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Further asymptotic size Ramsey results obtained via linear programming

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Abstract

Recently, the author (SIAM J. Discrete Math. 16 (2003) 99–113) has asymptotically computed (via linear programming) size Ramsey numbers involving complete bipartite graphs. Here an attempt is made to extend this method to a larger class of problems by considering the 'simplest' open case when one of the forbidden graphs is $S_{1,n}$ (the n-star $K_{1,n}$ with an added leaf). Although we obtain new non-trivial results such as, for example, $\hat{r}(K_{2,n}, S_{1,n}) = (9+o(1))n$ and $\hat{r}(K_{3,n}, S_{1,n}) = (16 + o(1))n$, even this 'simple' case remains open.

Keywords: Size Ramsey number; Complete bipartite graphs

1. Introduction

We say that a graph G *arrows* an r-tuple of graphs $(F_1, \ldots, F_r)$ if any r-colouring of $E(G)$ contains a copy of F_i in colour i for some $i \in [r] := \{1, \ldots, r\}$. The *size Ramsey number* $\hat{r}(F_1, \ldots, F_r)$ is the smallest number of edges that an arrowing graph can have.

The author [3] has recently introduced a linear programming approach to size Ramsey numbers involving bipartite graphs, wherein the graph problem is approximated by a certain MIP (a system of linear inequalities in real and integer variables), see Section 2.

The author has essentially solved the corresponding optimisation problem if each forbidden graph is a complete bipartite graph and obtained various concrete results such

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as, for example, a proof of the conjecture of Faudree and Sheehan [2] that $\hat{r}(K_{2,n}, K_{2,n}) = 18n - 15$ for all large n. ($K_{m,n}$ is the complete bipartite graph with parts of size m and n.)

Unfortunately, the MIP seems very difficult to solve in other cases (mainly because of integer variables). Here an attempt is made to attack a larger class of problems by representing a forbidden graph as, roughly speaking, a union of complete bipartite graphs and applying the results from [3]. Here we concentrate on the 'simplest' open case as follows.

Let $S_{k,n}$, $n \geqslant k$, be obtained from the *n-star* $K_{1,n}$ by adding an extra vertex which is connected to some k leaves of the star. (For example, $S_{n,n} = K_{2,n}$.) The graph $S_{1,n}$ was first studied (in the context of size Ramsey numbers) by Bielak [1] who showed that $\hat{r}(S_{1,n}, S_{1,n}) = 4n - 2$, $n \geqslant 3$.

In this paper, we investigate the size Ramsey number of a complete bipartite graph versus $S_{k,n}$. We introduce an approach for establishing lower bounds. In some cases, we are able to determine the asymptotics of the corresponding size Ramsey function. But, in general, the problem remains open. For example, we do not even know the asymptotics of $\hat{r}(K_{2,t_1 n}, S_{1,t_2 n})$ except for $t_1 \geqslant 2t_2$ or $t_1 = t_2$ (although we have rather close upper and lower bounds). Nevertheless, we believe that our method has a potential for producing new interesting results.

2. Background

Let us briefly describe the ideas and results from [3]. For simplicity, we restrict ourselves to the case $r = 2$, that is, we have only two colours (which we call *blue* and *red*).

Suppose that we are interested in $\hat{r}(K_{s_1,t_1 n}, F_n)$ as $n \to \infty$, where $s_1 \in \mathbb{N}$ and $t_1 \in \mathbb{R}_{>0}$ are fixed and F_n may vary with n but we know that $F_n \subset K_{s,tn}$ for some fixed s, t. Of course, $t_1 n$ is not always an integer but we will treat it as if it is—this will have no effect on the asymptotic calculations.

A trivial observation is that $\hat{r}(K_{s_1,t_1 n}, F_n) = \mathrm{O}(n)$. Indeed, it is easy to see that, for example, $K_{s_1+s-1,m} \to (K_{s_1,t_1 n}, K_{s,tn})$ if $m > (t_1 n - 1)\binom{s_1+s-1}{s_1} + (tn-1)\binom{s_1+s-1}{s}$. And, in fact, it is usually not hard to obtain an (asymptotically correct) upper bound. So let us concentrate on the harder task of proving a lower bound.

