Hypergraph Turán Problem: Some Open Questions

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1 Introduction

Here we present a selection of some open questions related to the hypergraph Turán problem.

Let $[n]$ denote the interval $\{1,\ldots,n\}$. For a set X and an integer k, let $\binom{X}{k}=\{Y\subseteq X : |Y|=k\}$ be the family of all k-subsets of X. By a *k-graph* F we understand a k-uniform set system, that is, F is a pair (V,E) where V is the set of vertices and $E\subseteq\binom{V}{k}$. For convenience, we will identify k-graphs with their edge set. Thus e.g. $|F|=|E(F)|$ denotes the *size* of F.

Let $\mathcal{F}$ be a family of k-graphs. A k-graph G is *$\mathcal{F}$-free* if G does not contain any member of $\mathcal{F}$ as a (not necessarily induced) subgraph. The *Turán function* is

$$\mathrm{ex}(n,\mathcal{F})=\max\{|G| : G\subseteq\tbinom{[n]}{k},\ G\text{ is }\mathcal{F}\text{-free}\}.$$

It goes back to the fundamental paper of Turán [Tur41]. The *Turán density* is

$$\pi(\mathcal{F})=\lim_{n\to\infty}\frac{\mathrm{ex}(n,\mathcal{F})}{\binom{n}{k}};$$

it not hard to show that the limit exists. If $\mathcal{F}=\{F\}$, then we abbreviate $\mathrm{ex}(n,\{F\})$ to $\mathrm{ex}(n,F)$, etc.

We refer the reader to the surveys by Füredi [Für91], Sidorenko [Sid95], and Keevash [Kee11].

2 Open Questions

2.1 Complete Hypergraphs

Let $K_m^k=\binom{[m]}{k}$ be the complete k-graph on m vertices. Erdős offered a money prize for determining $\pi(K_m^k)$ for at least one pair k,m with $m>k\geq 3$; the highest money value of the prize we found in the literature is \$3000 (Frankl and Füredi [FF84, Page 323]). It is still unclaimed.

Conjecture 1 (Turán [Tur41]) $\pi(K_4^3)=\frac{5}{9}$.

There are many different constructions that achieve the lower bound (see Brown [Bro83], Kostochka [Kos82], and Fon-der-Flaass [FdF88]), which is one of the reasons why this problem is so difficult. Successively better upper bounds were proved by de Caen [dC88], Giraud (see [CL01]), and Chung and Lu [CL01]. Razborov's [Raz10a] flag algebra approach suggests that $\pi(K_4^3) \leq 0.561...$ (and, if needed, this can be converted into a rigorous proof).

Fon-der-Flaass [FdF88] presented a construction of K_4^3-free graphs from digraphs. A weakening of Conjecture 1 is that Fon-der-Flaass' construction cannot beat $\frac{5}{9}$; some progress in this direction was made by Razborov [Raz10b].

Kalai [Kal85] (see also [Kee11, Section 11]) presented an interesting approach to $\pi(K_4^3)$.

Conjecture 2 (Turán) $\pi(K_m^3) = 1 - \left(\frac{2}{m-1}\right)^2$.

A construction that achieves the lower bound can be found in [Sid95, Section 7]. Mubayi and Keevash (see [Kee11, Section 9]) found a different construction (via digraphs).

Let us mention here another very interesting question for whose solution de Caen [dC94, Page 190] offered 500 Canadian dollars.

Conjecture 3 *Does $k(1-\pi(K_{k+1}^k))$ tend to ∞ as $k \to \infty$?*

We know that $1 \leq k(1-\pi(K_{k+1}^k)) \leq (\frac{1}{2} + o(1)) \ln k$ (see also Lu and Zhao [LZ09]).

Finally, we refer the reader to the survey by Sidorenko [Sid95] that discusses $\pi(K_m^k)$ for some other k, m.

2.2 K_4^3 Minus an Edge

Let K_4^- be obtained from K_4^3 by removing one edge.

Conjecture 4 (Mubayi [Mub03]) $\pi(K_4^-) = \frac{2}{7}$.

