Available online at www.sciencedirect.com
ScienceDirect
Discrete Mathematics 307 (2007) 1455–1462
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Trees are almost prime

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Received 24 October 2002; received in revised form 10 June 2003; accepted 18 November 2005
Available online 29 November 2006

Abstract

Let S_n denote the graph on $\{1, \ldots, n\}$ in which two numbers are adjacent if and only if they are coprime. Around 1980 Entringer conjectured that S_n contains every tree of order n as a subgraph.

Here we show that this conjecture is true for all $n \leqslant 50$. Further positive evidence is provided by our main result that S_n contains every tree of order $(1 - o(1))n$.

Keywords: Entringer's conjecture; Prime graphs; Trees

1. Introduction

Let S_n denote the graph with $[n] := \{1, \ldots, n\}$ as the vertex set in which two vertices are adjacent if and only if they are coprime (as numbers). For example, S_5 is isomorphic to K_5 minus one edge.

A graph G of order n is called *prime* if it is a subgraph of S_n, that is, if there is a bijection $l : V(G) \to [n]$ such that any two adjacent vertices of G are assigned coprime numbers. This notion was introduced by Entringer who made the following, still open, conjecture.

Conjecture 1 (*Entringer*). Every tree is prime.

One popular direction of research was to verify the conjecture for some special classes of trees; we refer the reader to the dynamic survey by Gallian [2] which contains a section on prime labelling. In particular, Fu and Huang [1] proved that every tree with $n \leqslant 15$ vertices is prime. Pikhurko [4] extended this result to all $n \leqslant 34$.

In Section 2 we show that Entringer's conjecture holds for all trees with at most 50 vertices. The proof does not utilise any computer search and can be verified by hand. It also gives a practical algorithm for finding a prime labelling of any tree of order $n \leqslant 50$.

As it was pointed out by one of the referees, the M.Sc. Thesis of Shu-Hua Lin [3] verifies the validity of Entringer's conjecture for all orders $n \leqslant 104$. The original and interesting method of [3] is based on Hall's Matching Theorem.

URL: http://www.math.cmu.edu/~pikhurko.

[1] This research was done when the author was supported by a Research Fellowship, St John's College, Cambridge.

0012-365X/$ - see front matter © 2006 Published by Elsevier B.V.
doi:10.1016/j.disc.2005.11.083

Unfortunately, Lin's work has not been published and it seems that a considerable amount of work is needed in order to convert it into a rigorous proof (checking all omitted cases).

A different line of attack on the conjecture was initiated by Salmasian [6]. His idea was to allow more general sets of integers for the label set. Namely, a graph G is called *S-prime*, where $S \subset \mathbb{N}$, if there is an injection $l : V(G) \hookrightarrow S$ such that any two adjacent vertices receive coprime labels.

Problem 2 (*Salmasian*). *What is $m(n)$, the smallest m such that every tree of order n is $[m]$-prime*?

Of course, $m(n) \geqslant n$ and Entringer's conjecture states that we have equality here. Salmasian [6] proved that $m(n) \leqslant 4n$ for $n \geqslant 50$ and remarked that, in fact, his method gives $m(n) \leqslant (3.289\ldots + \mathrm{o}(1))n$.

Here we show that $m(n) = (1 + \mathrm{o}(1))\,n$. The proof can be easily converted into an algorithm for finding appropriate prime injections with running time $\mathrm{O}(n^2)$. We believe that our method has a potential for proving Entringer's conjecture for all large n.

2. Entringer's conjecture for trees of order $\leqslant 50$

2.1. Preliminaries

We essentially use the method from [4] although we add some new ideas. The strategy is to label a part of the given tree by some of the largest labels and apply the induction assumption to the remaining (unlabelled) forest. By carefully choosing labels for the points of 'contact' we avoid any clashes there. We cannot always guarantee that the remaining labels form an interval of integers; this is why we have to introduce the following predicate for general $S \subset \mathbb{N}$.