Let n be large enough and let G be an arrowing graph with $e(G) = (1 + \mathrm{o}(1))\,\hat{r}(K_{s_1,t_1 n}, F_n)$. Define $L = \{x \in V(G) \mid d(x) \geqslant t_1 n\}$. Observe that $l = |L|$ is bounded by a constant not depending on n. Clearly, we may assume that $\bar{L} = V(G) \setminus L$ is an independent set. Also, if we remove all edges within L, the arrowing property is only slightly impaired: the obtained graph arrows $(K_{s_1,t_1 n - l + s_1}, F_n)$.

So, our task is to prove a lower bound on $e(G)$ given that

$$G \to (K_{s_1,t_1 n - l + s_1}, F_n)$$

and $G \subset K_{l,m}$, for some constant l. Clearly, to define G (up to an isomorphism) it is enough to give the sizes of the 2^l sets $G_A = \{x \in \bar{L} \mid \Gamma(x) = A\}$, $A \in 2^L$. We let $g_A = |G_A|/n$, $A \in 2^L$. Now, we operate with this sequence g_A of numbers rather than with the graph G. We try to get a lower bound on $e(G)/n = \sum_{A \in 2^L} g_A\,|A|$ given a certain 'approximation' (in terms of g_A) to the arrowing property. This 'approximation' states that a certain MIP has no solution. The MIP does not depend on n because of the scaling factor of $1/n$ in the definition of g_A. It is described in [3] and is indeed shown to give the asymptotic value of the corresponding size Ramsey number provided F_n is a 'nice' sequence.

Let us concentrate on the case when F_n is a complete bipartite graph $K_{s_2,t}$, where s_2 is fixed and either $t = t_2 n + o(n)$ for some fixed $t_2 > 0$ or $t \geqslant s_2$ is constant (then we let $t_2 = 0$). In this special case, our general MIP can be reduced to the following problem (cf. Claim 3 in the proof of Theorem 4.1 in [3]).

Problem 1. *Compute* $r = \inf \sum_{A \in 2^L} g_A\,|A|$ *over non-negative* g_A, $A \in 2^L$, *such that there is no sequence* $c_{A,B} \geqslant 0$, *indexed by pairs* (A,B) *of disjoint subsets of* L, *with*

$$\sum_{A \cup B = X} c_{A,B} = g_X \quad \forall X \in 2^L, \tag{1}$$

$$\sum_{A \supset S_1} c_{A,B} \leqslant t_1 \quad \forall S_1 \in \binom{L}{s_1}, \tag{2}$$

$$\sum_{B \supset S_2} c_{A,B} \leqslant t_2 \quad \forall S_2 \in \binom{L}{s_2}. \tag{3}$$

Let us discuss informally where (1)–(3) come from. Similarly to our graph G being described by the numbers g_A, a 2-colouring of $E(G)$ can be described by the sequence $c_{A,B}$, where $nc_{A,B}$ counts the number of $x \in G_{A\cup B}$ such that A (resp. B) is the blue (resp. red) neighbourhood of x. Identity (1) says that each $x \in G_X$ is counted precisely once. Let $S_1 \in \binom{L}{s_1}$. The number of $x \in \bar{L}$ which are connected by a blue edge to every vertex of S_1 is precisely $n \sum_{A \supset S_1} c_{A,B}$. (Of course, it is understood that the sum is taken over all disjoint subsets $A, B \subset L$.) Inequality (2) says that there is no blue $K_{s_1,t_1 n+1}$-subgraph (assuming $t_1 n \geqslant l$). If $t_2 > 0$, then (3) carries the analogous message. If $t_2 = 0$, then (3) forces every $c_{A,B}$ with $B \supset S_2$ be zero. This is reasonable: if we have no red $K_{s_2,t}$, where t is fixed, then each $c_{A,B}$ with $|B| \geqslant s_2$ is at most $(t-1)/n = o(1)$. Thus, (2) and (3) "approximate" the statement that the colouring c has neither blue $K_{s_1,t_1 n}$ nor red $K_{s_2,t}$. To compute the size Ramsey number, we have to minimise the size of G given that no such colouring exists—precisely what Problem 1 says.