The lower bound come by partitioning $[n]$ into six parts, taking certain 10 complete 3-partite 3-graphs, and then recursively repeating the same construction inside each part, see [FF84, Page 323].

The best known upper bounds come from flag algebra computations: Baber and Talbot [BT10] (by using the method of Razborov [Raz10a] and generating a larger SDP program than that in [Raz10a]) showed that $\pi(K_4^-) \leq 0.2871$.

2.3 Turán Function for Books

Let *the book* $B_{k,m}$ consist of m edges sharing $k-1$ common points plus one more edge that contains the remaining m points and is disjoint otherwise. Let us exclude the case $m \leq 1$ when $\pi(B_{k,m}) = 0$. The hypergraph problems for books turned out (relatively) more tractable. We know $\mathrm{ex}(n, B_{k,m})$ exactly for all large n, when $2 \leq m \leq k \leq 4$, see [Bol74, FF83, FPS03, KM04, FPS05, FPS06, FMP08, Pik08].

Also, Frankl and Füredi [FF89] determined $\pi(B_{k,2})$ for $k = 5, 6$; in both cases the lower bounds comes by blowing up a small design. The following question is still open (see Frankl and Füredi [FF89, Conjecture 1.5]):

Problem 5 *Determine* $\mathrm{ex}(n, B_{5,2})$ *and* $\mathrm{ex}(n, B_{6,2})$ *exactly for all large* n*.*

One difficulty in proving Problem 5 is that it is not clear how to prove *the stability property*, that is, that all almost extremal graphs have similar structure.

Conjecture 6 ([FMP08, BFMP10])

$$\begin{aligned} \pi(B_{5,5}) &= \frac{40}{81}, \\ \pi(B_{6,6}) &= \frac{1}{2}. \end{aligned}$$

The lower bounds come from a "bipartite" construction. It was proved in [BFMP10] that $\pi(B_{5,5}) \leq 0.534...$ and that the bipartite construction is not optimal for $\pi(B_{k,k})$ when $k \geq 7$.

The Turán density is unknown for $B_{5,3}$ and $B_{5,4}$ which is an interesting (and perhaps tractable) open problem.

2.4 Tight 5-Cycle

Mubayi and Rödl [MR02] have given bounds on $\pi(C_5^3)$, where C_5^3 is the tight 3-graph 5-cycle:

$$C_5^3 = \{123, 234, 345, 451, 512\}.$$

In particular, the lower bound $\pi(C_5^3) \geq 2\sqrt{3} - 3$ comes from the following construction: partition the vertex set into two parts A and B, take all triples that intersect A precisely in 2 vertices, and recursively repeat this construction within B. Finding the optimal ratio between $|A|$ and $|B|$ gives the required. Razborov's [Raz10a] flag algebra computations showed that $\pi(C_5^3) < 0.4683$ (note that $2\sqrt{3} - 3 = 0.4641...$). This makes the following conjecture plausible.

Conjecture 7 $\pi(C_5^3) = 2\sqrt{3} - 3$.

2.5 Tight 5-Cycle Minus an Edge

Let the 3-graph C_5^- be obtained from C_5^3 by removing one edge. An example of a C_5^--free 3-graph can be obtained by taking a complete 3-partite 3-graph and repeating this construction recursively within each of the three parts. This gives density $1/4$ in the limit.

Conjecture 8 $\pi(C_5^-) = 1/4$.

2.6 Ruzsa-Szemerédi Theorem and Relatives

Let $f^r(n,s,p)$ be the largest size of an r-graph G on n vertices such that no set of s vertices spans at least p edges. For example, the celebrated theorem of Ruzsa and Szemerédi [RS78] states that $f^3(n,6,3) = o(n^2)$.

Conjecture 9 (Erdős, Frankl, and Rödl [EFR86]) *For any $r \geq 3$ and $s \geq 4$ we have*

$$f^r(n, s(r-2)+3, s) = o(n^2).$$

In [SS05] it is proved that $f^r(n, s(r-2) + \lfloor \log_2 s \rfloor, s) = o(n^2)$. The first remaining open case is to prove the conjecture for $f^3(n,7,4)$ (probably very hard).