$\mathscr{P}(S)$: "*Every tree with $|S|$ vertices is S-prime*."

Thus we will show that $\mathscr{P}([i])$ holds for all $i \in [50]$. We will need the following easy auxiliary results which can be proved directly or found in [4].

Lemma 3. *Let p be a prime. If $\mathscr{P}([p-1])$ is true, then so are $\mathscr{P}([p])$, $\mathscr{P}([p+1])$, and $\mathscr{P}([p+2])$.*

Lemma 4. *Let T be a tree with at least 4 vertices. Then there is a vertex u such that either some set A of 3 vertices can be represented as a union of (one or more) components of $T - u$ or there is a neighbour v of u of degree $k + 1 \geqslant 3$ such that its other k neighbours $u_1, \ldots, u_k$ have degree 2 each and are incident to end-vertices $v_1, \ldots, v_k$ correspondingly.*

Remark. We call the first configuration a *stump* and the second a *k-fork*.

Of course, we will also use the validity of $\mathscr{P}([i])$, $i \in [34]$, established in [4].

2.2. The algorithm

As an example, let us show that $\mathscr{P}(S)$ is true, where $S = [29] \cup \{32, 33\}$, the case which we will need later. This is done by Table 1, namely, by the row marked '1–29, 32–33'. The entries there are to be interpreted in the following way.

If the given tree T of order $|S| = 31$ contains a stump, then we label $l(u) = 29$ and $l(A) = \{28, 32, 33\}$. (The latter is possible as 33 is coprime to both 28 and 32.) The remaining forest $T - A - u$ can be labelled by [27] because of $\mathscr{P}([27])$. As $l(u) = 29$ is coprime to any other element of S, we obtain the required labelling.

Table 1
Establishing $\mathscr{P}([35])$

S	$l(A)$	$l(u)$	k	$l(v)$	$(l(u_j))_{1 \leqslant j \leqslant k}$	$(l(v_j))_{1 \leqslant j \leqslant k}$
1–35	See text	31	$\geqslant 3$	32	35, 29, 33, $(35 - 2j)_{j \geqslant 4}$	34, $(34 - 2j)_{j \geqslant 2}$
1–29, 32–33	28, 32, 33	29	$\geqslant 2$	32	33, $(31 - 2j)_{j \geqslant 2}$	$(30 - 2j)_{j \geqslant 1}$

Table 2
Establishing $\mathscr{P}([36]\backslash\{30,31\})$

S	$l(A)$	$l(u)$	k	$l(v)$	$(l(u_j))_{1\leqslant j\leqslant k}$	$(l(v_j))_{1\leqslant j\leqslant k}$
1–29, 32–36	28, 33, 34	29	$\geqslant 2$	32	$35, 33, (33-2j)_{j\geqslant 3}$	$36, 34, (34-2j)_{j\geqslant 3}$
1–27, 32, 35–36	24, 35, 36	23	2	20	27, 21	22, 26
			$\geqslant 3$	32	$35, 27, 25, (29-2j)_{j\geqslant 4}$	$36, (30-2j)_{j\geqslant 2}$
1–19, 24–25, 32, 35–36	24, 35, 36	19	$\geqslant 2$	32	$35, 25, (23-2j)_{j\geqslant 3}$	$36, 24, (24-2j)_{j\geqslant 3}$
1–22, 25–27, 32	21, 22, 26	19	2	22	27, 21	26, 20
			$\geqslant 3$	32	$27, 25, 21, (25-2j)_{j\geqslant 4}$	$26, (26-2j)_{j\geqslant 2}$
1–18, 25, 32	18, 25, 32	17	$\geqslant 2$	32	$25, (19-2j)_{j\geqslant 2}$	$(20-2j)_{j\geqslant 1}$
1–18, 20, 25, 27, 32	20, 27, 32	17	$\geqslant 2$	32	$27, 25, (21-2j)_{j\geqslant 3}$	$(22-2j)_{j\geqslant 1}$
1–16, 18, 25	12, 18, 25	13	$\geqslant 2$	16	$25, 15, (17-2j)_{j\geqslant 2}$	$18, (18-2j)_{j\geqslant 2}$
1–11, 14–16	14, 15, 16	11	$\geqslant 2$	16	$15, (13-2j)_{j\geqslant 2}$	$14, (14-2j)_{j\geqslant 2}$