Of course, we would like to know how to compute the infimum r in Problem 1. Suppose first that $t \geqslant s_2$ is fixed. For a fixed $s \geqslant s_1 + s_2 - 1$, let

$$t'_s = \lim_{n\to\infty} \frac{\min\{m \mid K_{s,m} \to (K_{s_1,t_1 n}, K_{s_2,t})\}}{n}. \tag{4}$$

It is easy to show directly that

$$t'_s \leqslant t_1 \binom{s}{s_1} \binom{s - s_2 + 1}{s_1}^{-1}. \tag{5}$$

Indeed, let n be large. Suppose that we have an admissible 2-colouring of $G=K_{s,m}$. The number of $x\in\bar{L}$ which send at least s_2 red edges to L is at most $\binom{s}{s_2}(t-1)=\mathrm{O}(1)$. (Note that $|L|=s$.) Hence, $m-\mathrm{O}(1)$ vertices of $\bar{L}$ send at least $s-s_2+1$ blue edges each. This creates at least $(m-\mathrm{O}(1))\binom{s-s_2+1}{s_1}$ blue $K_{s_1,1}$-subgraphs, which cannot exceed $\binom{s}{s_1}(t_1 n-1)$. Now, (5) follows.

But we can deduce (5) from (1)–(3) as well. For $G=K_{s,m}$ each g_A is zero except $g_L=m/n$. Suppose that a solution $c_{A,B}$ to (1)–(3) exists. Inequality (3) implies that $c_{A,B}=0$ unless $|A|\geqslant s-s_2+1$. Average (2) over all $S_1\in\binom{L}{s_1}$ and use (1) to obtain

$$\binom{s}{s_1}t_1\geqslant\sum_{A\subset L}\binom{|A|}{s_1}c_{A,L\setminus A}\geqslant\binom{s-s_2+1}{s_1}\sum_{A\subset L}c_{A,L\setminus A}=\binom{s-s_2+1}{s_1}g_L,$$

which implies the required bound (5).

We hope that these expository calculations gave the reader the feeling of Problem 1. But let us continue. We obtain

$$r\leqslant r':=\min\{st'_s\,|\,s\geqslant s_1+s_2-1\}. \tag{6}$$

We claim that we have, in fact, equality in (6). Let g_A, $A\in 2^L$, be an arbitrary sequence satisfying Problem 1. For disjoint $A,B\subset L$ let $c_{A,B}=0$ unless $|B|=s_2-1$ in which case we let $c_{A,B}=g_{A\cup B}/\binom{|A\cup B|}{s_2-1}$. Clearly, (1) and (3) are satisfied. So (2) must be violated, that is, for some $S_1\in\binom{L}{s_1}$ we have

$$t_1<\sum_{A\supset S_1}c_{A,B}=\sum_{S\supset S_1}\frac{g_S\binom{|S|-s_1}{s_2-1}}{\binom{|S|}{s_2-1}}\leqslant t_1\sum_{S\supset S_1}\frac{g_S\,|S|}{|S|\,t'_{|S|}}\leqslant\frac{t_1}{r'}\sum_S g_S\,|S|. \tag{7}$$

Hence, $\sum_S g_S|S|>r'$, implying that we have equality in (6). This gives us an algorithm for computing $r=\lim_{n\to\infty}\hat{r}(K_{s_1,t_1n},K_{s_2,t})/n$. (Note that we need to check only finitely many values of s in (6) because, for example, $t'_s\geqslant t_1$ for any s.)

However, (7) itself (which is inequality (4.5) in [3]) is more important in this paper, so we restate it again: If n is large and $G\to(K_{s_1,t_1n},K_{s_2,t})$, then

$$1-\mathrm{o}(1)\leqslant\sum_{s=s_1+s_2-1}^{l}\frac{g_s}{t'_s}, \tag{8}$$

where $g_s:=\sum_{S\in\binom{L}{s}}g_S$.

What happens if $t_2>0$, that is, if t grows with n? Then the Farkas lemma shows that t'_s can be computed as the maximum of $\sum_{j=0}^{s}w_j$ over all non-negative real sequences $(w_j)_{j\in[0,s]}$ satisfying

$$\sum_{j=s_1}^{s}w_j\binom{j}{s_1}\leqslant t_1\binom{s}{s_1},$$

$$\sum_{j=0}^{s-s_2}w_j\binom{s-j}{s_2}\leqslant t_2\binom{s}{s_2}. \tag{9}$$

Most of what we have just said about $t_2 = 0$, in particular (8), remains valid for $t_2 > 0$ too. We refer the reader to [3, Theorem 4.1] for details.

