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Weakly Saturated Hypergraphs and Exterior Algebra

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Given an r-graph F, an r-graph G is called weakly F-saturated if the edges missing from G can be added, one at a time, in some order, each extra edge creating a new copy of F. Let w-sat(n, F) be the minimal size of a weakly F-saturated graph of order n. We compute the w-sat function for a wide class of r-graphs called *pyramids*: these include many examples for which the w-sat function was known, as well as many new examples, such as the computation of w-sat$(n, K_s + \overline{K_t})$, and enable us to prove a conjecture of Tuza.

Our main technique, developed from ideas of Kalai, is based on matroids derived from exterior algebra. We prove that if it succeeds for some graphs then the same is true for the 'cones' and 'joins' of such graphs, so that the w-sat function can be computed for many graphs that are built up from certain elementary graphs by these operations.

1. Introduction

An *r-graph* F is a pair $(V(F), E(F))$ (*vertices* and *edges*), where

$$E(F) \subset V(F)^{(r)} = \{A \subset V(F) : |A| = r\}.$$

By analogy with the case $r = 2$, the *size* of F is $e(F) = |E(F)|$, the *order* is $v(G) = |V(F)|$, and in the obvious way we define the notions of an *isomorphism*, a *subgraph*, the *complementary graph* $\overline{F}$, *etc.*

Given a family $\mathscr{F}$ of r-graphs, an r-graph G of order n is called *weakly $\mathscr{F}$-saturated* (denoted by $G \in$ w-SAT$(n, \mathscr{F})$) if we can consecutively add all missing edges to G so that, every time we add an edge, a new *F-subgraph* (a subgraph isomorphic to F), for some $F \in \mathscr{F}$, appears. In other words, there exists an ordering $E(\overline{G}) = \{E_1, \ldots, E_i\}$ such that, for every $j \leqslant i$, there is an $F \in \mathscr{F}$ such that the graph $G + E_1 + \cdots + E_j$ has an F-subgraph

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containing E_j as an edge. We call the corresponding ordering of $E(\overline{G})$ $\mathscr{F}$*-proper*. The principal problem we consider is the determination of

$$\text{w-sat}(n, \mathscr{F}) = \min\{e(G) : G \in \text{w-SAT}(n, \mathscr{F})\}.$$

Note that we do not require that G is $\mathscr{F}$-free, as this does not affect w-sat$(n, \mathscr{F})$. If $\mathscr{F}$ consists of a single forbidden graph F, we write w-SAT(n, F) for w-SAT$(n, \{F\})$, *etc*. The related notion of (strong) saturation (studied in, *e.g.*, [2, 9, 5, 12]) is not considered in this work.

The w-sat function was introduced by Bollobás [3]. Its computation seems a difficult task and few results are known. Usually, given $\mathscr{F}$, it is easy to find a correct example of $G \in$ w-SAT$(n, \mathscr{F})$ but it is hard to prove that G is indeed minimum. There does not seem to be any systematic approach to the latter task.

Here we present the *E-proof*, a sufficient condition for $G \in$ w-SAT$(n, \mathscr{F})$ being extremal. This is a deterministic procedure which uses matroids constructed via exterior algebra. (Examples when other matroids apply can be found in [7, 13].) The links with matroid theory are not surprising: loosely speaking, an $\mathscr{F}$-proper addition of edges corresponds to closure, and the notion of a minimum weakly saturated graph resembles that of a base. Applying the E-proof and other related methods (e/e$'$/*etc*.-proofs), we obtain various results as follows.

Let sequences $\mathbf{r} = (r_1, \ldots, r_t)$ of nonnegative integers and $S_1, \ldots, S_t$ of disjoint sets of sizes $\mathbf{s} = (s_1, \ldots, s_t)$ be given. We define $[t] = \{1, ..., t\}$ and, for $I \subset [t]$, we use shortcuts such as $r_I = \sum_{i \in I} r_i$ and $S_I = \cup_{i \in I} S_i$; also, we assume $r_0 = 0$, $S_0 = \emptyset$, *etc*. Then the *pyramid* $P = P(\mathbf{s}, \mathbf{r})$ is the r-graph, $r = r_{[t]}$, on $S = S_{[t]}$ such that $E \in S^{(r)}$ is an edge of P if and only if, for every $i \in [t]$, we have $|E \cap S_{[i]}| \geqslant r_{[i]}$.

Perhaps our most important result is the exact computation (via the E-proof) of the w-sat function for pyramids.

Theorem 1.1. *Suppose we are given two non-empty sequences* $\mathbf{s} = (s_1, \ldots, s_t)$ *and* $\mathbf{r} = (r_1, \ldots, r_t)$ *of integers such that* $s_i \geqslant r_i \geqslant 1$ *for* $i \in [t]$. *Then*

$$\text{w-sat}(n, P(\mathbf{s}; \mathbf{r})) = \sum_{\mathbf{r}'} \binom{n - s_{[t]} + r_t}{r'_{t+1}} \prod_{i \in [t]} \binom{s_i + r_{i-1} - r_i}{r'_i}, \quad n \geqslant s_{[t]}, \tag{1.1}$$

where the summation is taken over all sequences of nonnegative integers $\mathbf{r}' = (r'_1, \ldots, r'_{t+1})$ *such that* $r'_{[t+1]} = r_{[t]}$ *and, for some* $i \in [t]$, $r'_{[i]} > r_{[i-1]}$.

The notion of pyramid is quite general. For example, for $t = 1$ we have $P(s; r) = K_s^r$ and we obtain the formula

$$\text{w-sat}(n, K_s^r) = \binom{n}{r} - \binom{n - s + r}{r}, \quad n \geqslant s \geqslant r \geqslant 1, \tag{1.2}$$

conjectured by Bollobás [3] and proved by Lovász [10], Frankl [6], and Kalai [7]. On the other hand, we obtain new results even for 2-graphs: $P(s, t; 1, 1) = K_s^2 + \overline{K_t^2}$ and we have

$$\text{w-sat}(n, K_s^2 + \overline{K_t^2}) = (s - 1)n - \binom{s}{2} + \binom{t}{2}, \quad n \geqslant s + t, \; s, t \geqslant 1.$$

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Also, $P(r-l+1,l;r-l+1,l-1)$ is the only graph in $\mathscr{H}_r(r+1,l)$, where the *uniform family* $\mathscr{H}_r(m,l)$ consists of all r-graphs on m vertices with l edges. Hence, (1.1) implies the following result conjectured by Tuza [14, Conjecture 7].

$$\text{w-sat}(n,\mathscr{H}_r(r+1,l)) = \binom{n-r+l-2}{l-2}, \quad n \geqslant r+1 \geqslant l \geqslant 2.$$

The case $l=3$ of Tuza's conjecture was proved by Erdős, Füredi and Tuza [5].

An advantage of the E-proof is that it might help in guessing answers. For example, a correct example of $G \in \text{w-sat}(n, P(\mathbf{s};\mathbf{r}))$ was obtained by the author by first guessing the more transparent construction (3.2) which plays the key role in the proof.

The *cone* $\text{cn}(F)$ of an r-graph F is obtained by adding to F a new vertex v and all r-edges containing v:

$$E(\text{cn}(F)) = E(F) \cup \left\{ D \cup \{v\} : D \in V(F)^{(r-1)} \right\}.$$

For a family $\mathscr{F}$ of graphs we define $\text{cn}(\mathscr{F}) = \{\text{cn}(F) : F \in \mathscr{F}\}$. Our more general Theorem 4.4 implies that the pair $\text{cn}(G) \in \text{w-SAT}(n, \text{cn}(\mathscr{F}))$ admits E/e/e′-proof provided the pair $G \in \text{w-SAT}(n,\mathscr{F})$ does likewise and any $r-1$ vertices in any $F \in \mathscr{F}$ are covered by at least one edge. Of course, this result is only useful if we know many graphs admitting E/e/e′-proof. Various examples of such graphs are indicated in this work. For 2-graphs, they include complete graphs, stars, odd cycles, initial colex-segments of $[n]^{(2)}$, disjoint edges (more generally, almost every forest or tree). Therefore, we are able to compute the w-sat function for $\text{cn}^l(F) = K_l^2 + F$, where F is any of these graphs. Also, (1.2) can easily be deduced from our cone-result by observing that $\text{cn}(K_s^r) = K_{s+1}^r$.

In Section 5 we define the join-operator and prove that it 'preserves' E/e/r-proofs. (See Theorem 5.3 for the precise statement.) For example, Alon's [1] computation of the w-sat function for joins of complete graphs (another proof is presented by Yu [15]) is a special instance of our result.

Unfortunately, E/e/e′-proofs are not easy to handle and there are many concrete examples for which we could not make our methods work. Besides, E/e/e′-proofs do not provide the immediate characterization of all minimum graphs. But we hope that they will lead to further results.

