly larger than $\frac{4}{9} \times \frac{1}{6} = \frac{2}{27}$.

Proof of Claim. Let

$$Q(y, z) = P(1 - y - z, y, z) - \frac{2}{27} = \frac{y^2 z}{2} - \frac{z^3}{2} - \frac{yz\sqrt{y^2 + z^2}}{2} + z^2 - \frac{z}{2} + \frac{5}{54}.$$

Let us minimize $Q$ over

$$S = \{(y, z) \in \mathbb{R}^2 : y \geq 0, \ z \geq 0, \ y + z \leq 1\}.$$

The derivative

$$\frac{\partial Q(y, z)}{\partial y} = -\frac{z\left(y - \sqrt{y^2 + z^2}\right)^2}{2\sqrt{y^2 + z^2}},$$

is non-positive. Hence, there is an optimal assignment with $y = 1 - z$. Note that

$$Q(1 - z, z) = \frac{5}{54} + \frac{(z^2 - z)\sqrt{1 - 2z + 2z^2}}{2}.$$

Let $I = [0, 1] \subseteq \mathbb{R}$ denote the closed unit interval. It is enough to show that $(5/54)^2 - (Q(1-z, z) - 5/54)^2$ is positive on $I$. The last expression is a polynomial and factorizes as $(18z^2 - 18z + 5)R(z)/2916$, where $R(z) = -81z^4 + 162z^3 - 99z^2 + 18z + 5$. Clearly, it remains to show that $R(z)$ is positive on $I$. The derivative $R'(z)$ has three simple roots $1/2$ and $(3 \pm \sqrt{5})/6$, all of which are in $I$. So the potential minima of $R$ on $I$ are restricted to values $R(0)$, $R(1/2)$, or $R(1)$. But each of these is positive. This proves Claim 14. $\square$

By Claim 14, we have that $|F|$ is strictly larger than $\left(\frac{2}{27} + o(1)\right)n^3 = \left(\frac{4}{9} + o(1)\right)\binom{n}{3}$, a contradiction.

*Case* 2. $(j(1), j(2), j(3)) = (2, 3, 1)$.

Thus $F$ contains the Turán pattern $T = T_{U_1, U_2, U_3}$ plus perhaps some extra edges. By Lemma 5, the 3-graph $T$ alone has at least $\left(\frac{4}{9} + o(1)\right)\binom{n}{3}$ edges. Since we have assumed that $|F| \leq \left(\frac{4}{9} + o(1)\right)\binom{n}{3}$, each of $F$ and $T$ has size $\left(\frac{4}{9} + o(1)\right)\binom{n}{3}$ and $|F \setminus T| = o(n^3)$. Thus $F$ can be converted into $T$ by changing $o(n^3)$ edges. By the second part of Lemma 5, $u_j = \left(\frac{1}{3} + o(1)\right)n$ for each $j \in [3]$. Therefore, $\delta_1(T, T_n) = o(n^3)$. Thus $F$ and $T_n$ are $o(n^3)$-close in the edit distance, a contradiction to our assumption.

Since $l = 1$ is impossible $\left(\text{otherwise } F = \binom{[n]}{3}\right)$, it remains to consider the case $l = 2$. By Claim 13 and the routine fact that the maximum of $x(1-x)(x^2 + (1-x)^2)^{1/2}$ for $x \in I$ is attained for $x = 1/2$, we have

$$|F| \geq \binom{n}{3} - \frac{u_1 u_2 \sqrt{u_1^2 + u_2^2}}{2} + O(n^2) \geq \frac{n^3}{6} - \frac{n^3}{8\sqrt{2}} + O(n^2),$$

which is strictly larger than $\left(\frac{4}{9} + o(1)\right)\binom{n}{3}$. This final contradiction proves Theorem 2.