3. Our approach

Suppose that we want to prove a lower bound on $\hat{r}(K_{s_1,t_1n}, S_{k,t_2n})$. Suppose first that $k \geqslant 2$. We have $K_{2,2}, K_{1,t_2n} \subset S_{k,t_2n}$. Hence, any $G \to (K_{s_1,t_1n}, S_{k,t_2n})$ also arrows $\mathscr{F}' := (K_{s_1,t_1n}, K_{1,t_2n})$ and $\mathscr{F}'' := (K_{s_1,t_1n}, K_{2,2})$. One can now conclude that

$$\hat{r}(K_{s_1,t_1n}, S_{k,t_2n}) \geqslant \max(\hat{r}(\mathscr{F}'), \hat{r}(\mathscr{F}'')). \tag{10}$$

However, (10) need not be sharp (e.g. when $\hat{r}(\mathscr{F}')=\hat{r}(\mathscr{F}'')$ but the extremal graphs are different). In some cases, a better bound can be obtained by unveiling the right-hand side of (10) one step further. Namely, we write down (8) twice: for the $\mathscr{F}'$-problem and for the $\mathscr{F}''$-problem. We try to prove a lower bound on $\sum_s sg_s$ (that is, we compute $z = \inf(\sum_{s=1}^{\infty} sg_s))$ given that *both* these inequalities hold (and that each $g_s \geqslant 0$).

Surprisingly, this approach works also for $k = 1$. We know (cf. Section 2) that it is enough (for asymptotic calculations) to restrict our attention to graphs that are subgraphs of $K_{s,m}$ with s being fixed. Let such a graph G arrow the pair $(K_{s_1,t_1n}, S_{1,t_2n})$. We claim that $G' \to (K_{s_1,t_1n}, K_{2,2})$, where G' is obtained from G by adding $\binom{s}{2}$ new vertices each one being connected to everything in $L \subset V(G)$. Suppose on the contrary that we have a $(K_{s_1,t_1n}, K_{2,2})$-free colouring of G'. Remove all vertices from G' that send at least two red edges to L. As we do not have a red $K_{2,2}$, there are at most $\binom{s}{2}$ such vertices. The 2-colouring of the remaining graph G'' is clearly $(K_{s_1,t_1n}, S_{1,t_2n})$-free but this contradicts the fact that G'' contains a subgraph isomorphic to G.

Thus our approach is applicable without any changes to the case $k = 1$ except that we have to subtract an O(1)-term from the right-hand side of (10), etc.

We remark here that using the methods from [3] it is possible to show that the ratio $\hat{r}(K_{s_1,t_1n}, S_{k,t_2n})/n$ tends to a limit when k, s_1, t_1, t_2 are fixed and $n \to \infty$. However, this requires going into laborious details so we skip the proof (especially that we do not use this fact in the paper).

4. $\hat{r}(K_{1,t_1n}, S_{k,t_2n})$

Here is a good illustration of how the proposed method works in practice.

Theorem 2. *Let $k \in \mathbb{N}$ and $t_1, t_2 \in \mathbb{R}_{>0}$ be fixed. Then*

$$\hat{r}(K_{1,t_1n}, S_{k,t_2n}) = (\max(4t_1, 3t_1 + t_2) + o(1))n. \tag{11}$$

Proof. We prove the lower bound as is prescribed in Section 3.

For $\mathscr{F}' = (K_{1,t_1n}, K_{1,t_2n})$ we have $t'_s = \max(\sum_{j=0}^{s} w_j)$ given that $\sum_{j=0}^{s} w_j \cdot j \leqslant t_1 \cdot s$ and $\sum_{j=0}^{s} w_j(s - j) \leqslant t_2 \cdot s$. Adding the last two inequalities we obtain that $t'_s \geqslant t_1 + t_2$, which is in fact sharp as the assignment $w_0 = t_2$, $w_s = t_1$ and $w_j = 0$, $j \in [s-1]$, demonstrates. Now, (8) says that

$$t_1 + t_2 - \mathrm{o}(1) \leqslant g_1 + g_2 + g_3 + \cdots \ . \tag{12}$$

For $\mathscr{F}'' = (K_{1,t_1 n}, K_{2,2})$ one has $t''_s = (s/(s-1))t_1$, $s \geqslant 2$. Hence,

$$t_1 - \mathrm{o}(1) \leqslant \tfrac{1}{2} g_2 + \tfrac{2}{3} g_3 + \tfrac{3}{4} g_4 + \cdots \ . \tag{13}$$