2. Proof techniques

In Section 2.2 we associate with every r-graph G its exterior matroid $\mathscr{E}_G$. As this definition relies on exterior algebra, we provide some background in Section 2.1; for a comprehensive introduction to the topic the reader can consult Marcus [11], for example. Then, in Section 2.3, we describe how we apply these notions to w-sat-type problems.

2.1. Exterior algebra

Let V be an n-dimensional real vector space with basis $\mathbf{e} = \{e_1,\ldots,e_n\}$. Its *exterior algebra* $\bigwedge V$ is a 2^n-dimensional vector space with the formal basis $(e_A)_{A\subset[n]}$. (We identify e_i with $e_{\{i\}}$, and $e_\emptyset$ with the scalar $1 \in \mathbb{R}$.) It comes equipped with the associative bilinear

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$\wedge$-product which is completely determined by $e_i \wedge e_j = -e_j \wedge e_i$, $i, j \in [n]$, and

$$e_{v_1} \wedge \cdots \wedge e_{v_k} = e_{\{v_1,\ldots,v_k\}}, \quad 1 \leqslant v_1 < \cdots < v_k \leqslant n.$$

Let $(e_A^*)_{A \subset [n]}$ be the dual basis of $(e_A)_{A \subset [n]}$. We naturally identify $\bigwedge(V^*)$ and $(\bigwedge V)^*$ so that $e_{v_1}^* \wedge \cdots \wedge e_{v_k}^*$ corresponds to $e_{\{v_1,\ldots,v_k\}}^*$, $1 \leqslant v_1 < \cdots < v_k \leqslant n$.

Let $\mathbf{f} = (f_1, \ldots, f_n)$ be another basis of V; in the obvious way we define f_A, f_A^*, *etc.* By $M = (\alpha_{ij})_{i,j \in [n]}$ we denote the $(n \times n)$-matrix satisfying $\mathbf{f}^* = M\mathbf{e}^*$, that is,

$$f_i^* = \alpha_{i1} e_1^* + \cdots + \alpha_{in} e_n^*, \qquad i \in [n].$$

Assume that $\mathbf{f}$ is in the *generic* position with respect to $\mathbf{e}$, that is, the entries of M are n^2 transcendentals algebraically independent over the rationals. Any equation we will consider can be reduced to the form $P = 0$ for some polynomial P in the αs with integer coefficients, and we agree that the statement is true if and only if P is the zero polynomial.

Let $\bigwedge^i V$ be the subspace of $\bigwedge V$ spanned by $(e_A)_{A \in [n]^{(i)}}$. For $h \in \bigwedge V$ its *support* is defined by $\mathrm{supp}(h) = \{A \subset [n] : e_A^*(h) \neq 0\}$. If we take the support in the basis $\mathbf{f}$, we emphasize this by adding a subscript: $\mathrm{supp}_{\mathbf{f}}(h) = \{A \subset [n] : f_A^*(h) \neq 0\}$. For $g^* \in \bigwedge V^*$, $h \in \bigwedge V$ denote $\langle g^*, h\rangle = g^*(h)$ and define the *left interior product* $g^* \llcorner h \in \bigwedge V$ by

$$\langle u^*, g^* \llcorner h\rangle = \langle u^* \wedge g^*, h\rangle, \quad \text{for all } u^* \in \textstyle\bigwedge V^*.$$

One can easily check that $\llcorner$ is a bilinear function such that $u^* \llcorner (g^* \llcorner h) = (u^* \wedge g^*) \llcorner h$ and, for the basis vectors, we have:

$$e_A^* \llcorner e_B = \begin{cases} \pm e_{B \setminus A}, & \text{if } A \subset B, \\ 0, & \text{if } A \not\subset B. \end{cases}$$

(The actual signs of ± 1-coefficients do not interest us.) Note that the cancellation $(g^* \wedge e_A^*) \llcorner (h \wedge e_A) = g^* \llcorner h$, which is not generally correct, can be applied if, for example, each $B \in \mathrm{supp}(h)$ is disjoint from A.

2.2. Construction

Let us describe how to construct the *exterior matroid* $\mathscr{E}_G$ of an r-graph G of order n. This construction is not new: for instance, Kalai [8] used it to construct symmetric matroids with a given growth polynomial. Also, in the special case $G = P(k, n-k; 1, 1)$, the matroid $\mathscr{E}_G$ is exactly Kalai's [7] *k-hyperconnectivity matroid* on $[n]^{(2)}$, which was applied to w-sat-type problems. These two papers by Kalai were the starting points for our research on exterior matroids.

Let $V(G) = [n]$ and let $Z \subset \bigwedge^r V$ be defined by the following linear relations:

$$Z = \{h \in \textstyle\bigwedge^r V : f_E^* \llcorner h = 0 \text{ for all } E \in E(G)\}. \tag{2.1}$$

Clearly, $\dim Z = \binom{n}{r} - e(G)$ and, in fact, Z is spanned by $\{f_E : E \in E(\overline{G})\}$.

We define the *exterior matroid* $\mathscr{E}_G$ on $[n]^{(r)}$ so that an r-graph F on $[n]$ is dependent if, for some coefficients c_E (not all zero), we have $\sum_{E \in E(F)} c_E e_E \in Z$, that is,

$$\sum_{E \in E(F)} c_E f_D^* \llcorner e_E = 0, \qquad D \in E(G). \tag{2.2}$$

By $M(G, F)$ we denote the $(e(G) \times e(F))$-matrix corresponding to (2.2). The columns of

$M(G, [n]^{(r)})$ provide a representation of $\mathscr{E}_G$. Note that the matroid $\mathscr{E}_G$ does not depend on the choice of generic $\mathbf{f}$. Also, $\mathscr{E}_G$ is a *symmetric* matroid, that is, for any permutation $\sigma : [n] \to [n]$, $A \subset [n]^{(r)}$ is $\mathscr{E}_G$-independent if and only if $\sigma'(A)$ is, where σ' is the induced action on $[n]^{(r)}$. Therefore, we can apply the notion of $\mathscr{E}_G$-dependence to an r-graph F with any vertex set. (If $v(F) > v(G)$, we add isolated vertices to G.)

The rank of $\mathscr{E}_G$ is $\mathrm{codim}(Z) = e(G)$. It is easy to show that G is a base of $\mathscr{E}_G$. Indeed, the determinant of $M(G, G)$ is a polynomial in the αs which assumes value 1 when M (and then $M(G, G)$) is the identity matrix. Therefore, the determinant is nonzero for generic M, which proves the claim.

2.3. Our approach

The following observation, due to Kalai [7], is crucial to our work. Suppose that $\mathscr{F}$ is an r-graph family and $\mathscr{M}$ is a matroid on $[n]^{(r)}$ such that, for every $F \in \mathscr{F}$ and for every embedding $V(F) \subset [n]$, the set $E(F) \subset [n]^{(r)}$ is an $\mathscr{M}$-*cycle*, that is, every edge $E \in E(F)$ is dependent on $E(F)\setminus\{E\}$. Let $G \in \text{w-SAT}(n, \mathscr{F})$ and let $E_1, \ldots, E_k$ be an $\mathscr{F}$-proper ordering of $E(\overline{G})$. Then, for every $i \in [k]$, there is an F-subgraph of $G_i = G + E_1 + \cdots + E_i$ which contains E_i, for some $F \in \mathscr{F}$. Thus, E_i lies in the $\mathscr{M}$-closure of G_{i-1}, which inductively implies that G spans $[n]^{(r)}$ in $\mathscr{M}$. Hence,

$$\text{w-sat}(n, \mathscr{F}) \geqslant R_{\mathscr{M}}([n]^{(r)}). \tag{2.3}$$

In this case we say that we can *m-prove* the inequality (2.3). If $\mathscr{M}$ is an exterior matroid or a representable matroid, then (2.3) is said to be *e-proved* or *r-proved*, correspondingly. If the lower bound in (2.3) is sharp, then we say that $\mathscr{F}$ admits an *m-proof* for n. In the obvious way we define an *e-proof* and an *r-proof*. (We use the indefinite article to emphasize that $\mathscr{M}$ is not given and, if a suitable $\mathscr{M}$ exists, it might be not unique.)

Furthermore, define

$$D_{\mathscr{M}}(\mathscr{F}) = \min\{D_{\mathscr{M}}(F) : F \in \mathscr{F}\},$$

where $D_{\mathscr{M}}(F) = \min_{F \subset [n]} (e(F) - R_{\mathscr{M}}(E(F)))$. The first edge E_1 added to G creates some forbidden $F \subset [n]$; clearly, $E(F) \setminus \{E\} \subset E(G)$. Therefore, there are $D_{\mathscr{M}}(F) - 1$ edges in G which are dependent on the remaining edges and we can improve (2.3) as follows:

$$\text{w-sat}(n, \mathscr{F}) \geqslant R_{\mathscr{M}}([n]^{(r)}) + D_{\mathscr{M}}(\mathscr{F}) - 1. \tag{2.4}$$

We say that (2.4) is *m′-proved*. If $\mathscr{M}$ is an exterior or representable matroid, then we respectively *e′-prove* or *r′-prove* (2.4). If (2.4) is sharp, then we obtain an *m′/e′/r′-proof*, correspondingly.