## 5. Exact result for the $\{G_0, G_3\}$-problem

First, we will obtain the conclusion of Theorem 1 under the additional assumptions that $G$ is close to a Turán pattern and its maximum degree is at most that of $T_n$:

**Theorem 6.** *There is $\varepsilon > 0$ such that the following holds. Let $G$ be a $\{G_0, G_3\}$-free 3-graph on $n \geq 1/\varepsilon$ vertices such that $|G| \leq t_n$, $\Delta(G) \leq \Delta(T_n)$, and $G$ is $\varepsilon n^3$-close in the edit distance to some $T_{V_0, V_1, V_2}$. Then $G$ is isomorphic to $T_n$.*

Then, in Section 5.2, we will show that Theorems 2 and 6 imply Theorem 1.

### 5.1. Proof of Theorem 6

Suppose on the contrary that Theorem 6 is false. Then for every $\varepsilon > 0$ there is a counterexample. In fact, there are infinitely many counterexamples (otherwise by Lemma 4, we would have eliminated all of them by making $\varepsilon$ sufficiently small). Thus we may assume that $\varepsilon \to 0$ and that $n$, the number of vertices, is arbitrarily large with respect to $1/\varepsilon$.

In order to make the proof more readable, we use the asymptotic notation where all terms depending on $\varepsilon$ are hidden. For example, $a = o(n)$ means that $|a| \le f(\varepsilon)n$ for some function $f(\varepsilon)$ that depends on $\varepsilon$ only and tends to 0 as $\varepsilon \to 0$.

We will use the following constants that are chosen in this order, each being a sufficiently small positive number depending on the previous ones:

$$c_1 \gg c_2 \gg c_3 \gg c_4 \gg c_5 \gg c_6 \gg c_7.$$

We do not try to optimize the inequalities that we derive in the course of the proof.

Let $G \subseteq \binom{[n]}{3}$ satisfy all assumptions of Theorem 6. Choose a *best-fit* partition, that is, a partition $[n] = V_0 \cup V_1 \cup V_2$ such that $|T \setminus G|$ is smallest possible, where $T = T_{V_0,V_1,V_2}$.

By our assumptions, there is a Turán pattern $T'$ with $|T' \setminus G| \le |T' \Delta G| \le \varepsilon n^3$. By the extremality of $T$, we have

$$|T \setminus G| \le |T' \setminus G| \le \varepsilon n^3 = o(n^3).$$

We conclude that

$$|T| \le |T \cap G| + |T \setminus G| \le |G| + \varepsilon n^3 \le t_n + o(n^3).$$

So, by Lemma 5, we have

$$v_i = (1/3 + o(1))n. \tag{6}$$

Let $B = T \setminus G$ and $S = G \setminus T$. We call triples in $B$ *bad* and triples in $S$ *superfluous*. Let $b = |B| \le \varepsilon n^3$ and $s = |S|$.

Since each $v_i$ is at least 4, we cannot remove any triple from $T$ without creating $G_0$. If $s = 0$, then $G \subseteq T$ and, in fact, $G = T$ by the $G_0$-freeness of $G$; thus $|T| = t_n$ and $G \cong T \cong T_n$ by Lemma 4, satisfying Theorem 6. So assume that $s > 0$. Also, $t_n \ge |G| = |T| + s - b \ge t_n + s - b$, so $s \le b$. Summarizing:

$$0 < s \le b = o(n^3). \tag{7}$$

Let $P = \{xy \in \binom{[n]}{2} : |B_{xy}| \ge n/20\}$ be the set of pairs of vertices that belong to at least $n/20$ bad triples. Let $p = |P|$.

*Claim* 1. $p \ge b/2n$.

Proof of Claim. Suppose on the contrary that $p < b/2n$. Let $L \subseteq B$ consist of bad triples that do not contain pairs in $P$. We have $|L| \ge b - p(n-2) \ge b/2$.