Let ($g_s \geqslant 0$) be a feasible solution to (12) and (13). If $g_s > 0$ for some $s \geqslant 3$, then the redefinition $g'_s = 0$ and $g'_2 = g_2 + 2(s-1)g_s/s$ does not decrease the right-hand side of (12) and (13) while $\sum jg_j$ changes by

$$\frac{4(s-1)}{s} g_s - sg_s = -\frac{(s-2)^2}{s} g_s < 0.$$

A moment's thought reveals that we have to minimise $g_1 + 2g_2$ provided $g_1, g_2 \geqslant 0$ and

$$t_1 + t_2 - \mathrm{o}(1) \leqslant g_1 + g_2,$$

$$2t_1 - \mathrm{o}(1) \leqslant g_2.$$

The (unique) optimal solution (disregarding the o(1)-terms) is $g_1 = 0$, $g_2 = 2t_1$ for $t_1 \geqslant t_2$ and $g_1 = t_2 - t_1$, $g_2 = 2t_1$ for $t_1 \leqslant t_2$, which gives the required.

Now let us show the upper bound.

If $t_1 \geqslant t_2$, then $G = K_{2,\,2t_1 n+k-2}$ is an example of an arrowing graph. Indeed, take any blue–red colouring without a blue $K_{1,t_1 n}$. Then at least $2t_1 n + k - 2 - 2(t_1 n - 1) = k$ vertices of V_2, the larger part of G, send two red edges to $V_1 = \bar{V}_2$. The red degree of an $x \in V_1$ is at least $2t_1 n + k - 2 - (t_1 n - 1) \geqslant t_2 n$, which gives a red $S_{k,t_2 n}$, as required.

If $t_1 \leqslant t_2$, let $E(G)$ consist of the edges $\{x_1, i\}$, $i \in [a]$ and $\{x_2, j\}$, $j \in [b]$, where $a = (t_1 + t_2)n + k - 2$ and $b = 2t_1 n + k - 2$. Take any 2-colouring without a blue $K_{1,t_1 n}$. There is a red $K_{1,t_2 n+k-1}$ centred at x_1. The set C of red neighbours of x_1 that lie in $[b]$ has cardinality at least $t_2 n + k - 1 - a + b = t_1 n + k - 1$. As we do not have a blue $K_{1,t_1 n}$, at least k edges connecting x_2 to C are red, giving a red $S_{k,t_2 n}$. □

It seems that one could prove the lower bound in Theorem 2 via direct reasoning: compare with Bielak's [1] proof that $\hat{r}(S_{1,n}, S_{1,n}) = 4n - 2$. However, the proof of the lower bound in [1] occupies four full pages (although one has to take into account that Bielak computes this function *exactly*). Also, a nice feature of our method is that it gives some hints on how a minimum arrowing graph might look like.

5. $\hat{r}(K_{2,t_1 n}, S_{k,t_2 n})$

Unfortunately, already this case does not yield to our method. The lower and upper bounds differ in general, so we give only some sketches.

Let us do the lower bound first. Routine computations show that, for $\mathscr{F}' = (K_{2,t_1 n}, K_{1,t_2 n})$, we have $t'_2 = t_1 + 2t_2$,

$$t'_3 = \min(3(t_1 + t_2)/2, t_1 + 2t_2), \tag{14}$$

$$t'_4 = \begin{cases} 2t_1 + \frac{4}{3} t_2, & 3t_1 \leqslant t_2, \\ \frac{6}{5} t_1 + \frac{8}{5} t_2, & \frac{1}{2} t_1 \leqslant t_2 \leqslant 3t_1, \\ t_1 + 2t_2, & t_2 \leqslant \frac{1}{2} t_1. \end{cases} \tag{15}$$

It seems that the coefficients t'_s, $s \geqslant 5$, have no effect on the obtained lower bound; however, a proof of this would require a large amount of symbolic computation, which we skip. Therefore, the reader should regard the stated lower bound (16) as our well-supported guess as to what our approach gives.