As we have already mentioned, it is usually easy to find a correct example of $G \in \text{w-SAT}(n, \mathscr{F})$ but hard to prove its extremality. Even if there exists an m-proof, it is not at all obvious how to search for a suitable matroid. However, in our approach we suggest considering the exterior matroid of G as a candidate for $\mathscr{M}$. If each graph in $\mathscr{F}$ is an $\mathscr{E}_G$-cycle, then we conclude by (2.3) that $\text{w-sat}(n, \mathscr{F}) \geqslant e(G)$, that is, G is extremal. In this case we say that the pair $(\mathscr{F}, G)$ admits the *E-proof*. Hence, the E-proof can be viewed as a sufficient criterion for $G \in \text{w-SAT}(n, F)$ to be of the minimal size that does not require any choice of a matroid. When G is understood from the context, we simply say '$\mathscr{F}$ admits the E-proof'.

It is easy to see that an r-graph F is an $\mathscr{E}_G$-cycle if and only if there is an $h \in Z$ with $\mathrm{supp}(h) = E(F)$. To verify this condition we have to find a solution $(c_E)_{E\in E(F)}$ with all entries nonzero of the system (2.2).

Let us prove one trivial lemma which, when combined with the results of Sections 4 and 5, has nontrivial consequences.

Lemma 2.1. *Let $K = lK_r^r$ be the union of l disjoint r-edges. Then $\mathscr{E}_K$ is the* uniform matroid *of rank l, that is, an r-graph F is independent in $\mathscr{E}_K$ if and only if $e(F) \leqslant l$.*

In particular, for any family $\mathscr{F}$ of r-graphs and for any n with $\binom{n}{r} \geqslant l$, we can e-prove that w-sat$(n,\mathscr{F}) \geqslant l$*, where $l = \min\{e(F) : F \in \mathscr{F}\} - 1$.*

Proof. We show, by induction on l, that any r-graph H of size l is $\mathscr{E}_K$-independent. Assume that $E = [r]$ is an edge in both these graphs. One can see that

$$\det(M(K,H)) = \pm\alpha_{11}\cdots\alpha_{rr}\det(M(K',H')) + (\text{other terms}),$$

where H' and K' are obtained, respectively, from H and K by removing E, and none of the 'other terms' contains $\alpha_{11}\cdots\alpha_{rr}$ as a factor. By induction, we conclude that $\det(M(K,H)) \neq 0$, and all claims follow. □

3. Specific classes

Here we obtain various results for some particular forbidden families.

3.1. Pyramids

Here we calculate w-sat$(n, P(\mathbf{s},\mathbf{r}))$ by showing that pyramids admit the E-proof. Note that we obtain the exact answer for *all feasible* values of the parameters n, $\mathbf{r}$ and $\mathbf{s}$.

Let $P = P(\mathbf{s},\mathbf{r})$. Assume $s_i \geqslant r_i \geqslant 1$ for $i \in [t]$ as it is not hard to see that any non-empty pyramid has such a representation.

Let us, for any $n \geqslant s = s_{[t]}$, provide a construction of $G \in$ w-SAT(n,P). Partition $[n] = A_1 \cup \cdots \cup A_{t+1}$ so that $a_i = |A_i| = s_i + r_{i-1} - r_i$, $i \in [t]$; thus

$$a_{t+1} = |A_{t+1}| = n - \sum_{i=1}^{t}(s_i + r_{i-1} - r_i) = n - s + r_t.$$

We also assume that our partition is *consecutive*, that is, in $[n]$, any element of A_i comes before any element of A_j whenever $i < j$. Let $E \in [n]^{(r)}$ be an edge of G if and only if for some $i \in [t]$ we have $|E \cap A_{[i]}| > r_{[i-1]}$. Equivalently, the complement of G is isomorphic to $P(a_{t+1},\ldots,a_1; r_t,\ldots,r_1,0)$, so, for example, any r-tuple intersecting A_1 is in $E(G)$.

Lemma 3.1. $G \in$ w-SAT(n,P).

Proof. Order the missing edges in any way so that the sequences

$$\left(|A_{[1]} \cap E|, \ldots, |A_{[t+1]} \cap E|\right), \quad E \in E(\overline{G}),$$

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are non-increasing in the lexicographic order. (Thus, we start with $(0, r_1, \ldots, r_t)$ and end with $(0, \ldots, 0, r)$.) Let us show that this ordering is P-proper. Consider the moment when we add some edge $E \in E(\overline{G})$. Let $E_i = E \cap A_{i+1}$, $i \in [t]$. Also, let $E = R_1 \cup \cdots \cup R_t$ and $[n] \setminus E = T_1 \cup \cdots \cup T_{t+1}$ be the consecutive partitions with $|R_i| = r_i$ and $|T_i| = s_i - r_i$, $i \in [t]$.

Let us show that $E_{[i]} \subset R_{[i]}$ and $T_{[i]} \subset A_{[i]} \setminus E_{[i-1]}$, $i \in [t]$. As all partitions in question are consecutive, it is enough to verify the sizes. By the definition of G we have $|E_{[i]}| = |E \cap A_{[i+1]}| \leqslant r_{[i]}$. Also,

$$|A_{[i]} \setminus E_{[i-1]}| \geqslant |A_{[i]}| - r_{[i-1]} = \sum_{j=1}^{i} (s_j + r_{j-1} - r_j) - r_{[i-1]} = |T_{[i]}|,$$

and the claim follows.

Let $S_i = T_i \cup R_i$, $i \in [t]$. We claim that E creates a forbidden subgraph P on the set $S = S_{[t]}$. For every $i \in [t]$ we have $|E \cap S_i| = |R_i| = r_i$, so $E \in E(P)$.

Suppose, on the contrary, that there exists $D \in E(P)$ coming after E. Let us show by induction on i that, for every $i \in [0, t]$, we have

$$D \cap S_{[i]} = E \cap S_{[i]} \quad \text{and} \quad D \cap A_{[i+1]} = E \cap A_{[i+1]}, \tag{3.1}$$

which would be a contradiction to the assumption $D \neq E$. As $D, E \in E(\overline{G})$ are disjoint from A_1, the claim is true for $i = 0$. Let $i \in [t]$. As $T_{[i]} \subset A_{[i]}$, we conclude, by the inductive assumption, that $D \cap T_{[i]} = E \cap T_{[i]} = \emptyset$. As $S_{[i]} = T_{[i]} \cup R_{[i]}$, we have $D \cap S_{[i]} \subset R_{[i]}$. On the other hand, $D \in E(P)$, so $|D \cap S_{[i]}| \geqslant r_{[i]}$, which implies

$$D \cap S_{[i]} = R_{[i]} = E \cap S_{[i]},$$

and the first part of (3.1) is proved. Now,

$$D \cap A_{[i+1]} \supset R_{[i]} \cap A_{[i+1]} \supset E_{[i]} \cap A_{[i+1]}.$$

By induction, $D \cap A_{[i]} = E \cap A_{[i]}$ and, as D was added later than E, we must have $|D \cap A_{[i+1]}| \leqslant |E \cap A_{[i+1]}|$, which proves (3.1) completely. □

Theorem 3.2. *The pair (P, G) admits the E-proof.*

Proof. We have to show that P is an $\mathscr{E}_G$-cycle. Let us consider

$$h = h_1 \wedge \cdots \wedge h_t, \quad \text{where} \quad h_i = f^*_{A_{[i]}} \llcorner e_{S_{[i]}} \in \textstyle\bigwedge^{r_i} V, \quad i \in [t]. \tag{3.2}$$

Each $E \in \text{supp}(h)$ is of the form $E'_1 \cup \cdots \cup E'_t$, for some $E'_i \in \text{supp}(h_i)$, $i \in [t]$. Clearly, $|E'_i| = r_i$ and $E'_i \subset S_{[i]}$. Therefore, $|E \cap S_{[i]}| \geqslant |E'_{[i]}| = r_{[i]}$, so $\text{supp}(h) \subset E(P)$. Similarly, $\text{supp}_{\mathbf{f}}(h_i)$ lives within $A_{[i+1,t+1]}$, $i \in [t]$, which implies that $\text{supp}_{\mathbf{f}}(h) \subset E(\overline{G})$.