Let $l$ be the number of pairs $(D, E) \in L \times S$ such that $|D \cap E| = 2$. Every bad triple $xyz \in L$ contributes at least $\left(\frac{1}{3} + o(1)\right)n$ to $l$. Indeed, if $x, y \in V_i$ and $z \in V_i \cup V_{i+1}$ for some $i \in \mathbb{Z}_3$, then for every $w \in V_{i-1}$ at least one of $wxy$, $wxz$, $wyz$ belongs to $S$ (otherwise the quadruple $wxyz$ spans a copy of $G_0$ in $G$). Thus $l \ge (b/2) \left(\frac{1}{3} + o(1)\right) n$. On the other hand, the 2-shadow of $S$ has at most $3s$ pairs, so some pair $xy$ is covered by at least $l/(3s)$ triples of $L$. Thus, by (7),

$$|B_{xy}| \ge |L_{xy}| \ge \frac{(b/2)\left(\frac{1}{3}+o(1)\right)n}{3b} > \frac{n}{20}.$$

Thus $xy \in P$, which contradicts the definition of $L$. $\quad\square$

*Claim* 2. There is a vertex $x$ with $|B_x| \ge c_2 n^2$.

Proof of Claim. Each pair in $P$ can either lie inside some part $V_i$ or connect two parts. We distinguish two cases depending on where the majority of pairs in $P$ go.

*Case* 1. $|P \cap (\cup_{i=0}^2 \binom{V_i}{2})| \geq p/2$.

Without loss of generality, suppose that $|P^0| \geq p/6$, where $P^0 = P \cap \binom{V_0}{2}$. Define

$$P' = \{xy \in P^0 : |B_{xy} \cap V_1| \geq n/40\}.$$

*Case* 1.1. $|P'| \geq p/12$.

For each quadruple $uwxy$ with $xy \in P'$ and $u, w \in B_{xy} \cap V_1$ (at least $(p/12) \times \binom{n/40}{2}$ choices), $uwx$ or $uwy$ is superfluous (otherwise $G[uwxy] \cong G_0$). Therefore, some triple, say $uwx \in S$ with $x \in V_0$, appears for at least

$$\frac{(p/12) \times \binom{n/40}{2}}{s} \geq \frac{(b/(24n)) \times \binom{n/40}{2}}{b} \geq c_1 n$$

choices of $y$, where we used (7) and Claim 1. This vertex $x$ is in at least $\frac{1}{2} \times c_1 n \times (n/20) \geq c_2 n^2$ bad triples, as required.

*Case* 1.2. $|P'| < p/12$.

For each pair $xy \in P^0 \setminus P'$ (at least $p/12$ choices), $u \in V_0 \cap B_{xy}$ (at least $|B_{xy}| - n/40 \geq n/40$ choices), and $w \in V_1 \setminus B_{xy}$ (at least $(1/3 - 1/40 + o(1))n$ choices), we have $wxy \in G$ and $uxy \notin G$. Thus, in order to avoid $G_3$, we have that $uwx$ or $uwy$ is in $B$. By averaging, some triple, say, $uwx \in B$ with $w \in V_1$ appears for at least $c_1 n$ choices of $y$ in this way. Out of these $c_1 n$ $P$-pairs connecting such vertices $y$ to $u$, $x \in V_0$, at least half go to the same vertex, which necessarily has $B$-degree at least $c_2 n^2$.

*Case* 2. More than half of the edges of $P$ connect two different parts $V_i$.

Without loss of generality, suppose that at least $p/6$ pairs of $P$ connect $V_0$ to $V_1$. Let the 2-graph $P^{01}$ consist of these pairs. Note that any bad triple $xyz$ with $xy \in P^{01}$ satisfies $z \in V_0$. Define

$$P'' = \{xy \in P^{01} : |V_2 \setminus S_{xy}| \geq n/6\}.$$

*Case* 2.1. $|P''| \geq p/12$.