If T contains a k-fork, then we label $l(u)=29$, $l(v)=32$, $l(u_1)=33$, $l(u_j)=31-2j$, $2\leqslant j\leqslant k$, and $l(v_j)=30-2j$, $j\in[k]$. The unused labels $[29-2k]$ form an interval, so we can label the remaining vertices with them.

The above considerations establish $\mathscr{P}([29]\cup\{32,33\})$, as it was claimed.

2.3. $\mathscr{P}([35])$ *is true*

Let T be a tree with 35 vertices. This case is somewhat exceptional so we describe it separately.

Suppose that T contains a k-fork. If $k=2$, then we label $A:=\{v,u_1,v_1,u_2,v_2\}$ by 31, 34, 33, 32, 35, respectively and the forest $T':=T-A$ by [30]. The case $k\geqslant 3$ fits into the general scheme of the proof and the appropriate labelling is given in Table 1.

So, suppose by Lemma 3 that there is a stump. Let $A=\{a,b,c\}$. It is easy to see that up to a symmetry there are two cases to consider.

Case 1: $\{a,b\},\ \{a,c\}\notin E(T)$.

Label u,a,b,c by 31, 30, 35, 34, respectively. Now the remainder of T can be labelled by $B:=[29]\cup\{32,33\}$ because the validity of $\mathscr{P}(B)$ was shown in Section 2.2.

Case 2: $\{b,u\},\{c,u\}\notin E(T)$.

Claim 1. *$T':=T-A$ admits a prime labelling by* [32] *in which the vertex u has a label different from* 30.

Proof of Claim. Let x be an end-vertex of T' and let y be its neighbour. Label $l(x)=32$, $l(y)=31$, and the remainder of T' by [30]. If $l(u)=30$, swap the labels of x and u. $\square$

Take the labelling l of Claim 1. Note that, whatever the value of $l(u)\in[29]\cup\{31,32\}$ is, it is coprime with at least one of 33, 34, 35. As the numbers 33, 34, 35 are pairwise coprime, we can assign them arbitrarily to A and achieve that $l(u)$ and $l(a)$ are coprime, obtaining the required labelling.

2.4. $\mathscr{P}([36])$ *is true*

Let T be a tree of order 36. Label an end-vertex of T by 30 and its neighbour by 31. It is enough to show that $\mathscr{P}([36]\backslash\{30,31\})$ holds. This is done by Table 2. Its meaning is identical to that of Table 1. One difference here is that after colouring a stump or a fork, the remaining labels do not always form an interval, so we have to prove the validity of the predicate $\mathscr{P}$ for this new set as well. As the result, our table has a few rows. The easy but lengthy verification is left to the reader.

2.5. *The remaining* $n\leqslant 50$

As 37 is prime, we conclude by Lemma 3 that every tree with at most 39 vertices is prime. Table 3 establishes the validity of $\mathscr{P}([40])$. As 41 and 43 are prime, $\mathscr{P}([i])$ holds for any $i\leqslant 45$. The proof of $\mathscr{P}([46])$ is encoded in Table 3 as well. As 47 is prime, we deduce that any tree of order at most 49 is prime.