So, to prove the theorem, it is enough to show that for any $E \in E(P)$ we have $P_E = \langle e^*_E, h \rangle \neq 0$. To do so, we can assume that S is an initial segment in $[n]$ and every element of S_i comes before every element of S_j whenever $i < j$. Furthermore, we can assume that $E_i = E \cap S_i$ is a final segment of S_i. Note that $A_{[i]} \subset S_{[i]} \subset A_{[i+1]}$, and $R_i = S_{[i]} \setminus A_{[i]}$ consists of the last r_i elements of S_i, $i \in [t]$. Clearly, $|E| = |R|$, where $R = R_{[t]}$, so let $g : E \setminus R \to R \setminus E$ be the order-preserving bijection.

As P_E is a polynomial in the αs, to show that $P_E \neq 0$, it is enough to demonstrate a particular example of the αs (or $\mathbf{f}^*$) such that $P_E \neq 0$. Define

$$f_x^* = \begin{cases} e_x^* + e_{g(x)}^*, & x \in E \setminus R, \\ e_x^*, & \text{otherwise.} \end{cases} \tag{3.3}$$

Let $i \in [t]$. Denote $W_i = A_{[i]} \setminus (E \setminus R)$ and

$$X_i = \{x \in A_{[i]} \setminus W_i : g(x) \in A_{[i]}\},$$
$$Y_i = \{x \in A_{[i]} \setminus W_i : g(x) \notin S_{[i]}\},$$
$$Z_i = \{x \in A_{[i]} \setminus W_i : g(x) \in S_{[i]} \setminus A_{[i]}\}.$$

As $A_{[i]} \subset S_{[i]}$, we have a partition $A_{[i]} = W_i \cup X_i \cup Y_i \cup Z_i$. As $f_x^* = e_x^*$ for $x \in W_i$ and $g(x) \in W_i$ for $x \in X_i$, we have

$$f_{A_{[i]}}^* = \pm e_{W_i}^* \wedge f_{X_i \cup Y_i \cup Z_i}^* = \pm e_{W_i \setminus g(X_i)}^* \wedge \big(\textstyle\bigwedge_{x \in X_i} (e_x^* + e_{g(x)}^*) \wedge e_{g(x)}^*\big) \wedge f_{Y_i \cup Z_i}^*$$
$$= \pm e_{W_i \setminus g(X_i)}^* \wedge \big(\textstyle\bigwedge_{x \in X_i} e_x^* \wedge e_{g(x)}^*\big) \wedge f_{Y_i \cup Z_i}^* = \pm e_{W_i \cup X_i}^* \wedge f_{Y_i \cup Z_i}^*.$$

Next, $g(Y_i) \cap S_{[i]} = \emptyset$ and $g(Z_i) \subset S_{[i]} \setminus A_{[i]} = R_i$. Hence

$$h_i = \pm\big(e_{W_i \cup X_i}^* \wedge \big(\textstyle\bigwedge_{x \in Y_i} (e_x^* + e_{g(x)}^*)\big) \wedge f_{Z_i}^*\big) \llcorner e_{S_{[i]}}$$
$$= \pm \left(e_{W_i \cup X_i \cup Y_i}^* \wedge f_{Z_i}^*\right) \llcorner e_{S_{[i]}} = \pm f_{Z_i}^* \llcorner e_{Z_i \cup R_i}.$$

For $i \in [t]$ we have $|E_{[i-1]}| \geqslant |R_{[i-1]}|$, and one of E_i and R_i is a subset of the other, so, for each $x \in E_i \setminus R$, $g(x)$ lies in $R_j = S_{[j]} \setminus A_{[j]}$ and $x \in Z_j$ for some $j \in [i+1, t]$. Therefore, $Z_{[t]} = E \setminus R$.

When we compute $P_E = \pm\langle e_E^*, \wedge_{i \in [t]} (f_{Z_i}^* \llcorner e_{Z_i \cup R_i})\rangle$ by expanding further each h_i in the $\mathbf{e}$-basis, we obtain h as a sum of terms each of the form e_D, for some $D \in [n]^{(r)}$. By definition, $\langle e_E^*, e_D\rangle = 0$ unless $E = D$. Consider some $x \in Z_i$. As $x \notin R$ and $Z_1, \ldots, Z_t$ are disjoint, no element of $\mathrm{supp}(h_j)$ can contain x unless $j = i$. Computing h_i, we have

$$h_i = \pm\big(f_{Z_i \setminus \{x\}}^* \wedge e_{g(x)}^*\big) \llcorner e_{Z_i \cup R_i} \pm \big(f_{Z_i \setminus \{x\}}^* \wedge e_x^*\big) \llcorner e_{Z_i \cup R_i},$$

and no element in the $\mathbf{e}$-support of the second summand can contain x. Thus we can harmlessly replace f_x^* by $e_{g(x)}^*$. (Clearly, this does not affect h_j for $j \neq i$.) Now,

$$P_E = \pm\big\langle e_E^*, \wedge_{i \in [t]} \big(e_{g(Z_i)}^* \llcorner e_{Z_i \cup R_i}\big)\big\rangle = \pm\big\langle e_E^*, e_{Z_{[t]} \cup R_{[t]} \setminus g(Z_{[t]})}\big\rangle = \pm\langle e_E^*, e_E\rangle = \pm 1.$$

Thus P_E is nonzero and the theorem is proved. □

Now the promised Theorem 1.1 clearly follows.

3.2. Uniform families

Let us consider uniform families studied for example by Erdős, Füredi and Tuza [5]. By definition, $G \in \text{w-SAT}(n, \mathscr{H}_r(m,l))$ if we can consecutively add all missing edges to G so that every time we create a new subgraph with at most m vertices and at least l edges. In [13] we present a general construction of a weakly $\mathscr{H}_r(m,l)$-saturated graph of order n, which we conjecture to be extremal, and prove some partial cases of our conjecture. Here we obtain further results confirming it: we compute $\text{w-sat}(n, \mathscr{H}_r(m,l))$ asymptotically for $l > \binom{m}{r} - m + r$ and exactly for $r = 2$.

First, we need one simple preliminary result.

Lemma 3.3. *Let G be an r-graph of order n and size at least $\binom{n}{r} - n + m$, where $n > m > r \geqslant 2$. Then every $E \in E(G)$ is contained in a complete subgraph of order m.*

Proof. Given $E \in E(G)$, remove from each missing edge one (arbitrary) vertex not belonging to E. We are left with at least m vertices spanning a complete subgraph which contains E. □

Theorem 3.4. *Let $l = \binom{m}{r} - k$, $c = m - k - r$, and $\mathscr{H} = \mathscr{H}_r(m, l)$. If $m > r + k$, then*

$$\text{w-sat}(n, \mathscr{H}) = c\binom{n}{r-1} + O(n^{r-2}). \tag{3.4}$$

Furthermore, if $r = 2$, then we have an e′-proof that

$$\text{w-sat}(n, \mathscr{H}_2(m, l)) = c(n - m) + l - 1, \quad n \geqslant m. \tag{3.5}$$

Proof. The r-graph G on $[n]$ consisting of all edges intersecting $[m - r + 1]$ in at least 2 vertices plus all edges intersecting $[c]$ is weakly $\mathscr{H}$-saturated: we can add first all missing edges intersecting $[m - r + 1]$ and then the remaining ones in any order. Computing $e(G)$ we obtain the upper bounds in (3.4) and (3.5).

On the other hand, in any $F \in \mathscr{H}$, any edge lies within a K^r_{m-k}-subgraph by Lemma 3.3. However, by Theorem 3.2, K^r_{m-k} is a cycle in $\mathscr{E}_P$, the exterior matroid of $P = P(c, n - c; 1, r - 1)$, so each $F \in \mathscr{H}$ is an $\mathscr{E}_P$-cycle. By (2.3), $\text{w-sat}(n, \mathscr{H}) \geqslant R_{\mathscr{E}_P}([n]^{(r)}) = e(P)$, which e-proves the required lower bound in (3.4).

Finally, let us e′-prove the lower bound in (3.5) for $r = 2$. Let $F \in \mathscr{H}$. As F has m vertices,

$$R_{\mathscr{E}_P}(F) \leqslant R_{\mathscr{E}_P}(K^2_m) \leqslant e(P(c, m - c; 1, 1)) = cm - \binom{c+1}{2}.$$

(The second inequality is true because $P(c, m - c; 1, 1) \in \text{w-SAT}(m, K^2_{m-k})$ and K^2_{m-k} is an $\mathscr{E}_P$-cycle.) Therefore $D_{\mathscr{E}_P}(F) \geqslant l - cm + \binom{c+1}{2}$ and we obtain

$$\text{w-sat}(n, \mathscr{H}) \geqslant R_{\mathscr{E}_P}(K_n) + D_{\mathscr{E}_P}(\mathscr{F}) - 1 \geqslant c(n - m) + l - 1. \qquad \square$$

Remark. **(1)** It is trivial that, if $1 \leqslant l \leqslant \binom{m-1}{r} + 1$, then $\text{w-sat}(n, \mathscr{H}_r(m, l)) = l - 1$, $n \geqslant m$, which settles all remaining cases of Theorem 3.4 for $r = 2$.

