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Note

# Asymptotic evaluation of the sat-function for $r$-stars

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**Abstract**

Let $S^r_m$ be the $r$-uniform set system on $m$ vertices consisting of all $r$-tuples containing a given vertex. We determine the asymptotic behaviour of $\mathrm{sat}(n, S^r_m)$ for all $r$ and $m$ thus extending a result of Erdős, Füredi and Tuza. 

An $r$-graph $G$ is a pair $(V(G), E(G))$ where the *edge set* $E(G)$ is a family of $r$-subsets of the *vertex set* $V(G)$. In the usual way, we define the notion of subgraph, isomorphism, complementary graph $\overline{G}$, size $|G| = |E(G)|$, etc. A graph isomorphic to $G$ is called a $G$-graph. Given an $r$-graph $H$ we say that $G$ is *monotonically $H$-saturated* (another common name is *strongly $H$-saturated*) if the addition of any new $r$-tuple to $E(G)$ creates at least one new $H$-subgraph. If, besides, $G$ does not contain $H$ as a subgraph then $G$ is called *$H$-saturated*. Next, $\mathrm{sat}(n, H)$ ($\mathrm{m\text{-}sat}(n, H)$) is defined to be the minimal size of an $H$-saturated (monotonically $H$-saturated) graph on $n$ vertices.

The *star* $S^r_m$, $m > r \geqslant 2$, has $[m]$ as the vertex set and $\{E \in [m]^{(r)} : E \ni m\}$ as the edge set. In other words, $S^r_m$ has $m$ vertices and its $\binom{m-1}{r-1}$ edges are the $r$-tuples containing some fixed vertex which is called the *centre*.

The exact values of $\mathrm{sat}(n, S^r_m)$ are known only for $S^2_m$, any $m$, (see [2]) and for $S^3_4$ (see [1]).

The asymptotic behaviour of $\mathrm{sat}(n, S^r_{r+1})$ was found by Erdős et al. [1, Theorem 2]. Exploiting their ideas we extend this result to all stars.

**Theorem 1.** *Let $m > r \geqslant 2$ and $S = S^r_m$. Then*

$$\frac{m-r}{2}\binom{n}{r-1} \geqslant \mathrm{sat}(n,S) \geqslant \mathrm{m\text{-}sat}(n,S) \geqslant \frac{m-r}{2}\binom{n}{r-1} - \mathrm{O}(n^{r-4/3}). \qquad (1)$$

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**Proof.** Let us provide a construction of an $S$-saturated graph $G = G^r_{m,n}$ proving the upper bound. Partition the vertex set $[n]$ into $n' = \lceil n/(m-r+1) \rceil$ *blocks* $B_1, \dots, B_{n'}$ of size $m-r+1$ each except possibly the last one. The edge set is

$$E(G) = \{F \in [n]^{(r)}\colon |F \cap B_j| \geqslant 2,\ j = \min\{i \in [n']\colon F \cap B_i \neq \emptyset\}\}.$$

Thus every edge of $G$ has at least two common points with some $B_j$ and intersects no $B_i$ with $i < j$.

Let us show that $S \not\subset G$. Suppose not and we have an $S$-subgraph $S' \subset G$ centred at $x$. Let

$$j = \min\{i \in [n']\colon V(S') \cap B_i \neq \emptyset\}. \tag{2}$$

Choose an $r$-set $F \ni x$ so that it contains one vertex from $B_j$ and some $r-1$ vertices in $V(S') \backslash B_j$ which is possible since $|V(S') \backslash B_j| \geqslant r-1$. We obtain a contradiction as on one hand $F$ contains the centre $x$ and must belong to $S$ while on the other hand $F \notin E(G)$ by the definition.

Now, if we add any extra edge $F$ to $G$ then the set $Y = F \cup B_j$ spans a copy of $S$ centred at $x$ where $B_j$ is the first block intersecting $F$ and $\{x\} = F \cap B_j$. Indeed, every $F' \in Y^{(r)}$ containing $x$ either equals $F$ or intersects $B_j$ in at least two points and so belongs to $E(G)$.