Table 3
Establishing $\mathscr{P}([40])$ and $\mathscr{P}([46])$

S	$l(A)$	$l(u)$	k	$l(v)$	$(l(u_j))_{1\leqslant j\leqslant k}$	$(l(v_j))_{1\leqslant j\leqslant k}$
1–40	38, 39, 40	37	$k\leqslant 3$	38	39, 35, 33	40, 36, 34
			$k\geqslant 4$	32	39, $(39-2j)_{j\geqslant 2}$	40, 36, 34, 38, $(40-2j)_{j\geqslant 5}$
1–46	44, 45, 46	43	$\leqslant 7$	46	45, $(45-2j)_{j\geqslant 2}$	$(46-2j)_{j\geqslant 1}$
			$\geqslant 8$	32	45, $(45-2j)_{j\geqslant 2}$	46, 42, 40, 38, 36, 34, 44, $(46-2j)_{j\geqslant 8}$

Finally, let us do $\mathscr{P}([50])$. The assignment $l(u)=47$ and $l(A)=\{48, 49, 50\}$ deals with a stump, so assume that we have a k-fork. If $k=2$, then the assignment $l(u, v, u_1, v_1, u_2, v_2)=(47, 46, 45, 44, 49, 48)$, reduces the problem to verifying $\mathscr{P}([43]\cup\{50\})$. But the latter property is easy: label an end-vertex by 50 and its only neighbour by 43 (and the rest by [42]). For $k\geqslant 3$ we label the fork in the following way: $l(u)=47$, $l(v)=46$, $(l(u_j))=(49, (49-2j)_{j\geqslant 2})$ and $(l(v_j))=(50, 44, 48, (50-2j)_{j\geqslant 4})$. This works unless $k\geqslant 13$ in which case $l(u_{13})=23$ is adjacent to $l(v)=46$. But for $k\geqslant 13$ we just modify the above labelling by swapping the labels 32 and 46.

Theorem 5. *Every tree with at most* 50 *vertices is prime.*

Remark. We do not know whether this method can handle trees of order 51 as the analysis gets too messy.

3. Every tree is almost prime

Recall that $m(n)$ is the smallest m such that every tree of order n is $[m]$-prime.

Theorem 6. *There is an absolute constant c such that for any $n\geqslant 3$ we have*

$$m(n)\leqslant\left(1+\frac{c}{\sqrt{\log n\ \log\log n}}\right)n. \tag{1}$$

The remainder of this section is dedicated to proving Theorem 6. We will show that $c=2+o(1)$ suffices in (1) as n tends to the infinity. (By log we will always mean the logarithm base e.)

Let $\varepsilon>0$ be an arbitrary small constant. Let $c>0$ satisfy $c^2>4+65\varepsilon$. Let n be large, T be a tree of order n, and let

$$m\geqslant\left(1+\frac{c}{\sqrt{\log n\ \log\log n}}\right)n.$$

Our aim is to show that T is a subgraph of S_m. Let us briefly describe the main ideas behind the proof. We define

$$P:=\{x\in[m]: x>m/2,\ x \text{ is prime}\}, \tag{2}$$

and label some $X\subset V(T)$ by a subset of P. Any vertex of P is connected to everything else in S_m so we will not have any problem with the vertices of X. Then we show that we can find vertex-disjoint bipartite subgraphs $H_i'\subset S_m$, $i\in[r]$, covering, let us say $(1+\varepsilon/2)n$ vertices such that $\delta(H_i')\approx(1+\varepsilon/2)n/2r$ (that is, each H_i' is almost complete and balanced). The standard embedding algorithm shows that H_i' contains any forest which is a subgraph of K_{δ_i,δ_i}, where $\delta_i=\delta(H_i')$. By carefully choosing X in the first place, we ensure that the forest $T-X$ can be embedded into the disjoint union of K_{δ_i,δ_i}, $i\in[r]$, which concludes the proof.

Let us carry out the above programme. By reducing m, we can assume that $m\leqslant(1+\varepsilon/2)n$. It is well-known that $\pi(x)$, the number of primes not exceeding x, is

$$\pi(x)=(1+o(1))\frac{x}{\log x}. \tag{3}$$

Hence,

$$|P| = (1+\mathrm{o}(1))\left(\frac{m}{\log m} - \frac{m/2}{\log(m/2)}\right) = (1+\mathrm{o}(1))\,\frac{m}{2\log m},$$

where $P \subset V(S_m)$ is defined by (2).