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(2) In fact, the graph G constructed in the proof is weakly F-saturated, where $E(F)$ consists of the first l elements of $[m]^{(r)}$ in the colex order. So, Theorem 3.4 remains true if F is the only member of $\mathscr{F}$; this covers all possible cases for $r = 2$ except the trivial case $l = \binom{m-1}{2} + 1$.

3.3. Forests

By Lemma 2.1 we know that, if $\text{w-sat}(n, T) = e(T) - 1$, then T admits an e-proof for n. In fact, we can show that we have the E-proof if T is a forest.

Lemma 3.5. *Let F and H be any forests with $e(F) \leqslant e(H)$. Then F is independent in $\mathscr{E}_H$. In particular, if $G \in \text{w-SAT}(n, F)$ and $e(G) = e(F) - 1$, then the pair (F, G) admits the E-proof.*

Proof. We use induction on $l = e(H)$. It is enough to prove the claim for $e(F) = e(H)$. Assume that 1 is an end-vertex incident to the edge $E = \{1, 2\}$ in both F and H. Clearly,

$$\det(M(H, F)) = \pm\alpha_{1,1}\alpha_{2,2} \det(M(H - E, F - E)) + (\alpha_{1,1}\text{-free polynomial}).$$

By induction we conclude that $\det(M(H, F)) \neq 0$, which proves the lemma. □

If T contains, for example, vertices a, b, c of degrees $1, 1, 2$, respectively, such that $\{a, c\}, \{c, d\}, \{b, d\} \in E(T)$, for some vertex d, then adding the edges $\{d, x\}$ and $\{x, y\}$ to T, for any $x, y \notin V(T)$, we create each time a new graph isomorphic to T; this implies that $\text{w-sat}(n, T) = e(T) - 1$ with possible exceptions for some $n \leqslant 2m$, $m = v(T)$. Generating a random tree by, for example, taking all m^{m-2} vertex-labelled trees to be equiprobable, one can show that almost every tree contains the above '*abc*-configuration' and therefore admits the E-proof.

3.4. Complete bipartite graphs

Kalai [7] proved (in fact, e-proved) that $\text{w-sat}(n, K_{s,s}) = (s-1)n - \binom{s-1}{2}$ for $n \geqslant 2s + 1$. The value of $\text{w-sat}(n, K_{s,t})$, $t > s$, is in general unknown. We can show that $K_{s,t}$ is an $\mathscr{E}_G$-cycle, where $E(G)$ consists of elements of $[n]^{(2)}$ intersecting $A = [s-1]$ or lying within $B = [t-1]$. (This can be done by considering $h = (f_A^* \llcorner e_{[s]}) \wedge (f_B^* \llcorner e_{[s+1,s+t]})$.) This gives a better lower bound than the trivial inequality $\text{w-sat}(n, K_{s,t}) \geqslant \text{w-sat}(n, K_{s,s})$ for any $s \geqslant 2$ when t is sufficiently large. However, we do not have the matching upper bound, so we do not provide any further details here.

3.5. Cycles

It is not hard to prove directly the following result on cycles.

Lemma 3.6. *Let $n > m$. If m is odd, then $\text{w-sat}(n, C_m) = n - 1$ and all extremal graphs are trees of order n and diameter at least $m - 1$. If m is even, then $\text{w-sat}(n, C_m) = n$ and all extremal graphs are trees of order n of diameter at least $m - 1$ plus an extra edge creating an odd cycle.*

If $n = m$, then $\text{w-sat}(n, C_m) = n$ *and all extremal graphs are obtained from a Hamiltonian cycle by adding an edge creating an odd cycle and removing any other edge.* □

In fact, any odd cycle C_m admits an e-proof for $n > m$. Indeed, consider $\mathscr{E} = \mathscr{E}_{P(1,n-1;1,1)}$. Any edge of C_m is $\mathscr{E}$-dependent on the remaining ones because the path with m edges is weakly K_3^2-saturated and K_3^2 is an $\mathscr{E}$-cycle by Theorem 3.2. Clearly, $R_{\mathscr{E}}([n]^{(2)}) = n - 1$, so the claim follows. (If restricted to $[n]^{(2)}$, $\mathscr{E}$ is the usual *cycle matroid*.)

Even cycles admit an r-proof. Indeed, let $\mathscr{M}$ be Doob's [4] *even-cycle matroid* on $[n]^{(2)}$ which can be represented by mapping $\{i, j\} \in [n]^{(2)}$ to $e_i + e_j$, for some basis $\{e_1, \ldots, e_n\}$ of a real vector space V. It is easy to show that every even cycle is an $\mathscr{M}$-circuit, and $R_{\mathscr{M}}([n]^{(2)}) = n$, which implies our claim.

Probably, even cycles do not admit an e-proof, but we cannot prove this.

3.6. Disjoint edges

The case when lK_r^r, the union of l disjoint r-edges, is forbidden is rather easy.

Lemma 3.7. *For $n > lr$ we have* $\text{w-sat}(n, lK_r^r) = l - 1$ *and* $(l-1)K_r^r$ *(plus isolated vertices) is the only extremal graph. Hence, by Lemma 2.1, we have the E-proof here.* □

4. Cones

The definition of the cone of an r-graph was given in the introduction. In this section we prove that cones 'preserve' E/e/g′-proofs.

Lemma 4.1. *Suppose that any $r - 1$ vertices of an r-graph F are covered by at least one edge. If F is an $\mathscr{E}_G$-cycle, for some r-graph G, then* $\text{cn}(F)$ *is an $\mathscr{E}_{\text{cn}(G)}$-cycle.*

Proof. Suppose first that $v(G) \geqslant v(F)$. Let $G' = \text{cn}(G)$, $V(G) = [n-1]$ and $V(G') = [n]$. Identify the vertices of G' with the basis $\{e_1, \ldots, e_n\}$ of a vector space V'. Let Z' be the subspace of $\bigwedge^r V'$ and let $\mathscr{E}_{G'}$ be the exterior matroid on $[n]^{(r)}$ corresponding to G'.

We may assume that $F' = \text{cn}(F)$ is embedded into $[n]$ so that $V(F') \setminus V(F) = \{n\}$. We have to show that $E(F')$ is a cycle in $\mathscr{E}_{G'}$, that is, we have to find $h' \in Z'$ such that $\text{supp}(h') = E(F')$. Define $g_n^* = f_n^*$ and

$$g_i^* = f_i^* - \frac{\alpha_{in}}{\alpha_{nn}} f_n^*, \qquad i = 1, \ldots, n-1. \tag{4.1}$$

Recall that $\mathbf{f}^*$ is a generic basis of $(V')^*$ and $\alpha_{ij} = f_i^*(e_j)$, so $g_i^*(e_n) = 0$, $i \in [n-1]$, and this is the main point of our definition. The matrix $N = (g_i^*(e_j))_{i,j\in[n-1]}$ is clearly a generic matrix for a generic choice of the αs.

As F is an $\mathscr{E}_G$-cycle, the system of linear equations

$$g_D^* \llcorner \left(\sum_{E \in E(F)} c_E e_E \right) = 0, \qquad D \in E(G), \tag{4.2}$$

with respect to the undeterminants $(c_E)_{E\in E(F)}$, has a solution with all cs being nonzero for generic $\mathbf{g}$ (which is the case for generic $\mathbf{f}$). Apply elementary matrix transforms to

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write the system (4.2) in a diagonal form. For the free variables choose $\beta_1, \dots, \beta_k$ which (together with the αs) are algebraically independent over the rationals, and compute the other variables, each being a rational function of the αs and βs.

Let $h = \sum_{E \in E(F)} c_E e_E$ and $h' = f_n^* \llcorner (h \wedge e_n)$. To complete the theorem it is enough to show that $h' \in Z'$ and $\mathrm{supp}(h') = E(F')$.

Let $D \in E(G')$. We want to show that $f_D^* \llcorner h' = 0$. This is clearly the case if $D \ni n$. If $D \not\ni n$, then

$$\langle f_D^*, h' \rangle = \left\langle \bigwedge\nolimits_{i \in D} \left(g_i^* + \tfrac{\alpha_{in}}{\alpha_{nn}} f_n^*\right), f_n^* \llcorner (h \wedge e_n) \right\rangle = \langle g_D^* \wedge f_n^*, h \wedge e_n \rangle = f_n^*(e_n) \langle g_D^*, h \rangle = 0,$$

where we used (4.1), the relations $g_i^*(e_n) = 0$, $i \in [n-1]$, and the definition of h. Therefore $h' \in Z'$.