Therefore, we conclude that $G$ is $S$-saturated. To prove the desired upper bound $|G^r_{m,n}| \leqslant \frac{m-r}{2} \binom{n}{r-1}$ we observe, for $r=2$, that each vertex of the 2-graph $G^2_{m,n}$ has degree at most $m-r$ while, for $r \geqslant 3$, we use induction and the equality $|G^r_{m,n+1}| = |G^r_{m,n}| + |G^{r-1}_{m-1,n}|$.

Trivially, $\mathrm{sat}(n,S) \geqslant \text{m-sat}(n,S)$.

Finally, let $G$ be a minimum monotonically $S$-saturated graph on $V = [n]$. By the definition, the addition to $G$ of any edge $F \in E(\overline{G})$ creates at least one $S$-subgraph $S' \subset G + F$. Let $\mathscr{S}(F)$ be the set of all such subgraphs $S'$ created by $F$.

Let $\mathscr{F}(F)$ denote the set of edges in $\overline{G}$ which intersect $F \in E(G)$ in $r-1$ points and create a copy of $S$ containing $F$ as an edge. Formally,

$$\mathscr{F}(F) = \{F' \in E(\overline{G})\colon |F \cap F'| = r-1,\ \exists S' \in \mathscr{S}(F')\ F \in E(S')\}, \quad F \in E(G).$$

Also, we define

$$\begin{aligned} \mathscr{F}(G') &= \bigcup_{F \in E(G')} \mathscr{F}(F), && G' \subset G, \\ \partial F &= F^{(r-1)}, && F \in [n]^{(r)}, \\ \partial G' &= \bigcup_{F \in E(G')} \partial F, && \text{an } r\text{-graph } G'. \end{aligned}$$

As $G$ is monotonically $S$-saturated we conclude that

$$\mathscr{F}(G) = V^{(r)} \backslash E(G). \tag{3}$$

Choose an integer $k = k(n)$, to be specified later, such that $k \to \infty$ and $k/n \to 0$. On the vertex set $V$ we define two subgraphs $G_0,\ G_1 \subset G$; $G_0$ is a maximal subgraph of $G$ with $|\mathscr{F}(G_0)| \leqslant k|G_0|$ and $G_1$ consists of the edges of $G$ not in $G_0: E(G_1) = E(G) \setminus E(G_0)$. By the maximality of $G_0$, for every $F \in E(G_1)$ we have

$$|\mathscr{F}(F) \setminus \mathscr{F}(G_0)| > k. \tag{4}$$

From (3) and the proved upper bound in (1) we conclude that $|\mathscr{F}(G)| = \binom{n}{r} - |G| = \binom{n}{r} - \mathrm{O}(n^{r-1})$. Taking into the account that $\mathscr{F}(G) = \mathscr{F}(G_0) \cup \mathscr{F}(G_1)$ and $|\mathscr{F}(G_0)| \leqslant k|G_0| = \mathrm{O}(kn^{r-1})$ we obtain

$$|X| = \binom{n}{r} - \mathrm{O}(kn^{r-1}), \tag{5}$$

where $X = \mathscr{F}(G_1) \setminus \mathscr{F}(G_0)$.

Let $Z = V^{(r-1)} \setminus \partial G_1$. We claim that

$$|Z| = \mathrm{O}(k^{1/2} n^{r-3/2}). \tag{6}$$

Suppose not. Then the average value of $z(D) = |\{E \in Z: E \supset D\}|$ over all $D \in V^{(r-2)}$ is greater than $\mathrm{O}(k^{1/2}n^{1/2})$. For any $E, E' \in Z$ with $|E \cap E'| = r-2$ we have $F = E \cup E' \notin X$, because otherwise at least one of $E, E' \in \partial F$ is covered by an edge of $S' \in \mathscr{S}(F)$ which then is necessarily an edge of $G_1$ (as it intersects $F \in \mathscr{F}(G_1) \setminus \mathscr{F}(G_0)$ in $r-1$ vertices). Therefore, we have at least $\binom{r}{2}^{-1} \sum_{D \in V^{(r-2)}} \binom{z(D)}{2}$ $r$-sets not in $X$, which exceeds $\binom{n}{r-2} \mathrm{O}(kn)$ by the convexity of $\binom{x}{2}$. This contradicts (5) and proves the claim.