One possible direction here is to look at multiple hypergraphs (when the same r-tuple can appear a multiple number of times) and ask for $F^r(n,p,s)$ maximum size of an r-multi-hypergraph such that every s-set spans at most p edges. See Füredi and Kündgen [FK02] for results in the graph case ($r = 2$).

A related question is as follows. Let A, B, and C be disjoint sets each of size n. Let $M_1, \ldots, M_l$, be matchings, where each edge of M_i has one point in each of A, B, and C. The forbidden configuration is: three edges abc, $a'b'c'$, $a''b''c''$ all in some M_i and one edge of the form $ab'c''$ in some other M_j (that is, the edge from M_j crosses the three edges of M_i). Additionally, we require that the union of all matchings M_i makes a simple (linear) 3-graph, call it M.

Conjecture 10 (Frankl-Rödl (see [ENR90])

$$|M| = o(n^2).$$

If true, this implies Roth's Theorem (every set of integers of positive upper density has a 3-term arithmetic progression). An obvious generalization is to consider the k-graph version where we have parts $A_1, ..., A_k$, each of size n, and instead of three edges in M_i we take k edges in M_i and another crossing edge as the forbidden configuration; as before, we require that the union M is a linear k-graph. Again, we believe that $|M| = o(n^2)$ and, if true, this would imply Szemerédi's Theorem. The case $k = 2$ is equivalent to the Ruzsa–Szemerédi Theorem that $f^3(n,6,3) = o(n^2)$.

The following conjecture seems to be related.

Conjecture 11 (Solymosi, Oberwolfach 2011) *Let F be a graph and $\alpha > 1$ be such that $\mathrm{ex}(n,F) = \Omega(n^\alpha)$. Then for any $\epsilon > 0$ there is n_0 so that if $n > n_0$ and a graph H is the edge-disjoint union of $m = \lceil \epsilon n^\alpha \rceil$ copies of F, then H contains another copy of F (i.e. has at least $m+1$ copies of F).*

The Removal Lemma implies that this is true when F is not bipartite.

2.7 Beyond the Turán Threshold

Erdős [Erd94] (see also [CG98, Page 93]) conjectured that for any $k \geq 4$ every 3-graph G with n vertices and $\mathrm{ex}(n, K_k^3)+1$ edges contains at least two copies of K_k^3. Even more strongly: it was conjectured that G must contain K_{k+1}^3 minus one edge.

Another interesting question is to estimate the number of copies of F that any k-graph $G \subseteq \binom{[n]}{k}$ with $\mathrm{ex}(n, F) + q$ edges must have. Some partial results (for those F for which we know $\mathrm{ex}(n, F)$) were obtained by Mubayi [Mub].

2.8 Tic-tac-toe Turán-type problem of Elekes

Let *tic-tac-toe* T be the 3-graph of order 9 and size 6 where the edges correspond to the rows and columns of the 3×3-tic-tac-tow board.

Elekes [Col99, Problem 4] asked the following problem.

Problem 12 *Dis/prove that the maximum size of $\mathcal{F} \subset \binom{[n]}{3}$ not containing T nor two edges intersecting in two points is $o(n^2)$.*

2.9 Bipartite Links

Given a k-graph G and a vertex x, its *link*

$$G_x = \{Y : Y \not\ni x,\ Y \cup \{x\} \in G\}$$

is the collection of $(k-1)$-sets that together with x form edges of G. The following conjecture is attributed to Erdős and Sós in [FF84, Page 238].

Conjecture 13 *If every link of $G \subseteq \binom{[n]}{3}$ is bipartite, then $|G| \leq (\frac{1}{4} + o(1))\binom{n}{3}$.*

Here are some possible constructions. Take a random tournament of order n and take all those triples which span directed 3-cycle. Now, let x be any vertex. The partition $N_{out}(x) \cup N_{in}(x)$ is the required bipartition of G_x. Another construction that achieves $1/4$ is to take a complete 3-partite 3-graph and repeat this construction recursively inside each part. Yet another construction is to place all n points equally around a circle and let three points to form an edge if the triangle spanned by them contains the center of the circle.