For every choice of $xy \in P''$, $z \in B_{xy} \subseteq V_0$, and $w \in V_2 \setminus S_{xy}$ (at least $(p/12) \times (n/20) \times (n/6)$ choices), at least one of $wxz$ or $wyz$ is superfluous (to prevent $G[wxyz] \cong G_0$). By averaging, some triple, say, $wxz \in S$ with $w \in V_2$ appears for at least $c_1 n$ choices of $y$, implying that $x$ has the required $B$-degree.

*Case* 2.2. $|P''| < p/12$.

For every choice of $xy \in P^{01} \setminus P''$, say $x \in V_0$ and $y \in V_1$, and distinct $u, w \in S_{xy} \cap V_2$, we have $uwx \in B$ or $uwy \in S$ (to avoid $G_3$). One of these alternatives occurs at least half of the time. Averaging gives a triple (in $B$ or in $S$) that appears at least $c_1 n$ times this way. As above, this gives a vertex incident to at least $c_2 n^2$ edges of $B$. The claim is proved. $\square$

Fix some vertex $x$ with $|B_x| \geq c_2 n^2$. The following definitions and assumptions will apply to the rest of the section. Assume without loss of generality that $x \in V_0$. Partition the link 2-graphs $B_x$ and $S_x$ as $B^0 \cup B^1 \cup B^2$ and $S^0 \cup S^1 \cup S^2$, where

$$B^0 = B_x \cap \binom{V_0}{2},$$

$$B^1 = \{yz \in B_x : y \in V_0, \ z \in V_1\},$$

$$B^2 = B_x \cap \binom{V_2}{2},$$

$$S^0 = \{yz \in S_x : y \in V_1, \ z \in V_2\},$$

$$S^1 = S_x \cap \binom{V_1}{2},$$

$$S^2 = \{yz \in S_x : y \in V_0, \ z \in V_2\}.$$

For $i \in \mathbb{Z}_3$, let $b_i = |B^i|$ and $s_i = |S^i|$.

Let $A$ be a largest subset of $V_1$ with the property that

$$|S^1[A]| \geq \binom{|A|}{2} - c_3 n^2. \tag{8}$$

Since $A = \emptyset$ satisfies (8), $A$ exists. Let $\alpha = |A|/n$. Also, let

$$C = \{y \in V_0 : |B_y^1| \geq c_4 n\},$$

where $B_y^1 = (B^1)_y$ is the set of neighbors of $y$ in the 2-graph $B^1$. Let $\gamma = |C|/n$.

Let us state a few easy inequalities relating some of the parameters that have just been defined.

By the definition of $A$, we have

$$s_1 \geq \binom{\alpha n}{2} - c_3 n^2. \tag{9}$$

Also, let us show that

$$b_1 \leq \alpha \gamma n^2 + c_4 n^2 + o(n^2). \tag{10}$$

The vertices in $V_0 \setminus C$ are incident to at most $|V_0| \times c_4 n < c_4 n^2$ edges of $B^1$. Let $C' = \{y \in C : |B_y^1| > \alpha n\}$. Clearly, $C \setminus C'$ is incident to at most $\alpha \gamma n^2$ edges of $B^1$. For every $y \in C'$, we have $|\overline{S^1}[B_y^1]| > c_3 n^2$ by the definition of $\alpha$; moreover, for every distinct $u, w \in B_y^1$ with $uw \notin S^1$, we have $uwy \in S$ (to avoid $G[uwxy] \cong G_0$). Thus $|C'| \times c_3 n^2 \leq |S|$. By (7), $|C'| = o(n)$, and (10) follows.