Let

$$g_1(E) = |\{F \in E(G_1): F \supset E\}|, \quad E \in \partial G_1.$$

Take any $F \in E(G_1)$. Let $\partial F = \{E_1, \ldots, E_r\}$. We claim that all but at most two of $g_1(E_i)$'s are larger than $k/6$. Suppose not, say $g_1(E_i) \leqslant k/6$, $i = 1,2,3$. Take $F' \in \mathscr{F}(F) \setminus \mathscr{F}(G_0)$ and any $S' \in \mathscr{S}(F')$ containing $F$ as an edge. Let $F' = E_i \cup \{x\}$, some $i \in [r]$, $x \in V \setminus F$. The star $S'$ contains $r-2$ edges of the form $E_j \cup \{x\}$, $j \neq i$. These edges cannot be in $G_0$ and so contribute at least 1 to $g_1(E_1) + g_1(E_2) + g_1(E_3)$. In total, each $\{x\} \cup E_j \in E(G_1)$ is counted at most twice. (Once it occurs then at most 2 edges of the form $\{x\} \cup E_i$ can belong to $E(\overline{G})$.) But this contradicts (4). The claim is proved.

Define

$$W = \{E \in \partial G_1: g_1(E) \leqslant m - r - 1\},$$

$$T = \{F \in E(G_1): W \cap \partial F \neq \emptyset\}.$$

We claim that $|W| = \mathrm{O}(k^{1/2} n^{r-3/2})$. Suppose not. Note that for $E, E' \in W$ with $|E \cap E'| = r-2$ we necessarily have $F = E \cup E' \notin X$ for otherwise in an $S' \in \mathscr{S}(F)$ centred at $x$, say $x \in E$, there are $m - r$ edges (necessarily in $E(G_1)$) different from $F$ and covering $E$. Thus there are at least $\binom{r}{2}^{-1} \sum_{D \in V^{(r-2)}} \binom{w(D)}{2}$ edges not in $X$, where $w(D) = |\{E \in W: E \supset D\}|$, $D \in V^{(r-2)}$. Using the convexity of the $\binom{x}{2}$-function as before we can argue that there are more than $\mathrm{O}(kn^{r-1})$ edges not in $X$ contradicting (5). The claim is established.

Every $E \in W$ is contained in at most $m-r-1$ edges $F \in E(G_1)$ so $|T| = \mathrm{O}(k^{1/2}n^{r-3/2})$. For every $F \in E(G_1)\setminus T$ we have $\sum_{E\in\partial F} 1/g_1(E) \leqslant 2/(m-r)+(r-2)6/k$. Note the following easy identity:

$$|\partial G_1| = \sum_{F\in E(G_1)\setminus T}\left(\sum_{E\in\partial F}\frac{1}{g_1(E)}\right) + \sum_{F\in T}\left(\sum_{E\in\partial F}\frac{1}{g_1(E)}\right)$$

$$\leqslant \left(\frac{2}{m-r} + \mathrm{O}(1/k)\right)|G_1| + r|T|.$$

We know, see (6), that $|\partial G_1| = \binom{n}{r-1} - \mathrm{O}(k^{1/2}n^{r-3/2})$. Hence,

$$\frac{m-r}{2}\binom{n}{r-1} - |G| = \mathrm{O}(k^{1/2}n^{r-3/2} + |G|/k) = \mathrm{O}(k^{1/2}n^{r-3/2} + n^{r-1}/k).$$

Taking $k = \lfloor n^{1/3} \rfloor$ we obtain the required. $\square$

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