Lemma 7. *Let H be a tree of order k. There is $x \in V(H)$ such that any component of $H - x$ has at most $k/2$ vertices.*

Proof. Start with an arbitrary $x \in V(H)$. Repeat the following as long as possible. If there is a component C of $H - x$ with more than $k/2$ vertices, replace x by x', where x' is the vertex of C connected to x. When we do so, $V(H)\backslash C$ becomes a component of $T - x'$; it has $k - |C| \leqslant k/2$ vertices, that is, we do not create any new component of order greater than $k/2$. On the other hand, the size of the largest component is now strictly smaller than $|C|$. Thus our procedure terminates, which proves the lemma. □

Now apply the following procedure $\lceil n^{1/2} \rceil$ times to a forest F starting with $F = T$: choose a component $C \subset V(F)$ of the maximum size and remove x from F, where x is the vertex given by Lemma 7 when applied to the tree $F[C]$ (the subgraph of F induced by $C \subset V(F)$). Let $X \subset V(T)$ consist of the removed vertices. Let $C_x \ni x$ be the component subdivided by $x \in X$.

Lemma 8. *Let $k = 2n/|X| = (2+\mathrm{o}(1))\,n^{1/2}$. Then any component of $T - X$ has less than k vertices.*

Proof. Suppose on the contrary that the claim is not true. Define the oriented graph A on X, where $(x, y) \in E(A)$ if the component C_y was created during the removal of x. It is easy to check that A is an *arborescence* whose root is the first removed vertex (that is, A is an oriented tree with all arcs pointing away from the root).

By the definition of X, each C_x, $x \in X$, has at least k elements. If x is a leaf of A, define $B_x = C_x$. If x is a vertex of A of out-degree 1, let $B_x = C_x \backslash C_y$, where y is the unique vertex with $(x, y) \in E(A)$. Note that $|C_y| \leqslant |C_x|/2$, so $|B_x| \geqslant k/2$. Otherwise, we let $B_x = \emptyset$. Clearly, the sets B_x, $x \in X$, are pairwise disjoint. Hence, $\sum_{x \in V(A)} |B_x| \leqslant n$ but this contradicts Lemma 9. □

Lemma 9. *For any arborescence A we have*

$$\sum_{x \in V(A)} b_x \geqslant k + \frac{(v(A)-1)k}{2}, \tag{4}$$

where $b_x = k$ if x is a leaf; $b_x = k/2$ if the out-degree of x is 1; and $b_x = 0$ otherwise.

Proof. We use induction on the order $v(A)$ of A, with the case $v(A) = 1$ being trivially true. If we have a vertex y of out-degree 1, then we remove y and the (unique) edges $(x, y), (y, z) \in E(A)$ but add the edge (x, z). (If y is the root of A, we just remove y.) Otherwise, we can find leaves x, y with $(z, x), (z, y) \in E(A)$ for some z. Removing y, we create at most one vertex of out-degree 1 (and no new leaves). In either case the left-hand side of (4) decreases by at least $k/2$ while $v(A)$ decreases exactly by 1, which proves the claim by induction. □

Let us summarise the situation so far: in the given tree T of order n we have found a set $X \subset V(T)$ of size $\lceil n^{1/2} \rceil$ such that each component of $T - X$ has less than $k := 2n/|X|$ vertices.

We label X bijectively by some (arbitrary) subset $P' \subset P$. (Note that $|X| \leqslant |P|$.) Let $M := [m]\backslash P'$ consist of available labels. As every vertex of P is connected to every other vertex in S_m, no label clashes at X can ever occur.

