Size Ramsey Numbers Involving Large Stars

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Abstract

We conjecture that for any fixed graph F with chromatic number $\chi(F) \geq 4$ the size Ramsey number $\hat{r}(K_{1n}, F)$ is $(1+o(1))\chi(F)(\chi(F)-2)n^2/2$ as $n \to \infty$ and present some partial results.

1 Introduction

Given two graphs F_1 and F_2, we say that a graph G *arrows* the pair (F_1, F_2), denoted by $G \to (F_1, F_2)$, if for any blue-red colouring of the edge set of G we have either a blue copy of F_1 or a red copy of F_2 (or both). The *size Ramsey number* $\hat{r}(F_1, F_2)$ is the minimal number of edges of a graph G such that $G \to (F_1, F_2)$.

Here we study $\hat{r}(K_{1n}, F)$ for a fixed F as n tends to infinity. (The *star* K_{1n} consists of n edges incident to a common vertex.) We make the following conjecture.

Conjecture 1 *For any fixed graph F of chromatic number $\chi(F) \geq 4$, we have*

$$\hat{r}(K_{1n}, F) = (1+o(1))\,\frac{\chi(F)(\chi(F)-2)}{2}\,n^2. \qquad (1)$$

It is easy to see that if F is a bipartite graph then $\hat{r}(K_{1n}, F) = o(n^2)$. The author [2] has recently shown that $\hat{r}(K_{1n}, F) = (1+o(1))n^2$ for any fixed 3-chromatic graph F. (That is, the case $\chi(F) = 3$ is perhaps exceptional.)

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2 Upper Bounds

The following lemma establishes the upper bound in Conjecture 1. Let $K_k(t)$ denote the complete k-partite graph with each part having t vertices.

Lemma 2 *Fix $\epsilon > 0$ and let A and B be two disjoint sets of sizes $m = \lceil (k-2+\epsilon)n \rceil$ and n respectively. Let G be the graph on $A \cup B$ consisting of all edges intersecting A.*

Then there exists a constant $c = c(\epsilon, k) > 0$ such that, if n is sufficiently large, any blue-red colouring of $E(G)$ without a red K_{1n} contains a blue $K_k(t)$ with $t > c \log n$.

PROOF. Given a colouring of $E(G)$ let G' be the blue subgraph. The set A spans at most $|A|n/2$ red edges, so the edge density of $G'[A]$ is strictly greater than $1 - \frac{1}{k-2}$. By the Erdős-Stone theorem [1], $G'[A]$ contains a $K_{k-1}(s)$-subgraph K with $s = \Theta(\log n)$.

Each vertex in $V = V(K)$ sends at least $(k-2+\epsilon)n - O(s)$ blue edges to $(A \cup B) \setminus V$. Let U consist of those vertices of $(A \cup B) \setminus V$ which send at least $(k-2+c_1)s$ blue edges to V, where $c_1 = c_1(\epsilon, k) > 0$ is a sufficiently small constant. The inequality

$$(k-1)s \cdot |U| + (k-2+c_1)s \cdot ((k-1+\epsilon)n - |U|) \geq (k-2+\epsilon)n \cdot (k-1)s - O(s^2)$$

shows that $|U| = \Theta(n)$. Each vertex of U covers at least one $K_{k-1}(s')$-subgraph of K, where $s' = \lceil c_1 s \rceil$. Hence, some such subgraph appears at least $|U| \cdot \binom{s}{s'}^{-(k-1)} > s'$ times, which gives us a blue $K_k(s')$, as required. $\square$

Of course, we can obtain more precise upper bounds for some specific F. For example, it is easy to see (e.g. by induction on k) that $K_{(k-2)n+1} + \overline{K}_n \to (K_{1,n}, K_k)$. Perhaps, this is best possible and we have, for any $k \geq 4$ and $n \geq 3$,

$$\hat{r}(K_{1n}, K_k) = \binom{(k-2)n+1}{2} + ((k-2)n+1)n.$$

3 Lower Bounds

Let $\mathcal{B}_j(n)$ be the family of all graphs G such that for any partition of $V(G)$ into j parts some part spans a graph of maximum degree at least n. The function $b_j(n) = \min\{e(G) \mid G \in \mathcal{B}_j(n)\}$ is of interest to us because $b_{k-1}(n) \leq \hat{r}(K_{1,n}, F)$ for any graph F with $\chi(F) = k$.

Here we prove an easy lower bound on $b_j(n)$. A simple lemma first.

Lemma 3 *If $G \in \mathcal{B}_j(n)$, then $\Delta(G) \geq jn$.*

PROOF. Consider a partition $V(G) = A_1 \cup \ldots \cup A_j$ which minimizes $s = \sum_{i=1}^{j} e(G[A_i])$. For some $i \in [j]$ there exists $x \in G[A_i]$ of degree at least n. If we move x to any other part, then s does not decrease, so x sends at least n edges to each part. Hence, $d(x) \geq jn$, as required. □

Theorem 4 *For any $k \geq 4$, we have $b_{k-1}(n) \geq \binom{k-1}{2} n^2$.*

PROOF. Given $G \in \mathcal{B}_{k-1}(n)$, let $A_1 = V(G)$. Repeat the following for $i = 1, 2, \ldots, k-2$. Given a set A_i such that $G[A_i] \in \mathcal{B}_{k-i}(n)$, let $B_i \subset A_i$ be a set of size n minimizing $e(G[A_{i+1}])$, where $A_{i+1} = A_i \setminus B_i$. Clearly, $G[A_{i+1}] \in \mathcal{B}_{k-i-1}(n)$. So, if $i < k-2$, we can repeat the step with next i.

Let $i \in [k-2]$. By Lemma 3, $G[A_{i+1}]$ contains a vertex x of degree at least $(k-i-1)n$. Now, every $y \in B_i$ sends at least $(k-i-1)n$ edges to A_{i+1}, because the exchange of x and y does not decrease $e(G[A_{i+1}])$.

Hence, $e(G) \geq \sum_{i=1}^{k-2} (k-i-1)n^2 = \binom{k-1}{2} n^2$, as required. □

In particular, for $k = 4$ we have $3n^3 \leq b_3(n) \leq 4n^2 + 2n$. Working harder on this simplest open case, the author was able to improve the lower bound to $b_3(n) \geq (3.69 + o(1))n^2$. The proof, built upon Theorem 4, is rather complicated and we do not have enough space to present it here. Anyway, we believe that the constant 3.69 can be improved and, hopefully, we will have proved better lower bounds by the time the conference starts.

References

[1] P. Erdős and A. H. Stone. On the structure of linear graphs. *Bull. Amer. Math. Soc.*, **52** (1946) 1087–1091.

[2] O. Pikhurko. Size Ramsey numbers of stars versus 3-chromatic graphs. Submitted to *Combinatorica*, 1999.