Clearly, Conjecture 13 can be rephrased as the Turán question for the obvious (infinite) family $\mathcal{B}$. By looking at two specific members of $\mathcal{B}$, namely, K_4^- and C_5^3 (defined above), Razborov [Raz10a] obtained 0.266 as an upper bound.

Problem 14 (Füredi, Oberwolfach 2004) *What happens if we require that $\chi(G_x) \leq k$ for any vertex x?*

2.10 Worst Graphs of Given Size

Sidorenko [Sid89] (see [Kee11, Section 6]) proved that for every k-graph F with f edges we have $\pi(F) \le \frac{f-2}{f-1}$. This gives $1/2$ if $f = 3$. If k is even, this is best possible, see [Fra90, KS05].

Problem 15 *Given odd $k \ge 3$, determine/estimate the smallest $\gamma = \gamma(k)$ such that $\pi(F) \le \gamma$ for every k-graph with 3 edges.*

For $k = 3$, we know that the answer is given by $F = K_4^-$ but we do not know $\pi(K_4^-)$ exactly. Perhaps, one may be able to describe all extremal graphs for $\gamma(k)$ without knowing its value. It is still open if $\gamma(5) < 1/2$, see [Kee11, Section 6].

2.11 Maximizing Lagrangian

The *Lagrangian* Λ_G of a k-graph $G \subseteq \binom{[n]}{k}$ is the maximum of

$$\lambda_G(x_1, \ldots, x_n) = \sum_{D \in G} \prod_{i \in D} x_i,$$

over all non-negative $x_i \ge 0$ with sum 1.

Conjecture 16 (Frankl and Füredi [FF89]) *or given k and $m = |G|$, the maximum of Λ_G is attained by the initial colex segment.*

See Talbot [Tal02] for a partial progress in this direction. If the conjecture is true, this would greatly simplify the proof in [FF89] and might have other interesting consequences.

2.12 Ramsey-Turán Problems

For a k-graph F, the *Ramsey-Turán function* $RT(n, F, l)$ is the maximum size of an F-free k-graph $G \subseteq \binom{[n]}{k}$ with independence number less than l (that is, every l-set spans at least one edge in G).

Various ranges of l lead to meaningful and interesting questions. A good starting point is the paper by Erdős and Sós [ES82] that contains a number of open Ramsey-Turán problems for hypergraphs.

The following two problems from [SS01] are also interesting.

Problem 17 *Find a function $f(n) \to \infty$, "not too small", for which*

$$RT(n, K_4^3, f(n)) = o(n^3).$$

Problem 18 *Is it true that for any r-graph H there is a* threshold*, that is, $f(n)$ such that*

$$RT(n, H, g(n)) = \begin{cases} o(n^3), & \text{if } g(n)/f(n) \to 0, \\ \Theta(n^3), & \text{if } g(n)/f(n) \to \infty. \end{cases}$$

Here a rare example of a Turán-type problem where the hypergraph case has been solved but which is still open for graphs. For a k-graph family $\mathcal{F}$, let the *Ramsey-Turán* density $\rho(\mathcal{F})$ be the supremum over all functions f such that $f(n) = o(n)$ of $RT(n, \mathcal{F}, f(n))/\binom{n}{k}$. It was shown in [MP08] that for every $k \geq 3$ there are two k-graphs F and G such that

$$\rho(\{F, G\}) < \min(\rho(F), \rho(G)). \tag{1}$$

The question whether there are two graphs satisfying (1) is still open [MP08, Problem 1]. For the Turán density, the analogous questions have been resolved ([Bal02, MP08]).

2.13 Co-Degree Density of Hypergraphs

The following problem was first systematically studied by Mubayi and Zhao [MZ07]. The *codegree* $C(G)$ of a k-graph G is the minimum size of the *link set*

$$G_A = \{x \in V(G) \setminus A : A \cup \{x\} \in G\}$$

over $(k-1)$-sets $A \subseteq V(G)$. For a family $\mathcal{F}$ of forbidden k-graphs, co-ex$(n, \mathcal{F})$ be the largest codegree in an $\mathcal{F}$-free k-graph on n vertices. Let

$$\gamma(\mathcal{F}) = \lim_{n\to\infty} \frac{\text{co-ex}(n, \mathcal{F})}{n}.$$

(It is shown in [MZ07] that the limit exists.)