Clearly, $\mathrm{supp}(h') \subset E(F')$. On the other hand, take any $E \in E(F')$. If $E \in E(F)$ then

$$\langle e_E^*, h' \rangle = \langle e_E^* \wedge f_n^*, h \wedge e_n \rangle = \langle e_E^*, h \rangle \cdot \langle f_n^*, e_n \rangle = c_E f_n^*(e_n) \neq 0,$$

because $n \notin E$. If $E \ni n$ then let $D_1, \dots, D_l$ be the edges of F containing $E' = E \setminus \{n\}$. By our assumption, $l > 0$. Let $D_i \setminus E = \{d_i\}$. Then

$$\begin{aligned} P_E &= \langle e_E^*, h' \rangle = \langle e_{E'}^* \wedge e_n^* \wedge f_n^*, h \wedge e_n \rangle = -\langle e_{E'}^* \wedge f_n^*, h \rangle \\ &= -\left\langle e_{E'}^* \wedge f_n^*, \sum_{E \in E(F)} c_E e_E \right\rangle = \sum_{i=1}^{l} \pm c_{D_i} \langle f_n^*, e_{d_i} \rangle = \sum_{i=1}^{l} \pm c_{D_i} \alpha_{n,d_i}. \end{aligned}$$

(The third equality is true as $\mathrm{supp}(h) = E(F) \subset [n-1]^{(r)}$.)

As every c_{D_i} is a rational function in the αs and βs so is P_E. To show that $P_E \neq 0$ for a generic $\mathbf{f}$, it is enough to demonstrate an example of $\mathbf{f}$ when $P_E \neq 0$. Let $\alpha_{in} = 0$, $i \in [n-1]$. Then system (4.2) reduces to

$$f_D^* \llcorner \left(\sum_{E \in E(F)} c_E e_E \right) = 0, \qquad D \in E(G). \tag{4.3}$$

By the algebraic independence of $(f_i^*(e_j))_{i,j \in [n-1]}$, if we perform the diagonalization for (4.3) in the same order as for (4.2), we will obtain the same set of free variables. Therefore, $(c_E)_{E \in E(F)}$ provides every solution for (4.3) when the βs range over the reals. Thus each c_E is nonzero (as F is an $\mathscr{E}_G$-cycle) and it can depend only on $f_i^*(e_j) = \alpha_{ij}$, $i, j \in [n-1]$, and the βs. Now it is obvious that $P_E = \sum_{i=1}^{l} c_{D_i} \alpha_{n,d_i}$ cannot be identically zero. This proves the lemma if $v(G) \geqslant v(F)$.

Otherwise, we can add $v(F) - v(G)$ isolated vertices to G to obtain H. By the above, $\mathrm{cn}(F)$ is a cycle in $\mathscr{E}_{\mathrm{cn}(H)}$, that is, each edge of $\mathrm{cn}(F)$ is dependent on the other edges. The latter claim is certainly true in $\mathscr{E}_{\mathrm{cn}(G)}$ which has more dependences than $\mathscr{E}_{\mathrm{cn}(H)}$ as $\mathrm{cn}(G) \subset \mathrm{cn}(H)$. □

Lemma 4.2. *If an r-graph F is independent in $\mathscr{E}_G$ and $v(F) \leqslant v(G)$, then $\mathrm{cn}(F)$ is independent in $\mathscr{E}_{\mathrm{cn}(G)}$.*

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Proof. We assume the same conventions as those appearing in the proof of Lemma 4.1 before (4.2). It is enough to prove our claim in the case $e(G) = e(F)$: if $e(G) > e(F)$ we can remove a G-edge with F still being $\mathscr{E}_G$-independent.

Let us show that the rank of $M'(G', F')$ is $e(F')$, where $M'(D, E) = \langle g^*_D, e_E \rangle$, $D \in E(G')$, $E \in E(F')$, which implies the lemma.

By our assumption, the square submatrix $M'(G, F) \subset M'(G', F')$ is nonsingular because the matrix N is generic. As $g^*_i(e_n) = 0$ for $i \in [n-1]$, we conclude that all entries of the submatrix $M'(G, F'')$ are zeros, where $E(F'') = E(F') \setminus E(F)$. Therefore, to prove the claim we have to show that the submatrix $M'(G'', F'')$ has the maximal possible rank $\binom{v(F)}{r-1}$, where $E(G'') = E(G') \setminus E(G)$.

For any $D' = D \cup \{n\} \in E(G'')$, $E' = E \cup \{n\} \in E(F'')$, we have

$$\langle g^*_{D'}, e_{E'} \rangle = g^*_n(e_n) \cdot \langle g^*_D, e_E \rangle,$$

because $g^*_i(e_n) = 0$, $i \in [n-1]$. (As n is the last element in D' and E', we do not have ± 1 in the formula.) Now,

$$M'(G'', F'') = g^*_n(e_n) \cdot M'\big([n-1]^{(r-1)}, V(F)^{(r-1)}\big)$$

has rank $\binom{v(F)}{r-1}$ because N is generic. □

Remark. It is not hard to show that, if F is not independent in $\mathscr{E}_G$, then $\mathrm{cn}(F)$ is not independent in $\mathscr{E}_{\mathrm{cn}(G)}$ for any r-graphs F and G. But we do not need this.

Lemma 4.3. *If* $G \in \text{w-SAT}(n-1, \mathscr{F})$*, then* $\mathrm{cn}(G) \in \text{w-SAT}(n, \mathrm{cn}(\mathscr{F}))$*. In particular,*

$$\text{w-sat}(n, \mathrm{cn}(\mathscr{F})) \leqslant \text{w-sat}(n-1, \mathscr{F}) + \binom{n-1}{r-1}.$$

Proof. Let $E_1, \ldots, E_m$ be an $\mathscr{F}$-proper ordering of $E(\overline{G})$. To show that $G' = \mathrm{cn}(G)$ is weakly $\mathrm{cn}(\mathscr{F})$-saturated, add these edges in the same order to G'. (Note that $E(\overline{G'}) = E(\overline{G})$.) Every E_i creates an F-subgraph in G, $F \in \mathscr{F}$, which, together with the extra vertex, creates a copy of $\mathrm{cn}(F)$ in G', so $G' \in \text{w-SAT}(n, \mathrm{cn}(\mathscr{F}))$. □

Theorem 4.4. *Let $\mathscr{F}$ be a family of r-graphs such that, in each $F \in \mathscr{F}$, any $r-1$ vertices are covered by at least one edge.*

If $(\mathscr{F}, G)$ admits the E-proof, then so does the pair $(\mathrm{cn}(\mathscr{F}), \mathrm{cn}(G))$.

If we can e-prove $\text{w-sat}(n-1, \mathscr{F}) \geqslant l$*, then we can e-prove*

$$\text{w-sat}(n, \mathrm{cn}(\mathscr{F})) \geqslant l + \binom{n-1}{r-1}. \tag{4.4}$$

In particular, if $\mathscr{F}$ admits an e-proof for $n-1$, then $\mathrm{cn}(\mathscr{F})$ *admits an e-proof for n. The analogous claim is true for the e$'$-technique.*

Proof. The claim about the E-proof follows from Lemmas 4.3 and 4.1.

Next, consider the e-technique. Take any G such that each $F \in \mathscr{F}$ is an $\mathscr{E}_G$-cycle and $R_{\mathscr{E}_G}(K^r_{n-1}) \geqslant l$. Adding extra vertices to G, we may assume $v(G) \geqslant n-1$. By Lemma 4.1, each graph in $\mathrm{cn}(\mathscr{F})$ is a cycle in $\mathscr{E}_{\mathrm{cn}(G)}$. By Lemma 4.2, $R_{\mathscr{E}_{\mathrm{cn}(G)}}(K_n^r) \geqslant l + \binom{n-1}{r-1}$, that is, we can e-prove (4.4), as required.

In the e′-case, choose G such that each $F \in \mathscr{F}$ is an $\mathscr{E}_G$-cycle, and

$$R_{\mathscr{E}_G}(K_{n-1}^r) + D_{\mathscr{E}_G}(\mathscr{F}) - 1 \geqslant l.$$

Now we proceed in the same way as in the e-case, except we have to show additionally that, for any $F \in \mathscr{F}$, we have $D_{\mathscr{E}_G}(F) \leqslant D_{\mathscr{E}_{\mathrm{cn}(G)}}(\mathrm{cn}(F))$.