Let us estimate $l$, the number of pairs $(E, D)$ with $E \in B^2$, $D \in S^0$, and $|E \cap D| = 1$ plus the number of pairs $(E, w)$ with $E \in B^2$, $w \in V_1$, and $E \cup \{w\} \in S$. On one hand, every $yz \in B^2$ contributes at least $v_1$ to $l$: for every $w \in V_1$ at least one of $wxy$, $wxz$, $wyz$ is in $S$ (to prevent $G[wxyz] \cong G_0$). On the other hand, each $D \in S^0$ contributes at most $v_2 - 1$ to $l$ while the number of pairs $(E, w)$ is at most $|S| = o(n^3)$. By (6), we have $b_2 n/3 \leq s_0 n/3 + o(n^3)$, i.e.

$$s_0 \geq b_2 + o(n^2). \tag{11}$$

Similarly to the above, let us estimate $l$, the number of pairs $(E, D)$ with $E \in B^0$, $D \in B^1$, and $|E \cap D| = 1$ plus the number of pairs $(E, w)$ where $E \in B^0$, $w \in V_1$, and $E \cup \{w\} \in B$. Each $yz \in B^0$ contributes at least $v_1$ to $l$: for every $w \in V_1$, at least one of $wxy$, $wxz$, $wzy$ is in $B$ (to avoid $G_3$). On the other hand, each $D \in B^1$ contributes at most $v_0 - 1$ to $l$ while there are at most $|B| = o(n^3)$ required pairs $(E, w)$. This implies that

$$b_1 \geq b_0 + o(n^2). \tag{12}$$

By (6) every vertex of $T$ (as well as of $T_n$) has degree $(4/9 + o(1)) \binom{n}{2}$. Since

$$\Delta(T_n) \geq \Delta(G) \geq |G_x| = |T_x| + |S_x| - |B_x|,$$

by the maximum degree assumption of Theorem 6, we conclude that

$$b_0 + b_1 + b_2 \geq s_0 + s_1 + s_2 + o(n^2). \tag{13}$$

The number of triples in $T \setminus G$ that contain $x$ is

$$|(T \setminus G)_x| = |B_x| = b_0 + b_1 + b_2. \tag{14}$$

If we change $T$ by moving $x$ to $V_1$, then $|(T \setminus G)_x|$ becomes $b_0 + \left( \binom{n/3}{2} - s_1 \right) + ((n/3)^2 - s_0) + o(n^2)$. By the best-fit property of $T$, this is at least (14), which implies that

$$b_1 + b_2 + s_0 + s_1 \leq \frac{n^2}{6} + o(n^2). \tag{15}$$

If we move $x$ to $V_2$, then $|(T \setminus G)_x|$ becomes $\left(\binom{n/3}{2} - s_1\right) + b_2 + ((n/3)^2 - s_2) + o(n^2)$. Again by the best-fit property of $T$, we have

$$b_0 + b_1 + s_1 + s_2 \leq \frac{n^2}{6} + o(n^2). \tag{16}$$

*Claim* 3. $s_0 \geq c_5 n^2$.

Proof of Claim. Let us suppose on the contrary that $s_0 < c_5 n^2$. By (11), we have $b_2 \leq c_5 n^2 + o(n^2)$. Thus

$$b_0 + b_1 = |B_x| - b_2 \geq c_2 n^2 - c_5 n^2 + o(n^2) \geq 3c_2 n^2/4 + o(n^2)$$

and by (12), we have $b_1 \geq 3c_2 n^2/8 + o(n^2)$. This, (10), and $\max(\alpha, \gamma) \leq 1/3 + o(1)$ imply that both $\alpha$ and $\gamma$ are at least $(3c_2/8 - c_4)/(1/3) + o(1) > c_2$.

Let $A$ be as in (8). Take $y \in V_0 \setminus C$. Let $A' = A \setminus B_y^1$. We have $|A'| \geq (\alpha - c_4)n$ and

$$|S^1[A']| \geq \binom{|A'|}{2} - c_3 n^2 \geq \binom{(c_2 - c_4)n}{2} - c_3 n^2 > c_2^2 n^2/3.$$

For every $wz \in S^1[A']$, we have $wyz \in S$ (to avoid $G_3$ on $wxyz$). Thus $|V_0 \setminus C| \times c_2^2 n^2/3 \leq |S| = o(n^3)$ and, by (6), $\gamma = 1/3 + o(1)$.