Czygrinow and Nagle [CN01] conjectured that $\gamma(K_4^3) = \frac{1}{2}$ (see [MZ07] for a few other related questions).

Mubayi and Zhao showed that the numbers $\gamma(\mathcal{F})$ are dense in $[0, 1)$.

Problem 19 (Mubayi and Zhao [MZ07]) *Characterize all possible values of $\gamma(\mathcal{F})$ for (possibly infinite) $\mathcal{F}$. Can every real $\alpha \in (0, 1)$ be realized?*

Similar questions can be asked about the *l-th codegree* $C_l(G)$, the minimum size of the link $(k-l)$-graph G_L over all l-sets L.

2.14 Weakly Triangle-Free Hypergraphs

Bollobás [Bol74] conjectured that the maximum size of a triangle-free k-graph of order n is achieved for the complete k-partite hypergraph. Shearer [She96] showed that this conjecture is false in general.

Call a hypergraph G *weakly triangle-free* if no edge contains strictly more than half of vertices from the symmetric difference of two edges. Mubayi and Pikhurko (unpublished) asked the following.

Problem 20 *Is it true that the maximum size of a weakly triangle-free k-graph on n vertices is attained for k-partite graph?*

2.15 Expanded K_4

To define the $2k$-graph K_4^{2k} take four disjoint k-sets $A_1, \ldots, A_4$ and let $A_i \cup A_j$ be edges. Thus we have $4k$ vertices and 6 edges. Let us call it *expanded* K_4.

Frankl [Fra90] showed that the Turán density $\pi(K_4^{2k}) \leq 2/3$. Keevash and Sudakov [KS05] showed that it is at most $2/3 - \varepsilon$ for some $\varepsilon > 0$.

Problem 21 *Determine the value of* $\pi(K_4^{2k})$.

Keevash [Kee11, Section 8] presents a new construction that shows $\pi(K_4^4) \geq 9/14$.

2.16 Hypergraphs Without Generalized 4-Cycle

Erdős [Erd77] stated the following problem. Determine $f_r(n)$, the maximum number of edges in r-graph on n vertices that does not contain four edges A, B, C, D with $A \cup B = C \cup D$ and $A \cap B = C \cap D = \emptyset$.

Füredi [Für84] proved that

$$\binom{n-1}{r-1} + \left\lfloor \frac{n-1}{r} \right\rfloor \leq f_r(n) < \frac{7}{2}\binom{n}{r-1}. \tag{2}$$

Even the case $r = 3$ is open. Improving on the previous upper bounds of Mubayi and Verstraëte [MV04], Pikhurko and Verstraëte [PV09] showed that that $f_3(n) \leq \frac{13}{9}\binom{n}{2}$.

Problem 22 *Improve on these bounds.*

2.17 Turán Function of Tight Cycles

The *tight 3-uniform cycle* C_t^3 is a 3-graph on t vertices $v_1, \ldots, v_t$ whose edges are $v_1v_2v_3, v_2v_3v_4, \ldots, v_tv_1v_2$. Let t be divisible by 3.

Problem 23 (Conlon (unpublished)) *Is it true that there is a constant c such that if a 3-uniform hypergraph on n vertices has $n^{2+c/t}$ edges then it contains a copy of C_t^3?*

If the answer is in the affirmative, this would imply better upper bounds on the size of a subgraph of the n-hypercube Q_n without a C_{4i+2}-cycle for all large fixed i, see Conlon [Con10]. Here, the *hypercube graph* Q_n has $2^{[n]} = \{X : X \subseteq [n]\}$ as the vertex set where two vertices (subsets of $[n]$) are adjacent if their symmetric difference consists of 1 element only.