For $\mathscr{F}'' = (K_{2,t_1 n}, K_{2,2})$ one easily obtains $t''_s = (s/(s-2))t_1$, $s \geqslant 3$.

Our symbolic computations showed that the $\mathscr{F}'$ and $\mathscr{F}''$ inequalities (disregarding t'_s, $s \geqslant 5$) imply that

$$2g_2 + 3g_3 + 4g_4 \geqslant o(1) + \begin{cases} 7t_1 + \dfrac{4t_2^2}{t_1 + t_2}, & t_1 \leqslant t_2, \\ 4t_2 + \dfrac{15t_1^2}{2t_1 + t_2}, & \dfrac{1}{2} t_1 \leqslant t_2 \leqslant t_1, \\ 8t_1, & t_2 \leqslant \dfrac{1}{2} t_1. \end{cases} \tag{16}$$

The best upper bounds that we have been able to find are as follows.

Theorem 3. *Let $k \in \mathbb{N}$ and $t_1, t_2 \in \mathbb{R}_{>0}$ be fixed. Then, as $n \to \infty$,*

$$\frac{\hat{r}(K_{2,t_1 n}, S_{k,t_2 n})}{n} \leqslant o(1) + \begin{cases} \frac{9}{2} t_1 + \frac{9}{2} t_2, & t_1 \leqslant t_2, \\ 9t_1, & \frac{21}{32} t_1 \leqslant t_2 \leqslant t_1, \\ \frac{24}{5} t_1 + \frac{32}{5} t_2, & \frac{1}{2} t_1 \leqslant t_2 \leqslant \frac{21}{32} t_1, \\ 8t_1, & t_2 \leqslant \frac{1}{2} t_1. \end{cases} \tag{17}$$

Proof. Fix an arbitrary $\varepsilon > 0$.

For $t_1 \leqslant t_2$ take $G = K_{3,3(t_1+t_2+\varepsilon)n/2}$. Let n be large and consider any blue–red colouring of G without a blue $K_{2,t_1 n}$. Identity (14), which for $t_1 \leqslant t_2$ reduces to $t'_3 = 3(t_1+t_2)/2$, implies by (4) that G contains a red $K_{1,t_2 n+2k-2}$, say $G[x_1, A]$. At most $t_1 n - 1$ vertices of A can send both blue edges to x_2, x_3, the two other high-degree vertices. This implies that the set $\{x_2, x_3\}$ receives at least $2k-1$ red edges from A, which clearly gives a red $S_{k,t_2 n}$.

If $\frac{21}{32} t_1 \leqslant t_2 \leqslant t_1$, we let $G = K_{3,(3t_1+\varepsilon)n}$. Take any colouring without a blue $K_{2,t_1 n}$. As $t''_3 = 3t_1$, we conclude that there is a red $K_{2,k}$, say $G[\{x_1, x_2\}, A]$. At most $t_1 n - 1$ vertices can send two blue edges to $\{x_1, x_2\}$, so the set $\{x_1, x_2\}$ receives at least

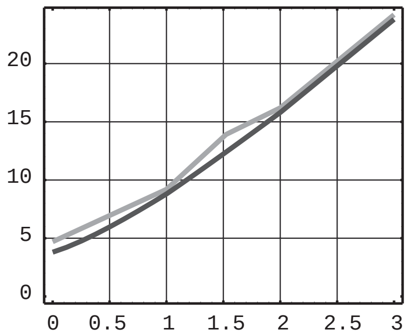

Fig. 1. $\hat{r}(K_{2,t_1 n}, S_{1,n})$.

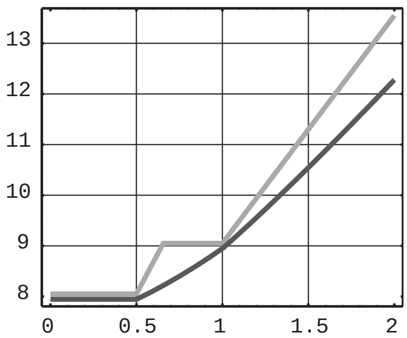

Fig. 2. $\hat{r}(K_{2,n}, S_{1,t_2 n})$.