Pick $y \in C$, $z \in B_y^1$, and $w \in V_2$. There are at least $\gamma n \times c_4 n \times v_2$ such triples. By (7) and the assumption on $s_0$, $o(n^3)$ choices satisfy $wyz \in S$ and at most $c_5 n^3$ choices satisfy $wz \in S^0$. For all remaining triples $wyz$, we have $wy \in S^2$ (to avoid $G_0$ on $wxyz$). Let $\bar{S} = \{wy : y \in C, \ w \in V_2, \ wy \notin S^2\}$ be the bipartite complement of $S^2[C, V_2]$. Since for each $wy \in \bar{S}$ there are at least $c_4 n$ choices of $z$, we have $|\bar{S}|c_4 n \leq c_5 n^3 + o(n^3)$. Thus e.g. $|\bar{S}| \leq (c_4 + o(1))n^2$. We conclude that

$$s_2 \geq |C| \times |V_2| - |\bar{S}| \geq (1/9 - c_4 + o(1))n^2.$$

By (11) and (13), we have

$$(1/9 - c_4)n^2 + s_1 \leq b_0 + b_1 + o(n^2). \tag{17}$$

Inequalities (9), (10) and (17), and $b_0 \leq \binom{v_0}{2}$ imply that

$$\frac{1}{9} - c_4 + \frac{\alpha^2}{2} - c_3 \leq \frac{1}{18} + \frac{\alpha}{3} + c_4 + o(1). \tag{18}$$

Inequalities $\frac{1}{9} + \frac{\alpha^2}{2} \leq \frac{1}{18} + \frac{\alpha}{3}$ and $0 \leq \alpha \leq \frac{1}{3}$ imply that $\alpha = 1/3$. It follows from (18) that, for example, $\alpha \geq 1/3 - c_2$, $b_0 \geq (1/18 - c_2)n^2$, $s_1 \geq (\alpha^2/2 - c_2)n^2$, and $b_1 \geq (\alpha/3 - c_2)n^2$. But this contradicts (16). The claim is proved. $\quad\square$

*Claim* 4. $s_1 \geq (1/18 - c_6^2)n^2$.

Proof of Claim. Let

$$V_1' = \left\{ y \in V_1 : \left| S_y \cap \binom{V_2}{2} \right| \leq c_7 n^2 \right\}. \tag{19}$$

By (7), $|V_1 \setminus V_1'| = o(n)$. In particular, the number of $S^0$-edges intersecting $V_1 \setminus V_1'$ is $o(n^2)$. By Claim 3 and (6), the average $S^0$-degree of a vertex in $V_1'$ is at least $(3c_5 + o(1))n$. Take a vertex $y \in V_1'$ whose $S^0$-degree is at least this average. Let $D = S_y^0$. For every distinct $u, w \in D$ with $uwy \notin G$, we have $uw \in B^2$ (to avoid $G_3$). Thus

$$|B^2[D]| \geq \binom{|D|}{2} - c_7 n^2. \tag{20}$$

Fix this $D$. Let $z \in V_1'$ be arbitrary. Let $D' = D \setminus S_z^0$. For every pair $uw \in \binom{D'}{2}$, we have $uw \notin B^2$ or $uwz \in S$ (to avoid $G_0$ on $uwxz$). Thus $\binom{|D'|}{2} \le (c_7 + c_7 + o(1))n^2$ by (19) and (20). Hence,

$$|S_z^0 \cap D| \ge |D| - (2\sqrt{c_7} + o(1))n$$

for every $z \in V_1'$. We conclude that

$$|S^0[D, V_1']| \ge (|D| - 2\sqrt{c_7}n) \times |V_1'| + o(n^2) \ge |D|\,|V_1'| - \sqrt{c_7}n^2. \tag{21}$$

