Size Ramsey Numbers and Integer Programming

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Abstract

We show that $\lim_{n\to\infty} \hat{r}(F_{1,n},\ldots,F_{q,n},F_{p+1},\ldots,F_r)/n$ exists, where the bipartite graphs $F_{q+1},\ldots,F_r$ do not depend on n while, for $1 \le i \le q$, $F_{i,n}$ is obtained from some bipartite graph F_i with parts $V_1 \cup V_2 = V(F_i)$ by duplicating each vertex $v \in V_2$ $(c_v + o(1))n$ times for some real $c_v > 0$.

In fact, the limit is the minimum of a certain mixed integer program. Using the Farkas Lemma we compute it when each forbidden graph is a complete bipartite graph, in particular answering a question of Erdős, Faudree, Rousseau and Schelp (1978) who asked for the asymptotics of $\hat{r}(K_{s,n},K_{s,n})$ for fixed s and large n. Furthermore, we prove (for all sufficiently large n) the conjecture of Faudree, Rousseau and Sheehan (1983) that $\hat{r}(K_{2,n},K_{2,n}) = 18n - 15$.

Complete proofs can be found in [3].

Key words: size Ramsey number, bipartite graphs, mixed integer programming, Farkas Lemma

1 Size Ramsey Numbers

Let $(F_1,\ldots,F_r)$ be an r-tuple of graphs which are called *forbidden*. We say that a graph G *arrows* $(F_1,\ldots,F_r)$ if for any r-colouring of $E(G)$, the edge set of G, there is a copy of F_i of colour i for some $i \in [r] := \{1,\ldots,r\}$. We denote this *arrowing property* by $G \to (F_1,\ldots,F_r)$.

The *size Ramsey number*

$$\hat{r}(F_1,\ldots,F_r) = \min\{e(G) \mid G \to (F_1,\ldots,F_r)\}$$

[1] Supported by a Research Fellowship, St. John's College, Cambridge, UK.

is the smallest number of edges that an arrowing graph can have.

2 Limit Theorem

It is not hard to see that, for fixed $s_1, \ldots, s_r \in \mathbb{N}$ and $t_1, \ldots, t_r \in \mathbb{R}_{>0}$, we have

$$\hat{r}(K_{s_1,\lfloor t_1 n\rfloor}, \ldots, K_{s_r,\lfloor t_r n\rfloor}) = O(n).$$

Here we will show that the limit

$$r(F_{1,n}, \ldots, F_{r,n}) = \lim_{n\to\infty} \frac{\hat{r}(F_{1,n}, \ldots, F_{r,n})}{n} \tag{1}$$

exists if each forbidden graph is either a fixed bipartite graph or a subgraph of $K_{s,\lfloor tn\rfloor}$ 'dilating uniformly' with n, which we will define below.

For example, it follows that $\hat{r}(K_{1,n}, F)/n$ tends to a limit for any fixed bipartite graph F.

3 Dilatation

We assume that bipartite graphs come equipped with a fixed bipartition $V(F) = V_1(F) \cup V_2(F)$, although graph embeddings need not preserve it. We denote $v_i(F) = |V_i(F)|$, $i = 1, 2$; thus $v(F) = v_1(F) + v_2(F)$.

For $A \subset V_1(F)$, we define

$$F^A = \{v \in V_2(F) \mid \Gamma_F(v) = A\},$$

where $\Gamma_F(v)$ denotes the neighbourhood of v in F. Clearly, in order to determine F (up to an isomorphism) it is enough to have $V_1(F)$ and $|F^A|$ for all $A \subset V_1(F)$. This motivates the following definitions.

A *weight* $\mathbf{f}$ on a set $V(\mathbf{f})$ a sequence $(f_A)_{A\in 2^{V(\mathbf{f})}}$ of non-negative reals. A bipartite graph F *agrees* with $\mathbf{f}$ if $V_1(F) = V(\mathbf{f})$ and $F^A = \emptyset$ if and only if $f_A = 0$, $A \in 2^{V(\mathbf{f})}$. A sequence of bipartite graphs $(F_n)_{n\in\mathbb{N}}$ is a *dilatation* of $\mathbf{f}$ (or *dilates* $\mathbf{f}$) if each F_n agrees with $\mathbf{f}$ and

$$|F_n^A| = f_A n + o(n), \quad \text{for all } A \in 2^{V(\mathbf{f})}.$$

It is not hard to see that any sequence of bipartite graphs described in the abstract is in fact a dilatation of some weight.

4 Conrete Forbidden Graphs

The limit value can in fact be obtained as the minimum of a certain mixed integer program (which does depend on n). The author has been able to solve the MIP when each $F_{i,n}$ is a complete bipartite graph. In particular, this answers a question by Erdős, Faudree, Rousseau and Schelp [1, Problem B] who asked for the asymptotics of $\hat{r}(K_{s,n}, K_{s,n})$. Also, the conjecture of Faudree, Rousseau and Sheehan [2, Conjecture 15] that

$$\hat{r}(K_{2,n}, K_{2,n}) = 18n - 15, \tag{2}$$

has been proved for all large n.

Unfortunately, the MIP is not well suited for practical calculations and the author was not able to compute the asymptotics for any other non-trivial forbidden graphs. But we hope that the introduced method will produce more results: although the MIP is hard to solve, it may well be possible that, for example, some manageable relaxation of it gives good lower or upper bounds.

Complete proofs of all above results can be found in [3].

References

[1] P. Erdős, R. J. Faudree, C. C. Rousseau, and R. H. Schelp, *The size Ramsey number*, Period. Math. Hung. **9** (1978), 145–161.

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