Let $(p_1, p_2, p_3, \ldots) := (2, 3, 5, \ldots)$ be the increasing sequence of all primes. Let

$$t = \left\lceil \frac{\log n}{(2+2\varepsilon)\,\log\,\log\, n} \right\rceil \quad \text{and} \quad r = \prod_{i=1}^{t} p_i.$$

As $p_t = (1+\mathrm{o}(1))\,t\log t$, we conclude that $r \leqslant p_t^t < n^{1/(2+\varepsilon)}$.

Let

$$R_i := \{x \in M : x \equiv i \pmod{2r}\}, \quad i \in [2r].$$

We have

$$\lceil m/2r \rceil \geqslant |R_i| \geqslant \lfloor m/(2r) \rfloor - |X| = \frac{m}{2r} - \mathrm{O}(n^{1/2}).$$

We can obviously group the components of $T - X$ into r subgraphs $F_1, \ldots, F_r \subset F$ so that $|v(F_i) - v(F_j)| \leqslant k$ for any $i, j \in [r]$. This in particular implies that

$$|v(F_i) - n/r| \leqslant k, \quad i \in [r].$$

Indeed, if, for example, $v(F_i) < n/r - k$, then $v(F_j) \leqslant n/r$ for any j, which gives at most $|X| + (r-1)n/r + (n/r - k) < n$ vertices in total, a contradiction.

Of course, each F_i is a forest, in particular a bipartite graph. We can find a bipartition $V(F_i) = V_{i1} \cup V_{i2}$ such that $||V_{i1}| - |V_{i2}|| \leqslant k$, that is, $||V_{ij}| - v(F_i)/2| \leqslant k/2$, $j = 1, 2$. Thus

$$||V_{ij}| - n/(2r)| \leqslant k = \mathrm{O}(n^{1/2}), \quad i \in [r], \; j \in [2].$$

Let $i \in [r]$. Clearly, we are done if we can embed F_i into $H := S_m[R_{2i-1}, R_{2i}]$, the bipartite graph consisting of all S_m-edges between R_{2i-1} and R_{2i}. Let

$$l := \max(|V_{i1}|, |V_{i2}|) \leqslant n/2r + \mathrm{O}(n^{1/2}),$$

$$h := \min(|R_{2i-1}|, |R_{2i}|) \geqslant m/2r - \mathrm{O}(n^{1/2}).$$

Let $\overline{H} := \overline{S}_m[R_{2i-1}, R_{2i}]$ denote the bipartite complement of H.

Lemma 10. *Let us consecutively remove a vertex of H of degree less than l. If, for some $q \in \mathbb{N}$, we have $2e(\overline{H}) < q(h - l + 1) - \lfloor q^2/4 \rfloor$, then this procedure terminates in less than q steps.*

Proof. Suppose on the contrary that we can run the algorithm for q steps. Let $Q \subset V(H)$ be the set of the q removed vertices. For every vertex of Q we can find at least $h - l + 1$ incident edges with each edge either leading to an earlier element of Q or belonging to $E(\overline{H})$. Hence,

$$e(H[Q]) + 2e(\overline{H}) \geqslant q(h - l + 1).$$

As $H[Q]$ is bipartite, we have $e(H[Q]) \leqslant \lfloor q^2/4 \rfloor$, which contradicts the assumption on $e(\overline{H})$. $\square$

Let us estimate $e(\overline{H})$.

Of course, for no prime $p \leqslant p_t$ can $x \in R_{2i-1}$ and $y \in R_{2i}$ be both divisible by p. Hence, when we count $e(\overline{H})$ we have to take into account the divisibility by primes p with $p_t < p \leqslant m$ only. At most $m/(2rp) + 2$ elements of R_{2i-1} and at most $m/(2rp) + 2$ elements of R_{2i} are divisible by p, which gives at most $(m/(2rp) + 2)^2$ corresponding edges in $\overline{H}$. Hence,

$$e(\overline{H}) \leqslant \sum_{p_t < p \leqslant m} \left(\frac{m}{2rp} + 2\right)^2 \leqslant \sum_{p_t < p \leqslant \frac{\varepsilon m}{5r}} \left(\frac{m}{2rp} + 2\right)^2 + \sum_{\frac{\varepsilon m}{5r} < p \leqslant m} \mathrm{O}(1) =: \Sigma_1 + \Sigma_2. \tag{5}$$