2.18 Subsets in Hypercubes

Johnson and Talbot [JT10, Question 13] asked if for any d and $\varepsilon > 0$ there is n_0 such that if A is a vertex subset of the hypercube Q_n with $n \geq 0$ and

$|A| \geq \varepsilon 2^n$ then we can fund a d-dimensional subcube that contains at least $\binom{d}{\lfloor d/2 \rfloor}$ elements from A.

Independently, Bollobás and Leader [BL] and Bukh (personal communication) observed that this is equivalent to the following hypergraph Turán problem. For $r \geq s > t$ let the r-graph $S^r K_s^t$ have vertex set $V = [s+t-r]$ and edge set

$$\{D \in \tbinom{V}{r} : D \supseteq [r-t]\}.$$

(Thus if we remove $[r-t]$ from every edge, we get a copy of K_s^t.) Then the following question is equivalent to that of Johnson and Talbot.

Problem 24 *Is it true that for any $s > t \geq 2$ we have $\lim_{r\to\infty} \pi(S^r K_s^t) = 0$?*

Even the case $s = 4$ and $t = 2$ is currently open.

Finally, there are some interesting open questions (some very old) that may be also related to the hypergraph Turán problem, where one asks about the maximum size of a vertex/edge set in the hypercube Q_n without inducing a copy of Q_d. The recent paper [JT10] gives some references and is a good starting point for learning more about these questions.

3 Further Open Questions

Here are some of the questions that came up during the AIM Workshop March 21–25, 2011. One of the objectives of the workshop was to find new approaches and points of view on the hypergraph Turán problem, so some of these questions may be rather easy (or even not well-defined).

1. (Vera Sós.) Let $\mathcal{C}$ consist of all tight 3-graph cycles. Estimate $\mathrm{ex}(n, \mathcal{C})$. A star shows that $\mathrm{ex}(n, \mathcal{C}) \geq \binom{n-1}{2}$.

2. (Jacob Fox.) Does there exist a single r-graph F such that $\pi(F)$ is transcendental? What about families of forbidden hypergraphs (finite or infinite)?

3. (Benny Sudakov.) Let H be an n-vertex 3-uniform hypergraph. A simple probabilistic argument shows that $\alpha(H) = \Omega(\frac{n}{\sqrt{d}})$, where d is average degree. Erdős [Erd81, Page 52] asked if this can be improved if we require that H does not contain K_4^3. Duke, Rödl, and Lefmann [DLR95] (see also Ajtai et al [AKP$^+$82]) showed that if H does not have two edges sharing a pair (i.e. K_4^3 minus two edges), then the conjecture is true.

4. (De Caen [dC94].) A 3-graph H is *c-sparse* if every set S of vertices spans at most $c|S|^2$ edges. Prove that for every c there is $f(n) \to \infty$ as $n \to \infty$ such that $\alpha(H) \geq f(n)\sqrt{n}$ for each c-sparse n-vertex 3-graph H. Mubayi asked if $\alpha(H) \gg \frac{n}{\sqrt{d}}$ for an $O(1)$-sparse H with average degree d.

5. (Oleg Pikhurko.) Find $\alpha < \beta < 1$ such that α is a non-jump but β is a jump for 3-graphs.

6. (Linyuan Lu and László Székely.) Is it true that $\pi(H) \le 1 - \frac{1}{d+1}$ for any r-graph H such that each edge intersects at most d other edges. Lu and Székely can prove that $\pi(H) \le 1 - \frac{1}{e(d+1)}$, using the Local Lemma. The intuition here is that the complete graph should be the worst.

   Sidorenko [Sid89] showed that for $\pi(H) \le \frac{f-1}{f-2}$, where f is the number of edges. Keevash [Kee05] improved this when $r \gg f$.

7. (Jacob Fox and Benny Sudakov.) Let $P_r(n, \epsilon, d)$ be the smallest L such that every r-graph G on n vertices has an edge partition $E_0 \cup \ldots \cup E_l$ such that $l \le L$, $|E_0| \le \epsilon n^r$ while for every $x, y \in V(G)$ and every $i \in [l]$ there is a tight path between x and y in E_i. This can viewed as partitioning an r-graph into "small worlds." Conjecture: for fixed $\epsilon > 0$ there is c_r such that for every n we have $P_r(n, \epsilon, 3) \le c_r \epsilon^{-r}$. Fox and Sudakov [FS10] gave a construction which shows that this is tight. They can prove $P_r(n, \epsilon, d) \le c_r \epsilon^{-2^r}$.