Define $D'' = \left\{z \in D : \left|B_z \cap \binom{V_1}{2}\right| \le c_7 n^2\right\}$. Then $D''$ contains all but $o(n)$ vertices of $D$. Pick $z \in D''$ whose $S^0$-degree is at least the average, which is at least $(1/3 - \sqrt{c_7}/(3c_5) + o(1))n$ by (21). For every distinct $u, w \in S_z^0$ we have $uwz \in B$ or $uw \in S^1$ (to avoid $G_3$ on $uwxz$). Thus the edges in the complement $\overline{S^1}$ of $S^1$ are restricted to pairs that intersect $V_1 \setminus S_z^0$ and to $B_z \cap \binom{V_1}{2}$. Thus

$$|\overline{S^1}| \le (\sqrt{c_7}/(3c_2) + c_7 + o(1))n^2 \le c_6^2 n^2/2,$$

proving the claim. □

**Claim 5.** $b_1 \ge (1/9 - 2c_6)n^2$.

**Proof of Claim.** Let $V_0' = \left\{z \in V_0 : \left|S_z \cap \binom{V_1}{2}\right| \le c_7 n^2\right\}$. By (6) and (7), $|V_0'| = (1/3 + o(1))n$. Take $z \in V_0'$. Let $D = V_1' \setminus B_z^1$, where $V_1'$ is defined by (19). For every distinct $u, w \in D$, we have $uwz \in S$ or $uw \notin S^1$ (to avoid $G_3$ on $uwxz$). Thus

$$\binom{|D|}{2} \le |\overline{S^1}| + |S_z| \le (c_6^2 + c_7 + o(1))n^2,$$

where we used Claim 4. Thus $|D| \le ((2c_6^2 + 2c_7)^{1/2} + o(1))n$. Since $z \in V_0'$ was arbitrary, it follows that $|\overline{B_1}[V_0, V_1]|$, the number of pairs connecting $V_0$ to $V_1$ that are not in $B^1$, is at most

$$|V_0 \setminus V_0'| \times v_1 + (2c_6^2 + 2c_7)^{1/2}n \times (n/3) + o(n^2) \le c_6 n^2,$$

giving the claim. □

Claims 3, 4, and 5 imply that $b_1 + s_0 + s_1 \ge (1/9 - 2c_6 + c_5 + 1/18 - c_6^2 + o(1))n^2$, contradicting (15). This final contradiction proves Theorem 6.

### 5.2. Proof of Theorem 1

Let $\varepsilon > 0$ be the constant returned by Theorem 6. Let $c = c(\varepsilon) > 0$ be the constant returned by Theorem 2 on input $\varepsilon$. Assume that $c \le \varepsilon$. Let us show that $n_0 = (1/c)^3$ suffices. Let $G$ be an arbitrary $\{G_0, G_3\}$-free 3-graph on $n \ge n_0$ vertices with at most $t_n$ edges.

Initially, define $G_n = G$ and $m = n$. If $\Delta(G_m) \le \Delta(T_m)$, then we stop. Otherwise, pick a vertex $x$ of $G_m$ of degree at least $\Delta(T_m) + 1$, let $G_{m-1} = G_m - x$ be obtained from $G_m$ by removing this vertex $x$ (and all edges that contain it), decrease $m$ by 1, and repeat.

When we stop, then $m \ge 2$ and we have

$$0 \le |G_m| \le t_m - (n - m) \le \binom{m}{3} + m - n < m^3 - n. \tag{22}$$

Thus $m > n^{1/3} \ge 1/c$. Theorem 2 implies that $G_m$ is $\varepsilon m^3$-close to $T_m$ in the edit distance. (Note that $|G_m| \le t_m$ by (22).) Since $m \ge 1/c \ge 1/\varepsilon$, Theorem 6 implies that $G_m \cong T_m$. By (22), we have $m = n$. Thus $G \cong T_n$, proving Theorem 1.

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