Graph Limits:

Some Open Problems

1 Introduction

Here are some questions from the open problems session that was held during the AIM Workshop *"Graph and Hypergraph Limits"*, Palo Alto, August 15-19, 2011. Two questions (omitted) were solved; workshop's report (available from AIM's website) contains more information about them.

One of the objectives of the session was to identify possible new directions, so some of these problems may be rather imprecise. We encourage the reader to contact the proposer of each problem with questions and comments.

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2 Some Basic Notation

Here we point out some of the notation that is used. We refer the reader to the survey by Lovász [Lov09] for further details. Also, see the collection [Lov08] of open questions by Lovász.

For an integer n, we define $[n] = \{1, \ldots, n\}$.

A *graphon* is a measurable bounded symmetric function $W : I^2 \to \mathbb{R}$, where $I = [0, 1]$ is the unit interval of reals. Let $\mathcal{W}$ be the set of all graphons and let $\mathcal{W}_I \subseteq \mathcal{W}$ consist of graphons with values in I.

Graphons $U, W \in \mathcal{W}$ are in the same equivalence class if there are measure preserving maps $\phi, \psi : [0, 1] \to [0, 1]$ such that $W^\phi = U^\psi$ almost everywhere, where $W^\phi(x, y) = W(\phi(x), \phi(y))$; see [BCL10, Corollary 2.2] for some equivalent definitions. Let $\tilde{W}$ be the equivalence class of a graphon W.

The *density* of a finite graph $F = (V, E)$ in $W \in \mathcal{W}$ is

$$t(F, W) = \int_{I^V} \prod_{ij \in E} W(x_i, x_j) \prod_{i \in V} dx_i.$$

3 Open Problems

3.1 Selecting a Representative (Sourav Chatterjee)

A *selection* is a map $T : \{\tilde{W} \mid W \in \mathcal{W}\} \to \mathcal{W}$ with $T(\tilde{W}) \in \tilde{W}$ for every W.

Question 1 *Is there a selection T which is continuous?*

There are various interpretations of this question. One possibility is to take the cut distance on the equivalence classes and the cut norm on the values of T.

Question 2 *Is there a measurable selection T?*

The paper by Janson [Jan10] might be related to the last question.

3.2 Topological Graphons (Balázs Szegedy)

A *topological graphon* is a triple (Ω, W, μ) such that

- Ω is a *Polish space* (a separable completely metrizable topological space);
- μ is a probability measure on Borel subsets of Ω of *full support* (i.e., every open set has positive measure);
- $W : \Omega \times \Omega \to [0, 1]$ is a Borel-measurable symmetric function;
- for every $x \in \Omega$ the function $W(x, \cdot) : \Omega \to [0, 1]$ is measurable and the corresponding map $\Omega \to L^1(\Omega)$ is continuous.

One can show (see [LS10, Theorem 3.1]) that every graphon has a topological representation.

A topological graphon is *compact* if Ω in the above definition is a compact space. An example is the half-graph, $W : I^2 \to I$, where $W(x, y)$ is 0 if $x + y \leq 1$ and 1 otherwise.

Question 3 *Does every extremal graphon problem have a solution represented by a compact topological graphon?*

For example, the question whether one can simultaneously satisfy constraints $t(F_i, W) = a_i$ for $i = 1, \dots, n$ is equivalent to the statement that the minimum

$$\min_W \sum_i (t(F_i, W) - a_i)^2 .$$

is zero. Thus, in particular, can minimization of this type be restricted to compact topological graphons?

3.3 Maximizing the Entropy (Sourav Chatterjee)

Suppose we are given graphs $F_1, \dots, F_m$ and numbers $a_1, \dots, a_m$ such that the system

$$t(F_i, W) = a_i, \quad \forall i \in [m], \tag{1}$$

has a graphon $W \in \mathcal{W}_I$ that satisfies it.

Given the constrains in (1), we maximize the *entropy function*

$$h(W) = -\int \Big(W(x,y)\log W(x,y) + (1-W(x,y))\log(1-W(x,y))\Big)dxdy$$

A maximizer W^* exists by the compactness of $\mathcal{W}_I$.