   This is related to a Turán problem. Let H be the following r-graph. The vertex set $V(H) = U \cup W$ where $U = \{u_1, \ldots, u_r\}$ and $W = \{w_1, \ldots, w_r\}$ are disjoint. The edge set consists of U, W, and $W \setminus \{w_i\} \cup \{u_i\}$ for $i \in [r]$. (Thus $e(H) = r + 2$.) Also, H has diameter 3 and H is r-partite. Question: estimate limsup/liminf of $\log \mathrm{ex}(n, H)/\log n$.

   Note: H is contained in the complete r-partite hypergraph with parts of size 2. So by Erdős [Erd64], $n^{r - \frac{1}{2^{r-1}}}$ edges are enough to find the complete r-partite hypergraph. Fox and Sudakov think that the exponent should be $r - \Omega(\frac{1}{r})$.

8. (Miklós Simonovits.) Is it true that $\mathrm{ex}(n, K^3_{2,2,2}) = \Omega(n^{11/4})$? The best known upper bound is $O(n^{11/4})$, due to Erdos [Erd64]. Katz, Krop, and Maggioni [KKM02] proved $\Omega(n^{8/3})$ as a lower bound, see also Gunderson, Rödl and Sidorenko [GRS99]. Mubayi [Mub02, Conjecture 1.4] conjectured that for $r > 1$ and $s_1 \le \cdots \le s_r$

$$\mathrm{ex}(n, K^r(s_1, \ldots, s_r)) = \Theta(n^{r-1/s}),$$

   where $s = \prod_{i=1}^{r-1} s_i$.

9. (József Balogh.) The *chromatic threshold* of F is the infimum of all d such that for all large n, every n-vertex F-free 3-graph with minimum degree $\ge d\binom{n}{2}$ has bounded chromatic number. See Balogh et al [BBH$^+$11] for more details.

   Let the 3-graph F_5^3 have vertices a, b, c, d, e and edges $\{a, b, c\}$, $\{b, c, d\}$, and $\{d, e, a\}$. What is the chromatic threshold for F_5^3? Conjecture: it is $\frac{6}{49}$.

10. (Mathias Schacht.) Let F be a 3-graph. Let $\tilde{\pi}(F)$ be the infimum over all d such that for every $\epsilon > 0$ there is $\delta > 0$ and n_0 such that for every F-free 3-graph H with $n \ge n_0$ vertices there is $U \subseteq V(H)$ with $|U| \ge \delta n$

and the density of $H[U]$ being at most $d + \epsilon$. This differs from the usual Turán function in that now every linear size subset is not too sparse.

See eg Rödl [Röd86, Page 133] for a construction that $\tilde{\pi}(K_4^3) \geq 1/2$. Is this tight?

Question: Are there any non-jumps (appropriately defined)?

Question: Is $\tilde{\pi}(K_4^3$ minus an edge$) = \frac{1}{4}$?

Question: For which 3-graphs F is $\tilde{\pi}(F) = 0$? It is known to be 0 all simple hypergraphs and, of course, for all 3-partite hypergraphs. Also, it is 0 for all blow-ups of simple hypergraphs. Are there any other cases when it is 0?

For a related version with fewer parameters (easier to state), one can eliminate δ and instead consider those U with $|U| \geq \frac{n}{\log n}$.

11. (Miklós Simonovits.) Let $\mathcal{F}$ be a family of graphs. For large n, we maximize the edge density of a graph G_n such that $|G_n| = n$, G_n is $\mathcal{F}$-free, and $\alpha(G_n) = o(n)$. Is it true that for any $\mathcal{F}$, there is a constant R such that there is an asymptotically extremal graph G_n admitting a partition $V(G) = V_1 \cup \ldots \cup V_r$ such that $r \leq R$ and the density between any two classes $d(V_i, V_j)$, $1 \leq i \leq j \leq r$, is $o(1)$, $1 + o(1)$, or $\frac{1}{2} + o(1)$?