Let us estimate the summands in (5). Define, $b_k := (1 + \varepsilon)^k p_t$ and

$$I_k := \{p : b_k < p \leqslant b_{k+1}, \; p \text{ is prime}\}, \quad k \in \mathbb{Z}_{\geqslant 0}.$$

As $m/(2rp)+2\leqslant(1+\varepsilon)n/(2rp)$ for $p\leqslant\varepsilon m/(5r)$, we obtain

$$\frac{\Sigma_1}{(1+\varepsilon)^2}\leqslant\sum_{p_t<p\leqslant\frac{\varepsilon m}{5r}}\frac{n^2}{4r^2p^2}<\sum_{k=0}^{\infty}\sum_{p\in I_k}\frac{n^2}{4r^2p^2}\leqslant\sum_{k=0}^{\infty}(\pi(b_{k+1})-\pi(b_k))\frac{n^2}{4r^2b_k^2}$$
$$=\sum_{k=0}^{\infty}\frac{(\varepsilon+o(1))n^2}{4r^2b_k\log b_k}\leqslant\frac{(\varepsilon+o(1))n^2}{4r^2p_t\log p_t}\sum_{k=0}^{\infty}\frac{1}{(1+\varepsilon)^k}\leqslant\frac{(1+\varepsilon+o(1))n^2}{4r^2p_t\log p_t}.$$

As $p_t=(1+o(1))\,t\log t$, we have

$$p_t\log p_t=(1+o(1))\frac{\log n\,\log\log n}{2+2\varepsilon}.$$

Also, we have

$$\Sigma_2=\sum_{\varepsilon m/5r<p\leqslant m}O(1)=O(\pi(m))=o(1)\times\frac{n^2}{r^2\log n\,\log\log n}.$$

Assuming that, for example, $\varepsilon<1$, we conclude that

$$e(\overline{H})\leqslant\frac{(1+16\varepsilon)n^2}{2r^2\log n\log\log n}. \tag{6}$$

We also have

$$h-l\geqslant\frac{m-n}{2r}+O(n^{1/2})\geqslant\frac{(c+o(1))n}{2r\sqrt{\log n\,\log\log n}}.$$

The condition of Lemma 10 is satisfied for

$$q:=\frac{cn}{r\sqrt{\log n\,\log\log n}}.$$

So, by removing less than q vertices of H we can obtain a graph $H'\subset H$ of minimum degree at least l. As $q<2h$, H' is not empty. We embed F_i into H' vertex by vertex, so that each new vertex x has at most one neighbour y among the previously considered vertices. If y exists, then the image of y in H' has at least l neighbours, of which at most $l-1$ have been occupied, so we can always find a place for x. If y does not exists, choose for x any vertex (in the corresponding part $R_{2i-1}\cap V(H')$ or $R_{2i}\cap V(H')$), which is possible as the parts have at least l vertices each.

This completes the proof of Theorem 6.

4. Some remarks

An interesting related problem is to compute $\mu(n)$, the smallest size of a non-prime graph of order n. Rao [5] conjectures that for $n\geqslant 8$

$$\mu(n)=\begin{cases}t+3, & n=2t,\\ t+5, & n=2t+1.\end{cases}$$

The upper bound on $\mu(n)$ comes from considering the disjoint union of a perfect matching and of 2 or 3 copies of K_3 (depending on the parity of n). This graph is not prime as it has no independent set of size $\lfloor n/2\rfloor$ so there is not enough room for even labels.

By joining the components of the above graph with extra edges, one can build a non-prime connected graph of order n and size at most $n+3$. This shows that there is hardly any essential way of strengthening Entringer's conjecture along these lines.

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