Note that, if we have F-edges $E_1, \ldots, E_d$ whose removal does not decrease the $\mathscr{E}_G$-rank of $E(F)$, then the system of equations (4.2) has a solution in which $c_{E_1}, \ldots, c_{E_d}$ can be chosen to be the free variables $\beta_1, \ldots, \beta_d$. Following the proof of Lemma 4.1 (note that F is an $\mathscr{E}_G$-cycle), one can let (c_E) be such a solution of (4.2) and observe that

$$\langle e_{E_i}^*, h' \rangle = \langle e_{E_i}^* \wedge f_n^*, h \wedge e_n \rangle = \langle e_{E_i}^*, h \rangle \cdot \langle f_n^*, e_n \rangle = \beta_i \alpha_{nn}, \quad i \in [d],$$

since $E_i \subset [n-1]$. This means that, choosing generic βs, we can obtain $h' \in Z'$ whose support is $E(\mathrm{cn}(F))$ with $e_{E_i}^*(h')$ being generic, which is precisely to say that $E_1, \ldots, E_d$ are $\mathscr{E}_{\mathrm{cn}(G)}$-dependent on the other edges of $\mathrm{cn}(F)$. Hence, $D_{\mathscr{E}_{\mathrm{cn}(G)}}(\mathrm{cn}(F)) \geqslant d$ and the claim follows. □

Remark. We cannot generally discard the covering condition in Lemma 4.1 or Theorem 4.4. (But note that we do not have any covering condition on G.) Consider, for example, $r = 2$ when the condition rules out isolated vertices. Let F be a triangle plus an isolated vertex and let G be a star $K_{1,n-2}$, $n \geqslant 5$. Then (F, G) admits the E-proof (see Section 3.1). But it is easy to see that $\text{w-sat}(n, \mathrm{cn}(F)) = 6 < e(\mathrm{cn}(G)) = 2n - 3$, and so $\mathrm{cn}(F)$ cannot be an $\mathscr{E}_{\mathrm{cn}(G)}$-cycle.

5. Joins

Here we indicate how to extend the idea of E/e/*etc.*-proofs to layered graphs and prove that *joins* 'preserve' E/e/r-proofs.

Let $t \in \mathbb{N}$ be fixed. A *layered set* $\mathbf{X}$ of *signature* $\mathbf{n} = (n_1, \ldots, n_t)$ is a sequence of t disjoint sets, $\mathbf{X} = (X_1, \ldots, X_t)$, such that $|X_i| = n_i$, $i \in [t]$. The components of $\mathbf{X}$ are called *layers*. Given $\mathbf{r} = (r_1, \ldots, r_t)$, a *layered* $\mathbf{r}$*-graph* $\mathbf{G}$ is a pair $(V(\mathbf{G}), E(\mathbf{G}))$ where $V(\mathbf{G})$ is a layered set and $E(\mathbf{G}) \subset V(\mathbf{G})^{(\mathbf{r})}$. In other words, every $\mathbf{r}$-graph $\mathbf{G}$ is an r-graph (usually, given $\mathbf{r}$, we denote $r = \sum_{i \in [t]} r_i$) which comes with a fixed partition of the vertex set into t layers such that every edge intersects the ith layer in exactly r_i vertices. For example, bipartite graphs are $(1,1)$-graphs; for $t = 1$ we obtain the usual notion of an r-graph.

In the obvious way we define (within the class of $\mathbf{r}$-graphs) the notions of the complement, a subgraph, *etc.* (All morphisms, *etc.*, between $\mathbf{r}$-graphs respect the layers.) For example, given an $\mathbf{r}$-graph $\mathbf{F}$, $\text{w-SAT}(\mathbf{n}, \mathbf{F})$ consists of all $\mathbf{r}$-graphs $\mathbf{G}$ on an $\mathbf{n}$-set such that we can add consecutively all missing $\mathbf{r}$-edges to $\mathbf{G}$, creating an $\mathbf{F}$-subgraph every time.

It is clear how to extend the notion of an m/r-proof to layered graphs. It is also possible to introduce the exterior matroid of an $\mathbf{r}$-graph $\mathbf{G}$ defined on an $\mathbf{n}$-set $\mathbf{X}$. Indeed, identify each X_i with a basis $\mathbf{e}_i = (e_{i,j})_{j \in [n_i]}$ of an n_i-dimensional vector space V_i and consider

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$\bigwedge\mathbf{V} = \bigotimes_{i\in[t]} \bigwedge V_i$. Let $\bigwedge^{\mathbf{r}}\mathbf{V}$ be the linear subspace of $\bigwedge\mathbf{V}$ spanned by the elements

$$h = h_1 \otimes \cdots \otimes h_t, \qquad h_i \in {\textstyle\bigwedge}^{r_i} V_i, \quad i \in [t].$$

Let $\mathbf{f}_i = (f_{i,j})_{j\in[n_i]}$ be another basis of V_i lying in generic position with respect to $\mathbf{e}_i$, $i \in [t]$. In the obvious way we define supports, *etc.* For any $\mathbf{r}$-subset $\mathbf{E} \subset \mathbf{X}$, let

$$\mathbf{f}_{\mathbf{E}} = \textstyle\bigotimes_{i\in[t]} f_{i,E_i} \quad \text{and} \quad \mathbf{e}_{\mathbf{E}} = \bigotimes_{i\in[t]} e_{i,E_i}.$$

Let the linear subspace $\mathbf{Z} \subset \bigwedge^{\mathbf{r}}\mathbf{V}$ corresponding to $\mathbf{G}$ be spanned by the elements $\{\mathbf{f}_{\mathbf{E}} : \mathbf{E} \in E(\overline{\mathbf{G}})\}$, and let $\mathbf{r}$-sets $\mathbf{E}_1, \ldots, \mathbf{E}_k$ be independent if no linear combination of $\mathbf{e}_{\mathbf{E}_1}, \ldots, \mathbf{e}_{\mathbf{E}_k}$ (except 0) belongs to $\mathbf{Z}$. The required matroid $\mathscr{E}_{\mathbf{G}}$ of rank $\operatorname{codim}(\mathbf{Z}) = e(\mathbf{G})$ is built.

Given t (usual) r_i-graphs F_i, $i \in [t]$, with disjoint vertex sets, their *join* (or *tensor product*) $\mathbf{F} = F_1 \otimes \cdots \otimes F_t$ is the layered $\mathbf{r}$-graph on the layered set

$$V(\mathbf{F}) = (V(F_1), \ldots, V(F_t))$$

such that an $\mathbf{r}$-subset $\mathbf{E} = (E_1, \ldots, E_t)$ is an edge of $\mathbf{F}$ if and only if $E_i \in E(F_i)$ for every $i \in [t]$. Thus $e(\mathbf{F}) = \prod_{i\in[t]} e(F_i)$.

Suppose that we are given t families $\mathscr{F}_i$ of r_i-graphs, $i \in [t]$. Define their *join* by

$$\mathscr{F} = \mathscr{F}_1 \otimes \cdots \otimes \mathscr{F}_t = \{F_1 \otimes \cdots \otimes F_t : F_i \in \mathscr{F}_i,\ i \in [t]\}.$$

Let these conventions apply to the remainder of this section. We need the following simple lemmas.

Lemma 5.1. *If $G_i \in$ w-SAT$(n_i, \mathscr{F}_i)$, $i \in [t]$, then $\mathbf{G} \in$ w-SAT$(\mathbf{n}, \mathscr{F})$, where $\overline{\mathbf{G}} = \overline{G_1} \otimes \cdots \otimes \overline{G_t}$. In particular,*

$$\text{w-sat}(\mathbf{n}, \mathscr{F}) \leqslant \prod_{i\in[t]} \binom{n_i}{r_i} - \prod_{i\in[t]} \left(\binom{n_i}{r_i} - \text{w-sat}(n, \mathscr{F}_i) \right).$$

□

Lemma 5.2. *If F_i is a cycle in $\mathscr{E}_{G_i}$, $i \in [t]$, then $\mathbf{F} = F_1 \otimes \cdots \otimes F_t$ is a cycle in $\mathscr{E}_{\mathbf{G}}$, where $\overline{\mathbf{G}} = \overline{G_1} \otimes \cdots \otimes \overline{G_t}$.*

Proof. Given $h_i \in Z_{G_i} \subset \bigwedge^{r_i} V_i$ with $\operatorname{supp}_{\mathbf{e}_i}(h_i) = E(F_i)$, $i \in [t]$, consider $h = h_1 \otimes \cdots \otimes h_t \in \bigwedge^{\mathbf{r}}\mathbf{V}$. □

Theorem 5.3. *Suppose that, for every $i \in [t]$, the pair $(\mathscr{F}_i, G_i)$ admits the E-proof. Then so does the pair $(\mathscr{F}, \mathbf{G})$, where $\overline{\mathbf{G}} = \overline{G_1} \otimes \cdots \otimes \overline{G_t}$.*

Suppose that, for each $i \in [t]$, we can e-prove that w-sat$(n_i, \mathscr{F}_i) \geqslant l_i$. *Then we can e-prove that*

$$\text{w-sat}(\mathbf{n}, \mathscr{F}) \geqslant \prod_{i\in[t]} \binom{n_i}{r_i} - \prod_{i\in[t]} \left(\binom{n_i}{r_i} - l_i \right). \tag{5.1}$$

In particular, by Lemma 5.1, if each $\mathscr{F}_i$ admits an e-proof for n_i, then $\mathscr{F}$ admits an e-proof for $\mathbf{n}$. *The analogous statement is true for the r-technique.*

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Proof. The claim about the E-proof follows from Lemmas 5.1 and 5.2.