$3t_1 n - (t_1 n - 1) > 2t_2 n$ red edges. Hence, x_1 or x_2 has red degree at least $t_2 n$, which gives a red $S_{k,t_2 n}$.

For $\frac{1}{2}t_1 \leqslant t_2 \leqslant \frac{21}{32}t_1$ we take $G = K_{4,(6t_1+8t_2+2\varepsilon)n/5}$. Again, if we do not have a blue $K_{2,t_1 n}$, then we find a red $K_{1,t_2 n+5(k-1)} = G[x_1, A]$. Assume that any $y \in V_1 \setminus \{x_1\}$ sends at most $k-1$ red edges to A (otherwise we have a red $S_{k,t_2 n}$). Let H be obtained by deleting x_1 and A from G. Observe that $(6t_1 + 3t_2)/5 \geqslant 3((t_1 - t_2) + t_2)/2$; so by (14)

$$H \supset K_{3,(6t_1+3t_2+\varepsilon)n/5} \to (K_{2,(t_1-t_2)n}, K_{1,t_2 n}).$$

A blue $K_{2,(t_1-t_2)n} = H[\{x_2, x_3\}, B]$ yields a blue $K_{2,t_1 n} \subset G$ because x_2 and x_3 have at least $|A| - 2(k-1) \geqslant t_2 n$ common blue neighbours in A. If we have a red $K_{1,t_2 n} = H[x_2, B]$, then the remaining two vertices $x_3, x_4 \in V_1$ have at least $|A \cup B| - 4(k-1) \geqslant 2t_2 n \geqslant t_1 n$ common blue neighbours in $A \cup B$. In any case we obtain a forbidden subgraph.

Finally, for $t_2 \leqslant \frac{1}{2}t_1$ we take $K_{4,(2t_1+\varepsilon)n}$ which has already been shown to have the arrowing property (take $t_2 = t_1/2$ in the previous case). □

Figs. 1 and 2 illustrate our upper and lower bounds. Not surprisingly, the obtained functions are homogeneous in t_1 and t_2. We present two different plots (obtained with *Mathematica*): in Fig. 1 we let $t_2 = 1$ (and vary t_1) while in Fig. 2 we fix $t_1 = 1$.

As we see, the bounds coincide when $t_2 \leqslant \frac{1}{2} t_1$ and when $t_1 = t_2$. Let us carefully prove the lower bound in these cases. For $t_2 \leqslant \frac{1}{2} t_1$ the lower bound of $8t_1$ can be deduced from the $\mathscr{F}''$-inequality alone (and hence from (10)) as follows:

$$1 - o(1) \leqslant \sum_{s \geqslant 3} \frac{s-2}{st_1} g_s \leqslant \frac{1}{8t_1} \sum_{s \geqslant 3} s g_s,$$

where the second inequality follows by comparing, for every $s \geqslant 3$, the coefficients at g_s. (The straightforward minimisation shows that $x^2/(x-2) \geqslant 8$ for any real $x \geqslant 3$.)

Let $t_1 = t_2$. We can assume $t_1 = t_2 = 1$. Inequality (10) gives only $(8 + o(1))n$ as a lower bound. So we recourse to the $\mathscr{F}'$-inequality as well. Multiplying it by 2 and noting that each $t'_s \geqslant 2$ we obtain

$$2 - o(1) \leqslant \frac{2}{3} g_2 + \frac{2}{3} g_3 + \frac{5}{7} g_4 + \sum_{s \geqslant 5} g_s. \tag{18}$$

Now add the $\mathscr{F}''$-inequality $1 - o(1) \leqslant \sum_{s \geqslant 3} ((s-2)/s) g_s$ to (18):

$$3 - o(1) \leqslant \frac{2}{3} g_2 + g_3 + \frac{17}{14} g_4 + \sum_{s \geqslant 5} \frac{2s-2}{s} g_s \leqslant \frac{1}{3} \sum_{s \geqslant 2} s g_s,$$

where the second inequality follows from the easy fact that $(2s-2)/s \leqslant s/3$ for $s \geqslant 5$. Hence, $\sum_{s \geqslant 2} s g_s \geqslant 9 - o(1)$ as required.