12. (John Goldwasser.) What is the Turán density for $F_{1,r}^3$, the 3-graph which is a star? Its vertices are $\{v_0, v_1, \ldots, v_r\}$. Edges are all $\{v_0, v_i, v_j\}$, with $1 \leq i < j \leq r$.

    If we take the complement of the Fano place and blow it up, it is $F_{1,4}$-free (since each link graph is 3-partite). We can repeat this recursively within each of the 7 parts, getting in limit edge density 1/2.

    If $r = 5$, one can base a construction on the projective plane of order 3, which has 13 points and 13 lines. Take all triples that are not collinear, blow this up, and do recursion inside each of the 13 parts.

    Similar works whenever we have a projective plane. Goldwasser conjectures that these contructions (when a projective plane exists) give the exact value of $\pi(F_{1,r})$.

13. (Fan Chung.) Let r, n and e be given. Find the maximum size of an r-graph which is contained in every r-graph on n vertices and e edges. This is still open in general; the cases $r = 2, 3$ are studied in Chung and Erdős [CE83, CE87].

14. (József Solymosi.) Let $E(F) = \{abc, cde, acg, cfh, ghi, bej\}$ and $E(H) = \{abc, cde, acg, cfh, ghi, bei\}$. (Thus H is obtained from F by identifying vertices i and j.) Prove that every simple n-vertex 3-graph without F or H has $o(n^2)$ edges. This is related to $(7, 4)$-problem and, if true, would have interesting applications.

15. (Dhruv Mubayi and Jacques Verstraëte.) What is the maximum number of edges in n-vertex r-graph if it has no 2-regular subgraph? If r is even, one can take all r-tuples containing some vertex point. This was shown to be optimal for all large n in [MV09].

    If r is odd, one improve the above construction by adding a matching disjoint from the special vertex. It is conjectured in [MV09, Conjecture 1] that $\binom{n-1}{r-1} + \lfloor \frac{n-1}{r} \rfloor$ is best possible for all large n.

16. (David Conlon.) Embed K_{2^n} in $\mathbb{R}^n$ so that its vertices make the n-cube. Ramsey's theorem implies that if we color all edges red/blue, then there is a *planar* monochromatic K_4 (that is, its 4 vertices lie in one plane).

    An open question is to determine the maximum size of a subgraph of K_{2^n} without a planar K_4. The trivial lower bound is $\frac{2}{3} + o(1)$ (just take a K_4-free subgraph). If it is is sharp, then this problem is probably of comparable difficulty to the Density Hales Jewett Theorem.

    One can also ask for a planar triangle plus 1 pendant edge: is the maximum edge density $\frac{1}{2} + o(1)$ here?

17. (Dhruv Mubayi.) Motivated by Conjecture 3, consider the following hypergraphs. Let $k \in \mathbb{N}$, and $s, t \in [2, k]$. Let $F^k_{s,t}$ be the hypergraph whose vertex set is $S \cup T$, where $S \cap T = \emptyset$, $|S| = s$, $|T| = t$, and $F^k_{s,t}$ consists of all edges containing S and all edges containing T. Notice that $F^k_{2,k-1} = K^k_{k+1}$.

    Problem 25 *Determine the rate of growth of* $1 - \pi(F^k_{i,k-1})$ *as* $k \to \infty$ *and* $i = i(k)$. *In particular, is* $\pi(F^k_{k-1,k-1}) = 1 - \Theta(\frac{\log k}{k})$*?*

18. (Dhruv Mubayi and Jacques Verstraëte.) Is there a 5-uniform hypergraph F such that for some $c > 0$ and all large n we have

    $$n^{3+c} \leq \mathrm{ex}(n, F) \leq n^{4-c}?$$

    Note that we forbid only one 5-graph (not a family). Also, Mubayi and Verstraëte (unpublished) can show that $n^{i+c} \leq \mathrm{ex}(n, F) \leq n^{i+1-c}$ is impossible for $i = 0, 1, 2$ (for a 5-graph F).

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