This is related to questions about counting. Namely, the logarithm of the number of graphs G on $[n]$ approximately satisfying (1) is $\binom{n}{2}h(W^*) + o(n^2)$ (see Chatterjee and Varadhan [CV11]).

Question 4 *Determine a maximizer W^* for non-trivial problems, e.g., with $m = 2$, $F_1 = K_2$ and $F_2 = K_3$.*

3.4 Exponential Random Graph Model with Edges and Triangles (Charles Radin)

This is related to the previous question. Given β_1 and β_2, define $\psi_n(\beta_1, \beta_2)$ so that the assignment

$$\mathbf{P}_{\beta_1,\beta_2}(G) = e^{n^2[\beta_1 t(K_2,G)+\beta_2 t(K_3,G)-\psi_n(\beta_1,\beta_2)]}$$

defines a probability distribution on graphs with vertex set $[n]$. Consider the limit

$$\psi(\beta_1, \beta_2) = \lim_{n\to\infty} \psi_n(\beta_1, \beta_2),$$

where β_1 and β_2 are fixed reals. We are interested in determining precisely those β_1 and β_2 at which $\psi(\beta_1, \beta_2)$ is analytic. This is essentially solved (by Chatterjee and Diaconis [CD11] together with Radin and Yin [RY11]) for $\beta_2 > 0$. The problem is generally open for the case $\beta_2 < 0$ which is more interesting. See Aristoff and Radin [AR11] for further results.

3.5 Counting C_4-Free Graphs (Miklós Simonovits)

Here is a problems about graphs with intermediate number of edges (between dense and sparse cases): $\Theta(n^{3/2})$.

Erdős, Kleitman, and Rothschild [EKR76] counted the number of triangle-free graphs on $[n]$. At the same time, a similar result for C_4-free graphs is unknown.

Question 5 (Erdős) *How many C_4-free graphs on $[n]$ are there?*

If one takes subgraphs of a maximum C_4-free graph, then one gets at least $2^{(\frac{1}{2}+o(1))n^{3/2}}$ different graphs. Kleitman and Winston [KW82] showed an upper bound $2^{O(n^{3/2})}$, i.e, we lose a constant factor in the exponent.

3.6 Percolation Thresholds for Sparse Graphs (Louigi Addario-Berry)

Let (G_n) with $v(G_n) = n$ converge to G in the local weak sense. For simplicity, let us assume that all graphs are vertex-transitive. Let $p_c(G)$ denote the critical percolation constant of the infinite graph G. Find sufficient conditions on the sequence (G_n) such that for $p > p_c(G)$, there exists $\alpha > 0$ such that

$$\liminf \mathbf{P}\Big(p\text{-percolation on } G_n \text{ yields a component of order} > \alpha n\Big) > 0$$

and for $p < p_c(G)$ and every $\alpha > 0$, the lim sup of the above probability is equal to 0.

This is a version of a question that appears in [BNP11].

3.7 Continuity with respect to Local Topology (Omer Angel)

Suppose that we have a sequence of infinite graphs (G_n), say all vertex-transitive and of uniformly bounded degree and that they converge with respect to local topology to a limit graph G.

Question 6 *When is p_c continuous, that is, when $\lim_{n\to\infty} p_c(G_n) = p_c(G)$?*

The answer is in the negative in general. For example: let G_n be the cycle C_n times the infinite path. Then $p_c(G_n) = 1$ but in the limit get a 2-dimensional object so $p_c(G) < 1$.

This question does not have to be specifically about percolation. We can ask about other models, e.g., Ising. One particularly nice parameter is the *connectivity constant*

$$\lambda(G) = \liminf_{n\to\infty} f_n^{1/n},$$

where f_n be the number of self-avoiding paths in G of length n starting at some x. (For a connected graph G, $\lambda(G)$ is independent of the initial vertex.)

Question 7 *Under what assumptions on (G_n) is the connectivity constant continuous?*

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