Theorem 4. *Let $k \in \mathbb{N}$ and $t_1, t_2 \in \mathbb{R}_{>0}$ be fixed and let $n \to \infty$. If $t_2 \leqslant \frac{1}{2} t_1$, then $\hat{r}(K_{2,t_1 n}, S_{k,t_2 n}) = (8t_1 + o(1))n$. Also, $\hat{r}(K_{2,n}, S_{k,n}) = (9 + o(1))n$.*

6. $\hat{r}(K_{3,n}, S_{k,n})$

Here is another 2-colour case when we were able to compute the asymptotics.

Theorem 5. *Let $k \geqslant 1$ be fixed and $n \to \infty$. Then*

$$\hat{r}(K_{3,n}, S_{k,n}) = (16 + o(1))n. \tag{19}$$

Proof. Let us show the lower bound. It is routine to compute a few first terms of the inequality corresponding to $\mathscr{F}' = (K_{3,n}, K_{1,n})$:

$$1 - o(1) \leqslant \frac{1}{4} g_3 + \frac{1}{4} g_4 + \frac{7}{25} g_5 + \frac{2}{7} g_6 + \frac{2}{7} g_7 + \frac{5}{17} g_8 + \sum_{s \geqslant 9} \frac{1}{t'_s} g_s. \tag{20}$$

Now multiply (20) by 3 and add it to the $\mathscr{F}''$ inequality $1 - o(1) \leqslant \sum_{s \geqslant 4} ((s-3)/s) g_s$. Observing that $t'_s \geqslant 2$ for any $s \geqslant 9$ we obtain

$$4 - o(1) \leqslant \frac{3}{4} g_3 + g_4 + \frac{31}{25} g_5 + \frac{19}{14} g_6 + \frac{10}{7} g_7 + \frac{205}{136} g_8 + \sum_{s \geqslant 9} \left(\frac{3}{2} + \frac{s-3}{s} \right) g_s \leqslant \frac{1}{4} \sum_{s \geqslant 3} s g_s.$$

(The second inequality follows by comparing the coefficients at g_s and using the fact that $\frac{3}{2}+(s-3)/s \leqslant \frac{1}{4}s$ for $s \geqslant 9$.) This implies the lower bound.

For the upper bound consider $G = K_{4,(4+\varepsilon)n}$. Take any blue–red colouring without a blue $K_{3,n}$. We know that $t'_4 = 4$. Hence, we have a red $K_{1,n+3k-3} = G[x_1, A]$. But to avoid a blue $K_{3,n}$, the three remaining high-degree vertices must send at least $3k-2$ red edges to A, which gives the required red $S_{k,n}$. □

Remark. Note that (10) gives only $(12+o(1))n$ as a lower bound.

Unfortunately, the author was not able to compute the asymptotics of $\hat{r}(K_{4,n}, S_{k,n})/n$. An upper bound of 25 comes from considering $K_{5,(5+\varepsilon)n} \to (K_{4,n}, K_{1,n+4k-4})$. Bound (10) gives only 20 for a lower bound. Our approach seems to improve this to $24\frac{2}{13}$. There is still a gap between bounds so we do not provide further details.

7. Final remarks

Our approach is applicable in more general settings. For example, it works for three or more colours (with the obvious modifications).

Also, some other forbidden graphs can be handled. Here is one example. Let $T_{k,n}$ be obtained from $K_{1,n}$ by adding k new vertices, each sending an edge to the same leaf x of $K_{1,n}$. It is not hard to see that a lower bound on $\hat{r}(K_{s_1,t_1 n}, T_{k,n})$ can be obtained, in the fashion of Section 3, by considering $\mathscr{F}' = (K_{s_1,t_1 n}, K_{1,n})$ and $\mathscr{F}'' = (K_{s_1,t_1 n}, K_{k,k})$.

But, as the reader has seen, we could not compute asymptotically even $\hat{r}(K_{2,t_1 n}, S_{1,n})$. We believe that our lower bounds in Section 5 (when they differ from the upper bounds) can be improved. An improvement in this and other cases might come from the fact that, besides (8), there are probably other restrictions that the sequence (g_i) of any $G \to (K_{s_1,t_1 n}, K_{s_2,t})$ must satisfy. As $(0,\ldots,0,t'_i,0,\ldots)$ is a feasible sequence, any extra (non-redundant) inequality cannot be linear in the g_i's. So far, we had no success in identifying any new and useful restrictions.

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