Now, consider the e-case. For $i \in [t]$, choose G_i such that each graph in $\mathscr{F}_i$ is an $\mathscr{E}_{G_i}$-cycle and the $\mathscr{E}_{G_i}$-rank of $K_{n_i}^{r_i}$ is at least l_i; let $H_i \subset K_{n_i}^{r_i}$ be an $\mathscr{E}_{G_i}$-independent subgraph of size l_i and order n_i. Let

$$\overline{\mathbf{G}} = \overline{G_1} \otimes \cdots \otimes \overline{G_t},$$
$$\overline{\mathbf{H}} = \overline{H_1} \otimes \cdots \otimes \overline{H_t}.$$

By Lemma 5.2, each $F_1 \otimes \cdots \otimes F_k \in \mathscr{F}$ is an $\mathscr{E}_{\mathbf{G}}$-cycle.

Let us show that $\mathbf{H}$ is independent in $\mathscr{E}_{\mathbf{G}}$. As each H_i is $\mathscr{E}_{G_i}$-independent, we can find a linear map $p_i : \bigwedge^{r_i} V_i \to Z_{G_i}$, which is the identity map on Z_{G_i} while $p_i(e_E) = 0$ if $E \in E(H_i)$, $i \in [t]$. Define

$$\mathbf{p} = p_1 \otimes \cdots \otimes p_t : \bigwedge^{\mathbf{r}} \mathbf{V} \to Z_{G_1} \otimes \cdots \otimes Z_{G_t},$$

that is, $\mathbf{p}(u_1 \otimes \cdots \otimes u_t) = p_1(u_1) \otimes \cdots \otimes p_t(u_t)$. Now, $\mathbf{p}$ is the identity map on $Z_{G_1} \otimes \cdots \otimes Z_{G_t} = Z_{\mathbf{G}}$, while $\mathbf{p}$ is zero on $e_{\mathbf{E}}$ for each $\mathbf{E} = E_1 \cup \cdots \cup E_t \in E(\mathbf{H})$. Hence, no nonzero linear combination of $e_{\mathbf{E}}$, $\mathbf{E} \in E(\mathbf{H})$ can lie in $Z_{\mathbf{G}}$, that is, $\mathbf{H}$ is independent in $\mathscr{E}_{\mathbf{G}}$. The size of $\mathbf{H}$ equals the right-hand side of (5.1), as required.

In the r-case, our task is to construct a matroid $\mathscr{M}$ on the set of $\mathbf{r}$-subsets of $\mathbf{X}$ such that every graph in $\mathscr{F}$ is an $\mathscr{M}$-cycle, should we be given appropriate matroids $\mathscr{M}_i$ on $Y_i = X_i^{(r_i)}$, $i \in [t]$.

Let $k_i : Y_i \to V_i$, for some vector space V_i, be a representation of the matroid $\mathscr{M}_i$, $i \in [t]$. Identify Y_i with a basis of some vector space W_i via $g_i : Y_i \hookrightarrow W_i$. Let $h_i : W_i \to V_i$ be the linear map extending k_i. Denote $Z_i = \ker(h_i) \subset W_i$. Clearly, $\operatorname{codim} Z_i = R_{\mathscr{M}_i}(Y_i) = e(G_i) \geqslant l_i$, where G_i is a base of $\mathscr{M}_i$.

Let $\overline{\mathbf{G}} = \overline{G_1} \otimes \cdots \otimes \overline{G_t}$. Identify the $\mathbf{r}$-subsets of $V(\mathbf{G})$ with a basis of $\mathbf{W} = \bigotimes_{i\in[t]} W_i$ by mapping $\mathbf{E} = (E_1, \ldots, E_t)$ into $g(\mathbf{E}) = \bigotimes_{i\in[t]} g_i(E_i)$. Let $\mathbf{Z} = \bigotimes_{i\in[t]} Z_i \subset \mathbf{W}$ and $p : \mathbf{W} \to \mathbf{W}/\mathbf{Z}$ be the projection.

Let $\mathscr{M}$ be the matroid represented by $p \circ g : V(\mathbf{G})^{(\mathbf{r})} \to \mathbf{W}/\mathbf{Z}$. Let us show that $\mathscr{M}$ r-proves (5.1).

As $g(V(\mathbf{G})^{(\mathbf{r})})$ is a basis for $\mathbf{W}$, we conclude that the rank of $\mathscr{M}$ is

$$\dim \mathbf{W} - \dim \mathbf{Z} = \prod_{i\in[t]} \binom{n_i}{r_i} - \prod_{i\in[t]} e(\overline{G_i}),$$

which is no smaller than the right-hand side of (5.1).

Thus, all we have to do is to check that any $\mathbf{F} = F_1 \otimes \cdots \otimes F_t \in \mathscr{F}$ is an $\mathscr{M}$-cycle. Fix an edge $\mathbf{E} = (E_1, \ldots, E_t) \in E(\mathbf{F})$. As F_i is an $\mathscr{M}_i$-cycle, we conclude that there are $c_{i,E} \in \mathbb{R}$, $E \in E(F_i) \setminus \{E_i\}$, and $z_i \in Z_i$ such that

$$g_i(E_i) = z_i + \sum_{D \in E(F_i)\setminus\{E_i\}} c_{i,D}\, g_i(D), \qquad i \in [t]. \tag{5.2}$$

If we take the tensor product of (5.2) over $i \in [t]$, we obtain on the left-hand side the element $g(\mathbf{E})$, while on the right-hand side we will have $z_1 \otimes \cdots \otimes z_t \in \mathbf{Z}$ plus some other tensor products. Next, in the remaining tensor products replace each z_i by the linear combination of $(g_i(D))_{D\in E(F_i)}$ derived from (5.2). Each term then becomes $\otimes_{i\in[t]} g_i(D_i)$ for some $D_i \in E(F_i)$, that is, it is of the form $g(\mathbf{D})$, $\mathbf{D} = (D_1, \ldots, D_t) \in E(\mathbf{F})$ and, moreover, we never have $\mathbf{D} = \mathbf{E}$. So we have a representation of $g(\mathbf{E})$ as a linear combination of an element of $\mathbf{Z}$ and of $g(\mathbf{D})$, $\mathbf{D} \in E(\mathbf{F}) \setminus \{\mathbf{E}\}$, precisely as required. The theorem is proved. □

Unfortunately, there does not seem to be a natural ⊗-operation for matroids (*cf.* Lovász [10]), so we do not know if joins preserve m-proofs.

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References

[1] Alon, N. (1985) An extremal problem for sets with applications to graph theory. *J. Combin. Theory Ser. A* **40** 82–89.
[2] Bollobás, B. (1965) On generalized graphs. *Acta Math. Acad. Sci. Hungar.* **16** 447–452.
[3] Bollobás, B. (1967) Weakly k-saturated graphs. In *Proc. Coll. Graph Theory*, Ilmenau, pp. 25–31.
[4] Doob, M. (1973) An interrelation between line graphs, eigenvalues, and matroids. *J. Combin. Theory Ser. B* **15** 40–50.
[5] Erdős, P., Füredi, Z. and Tuza, Z. (1991) Saturated r-uniform hypergraphs. *Discrete Math.* **98** 95–104.
[6] Frankl, P. (1982) An extremal problem for two families of sets. *Europ. J. Combin.* **3** 125–127.
[7] Kalai, G. (1985) Hyperconnectivity of graphs. *Graphs Combin.* **1** 65–79.
[8] Kalai, G. (1990) Symmetric matroids. *J. Combin. Theory Ser. B* **50** 54–64.
[9] Kászonyi, L. and Tuza, Z. (1986) Saturated graphs with minimal number of edges. *J. Graph Theory* **10** 203–210.
[10] Lovász, L. (1977) Flats in matroids and geometric graphs. In *Proc. 6th British Combin. Conf.* (P. J. Cameron, ed.), Academic Press, pp. 45–86.
[11] Marcus, M. (1973, 1975) *Finite Dimensional Multilinear Algebra, I, II*, Marcel Dekker, New York.
[12] Pikhurko, O. (1999) The minimum size of saturated hypergraphs. *Combin. Probab. Comput.* **8** 483–492.
[13] Pikhurko, O. Uniform families and count matroids. To appear in *Graphs Combin.*
[14] Tuza, Z. (1988) Extremal problems on saturated graphs and hypergraphs. *Ars Combinatoria* **25B** 105–113.
[15] Yu, J. (1993) An extremal problem for sets: A new approach via Bezoutians. *J. Combin. Theory Ser. A* **62** 170–175.