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Constructions of non-principal families in extremal hypergraph theory

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For Miklós Simonovits on his 60th birthday

Abstract

A family $\mathscr{F}$ of k-graphs is called *non-principal* if its Turán density is strictly smaller than that of each individual member. For each $k \geqslant 3$ we find two (explicit) k-graphs F and G such that $\{F, G\}$ is non-principal. Our proofs use stability results for hypergraphs. This completely settles the question posed by Mubayi and Rödl [On the Turán number of triple systems, J. Combin. Theory A, 100 (2002) 135–152].

Also, we observe that the demonstrated non-principality phenomenon holds also with respect to the Ramsey–Turán density as well.

Keywords: Turan density; Ramsey–Turan density; Stability; Non-principality

1. Introduction

In this paper, we prove the non-principality phenomenon for the classical extremal problems for k-uniform hypergraphs. The main motivation is to study the qualitative difference between the cases $k=2$ and $k \geqslant 3$, and our results for the Turán problem exhibit this difference. We also study this question in the context of Ramsey–Turán theory, introduced by Erdős and Sós. Although we prove the non-principality phenomenon for Ramsey–Turán problems when $k \geqslant 3$, the behavior for $k=2$ remains open. This is one of the few cases where an extremal problem for hypergraphs can be solved but not for graphs.

Given a family $\mathscr{F}$ of k-uniform hypergraphs (k-*graphs* for short), let

$$\mathrm{ex}(n, \mathscr{F}) := \max\{|G| : v(G) = n,\ \mathscr{F} \not\subset G\}$$

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be the maximum size of a k-graph G on n vertices which is $\mathscr{F}$*-free* (that is, for every $F \in \mathscr{F}$ we have $F \not\subset G$). It was observed by Katona, Nemetz, and Simonovits [13] that the ratio $\mathrm{ex}(n, \mathscr{F})/\binom{n}{k}$ is non-increasing with n. In particular, the limit

$$\pi(\mathscr{F}) := \lim_{n\to\infty} \frac{\mathrm{ex}(n, F)}{\binom{n}{k}}$$

exists; we call $\pi(\mathscr{F})$ the *Turán density* of $\mathscr{F}$. When $\mathscr{F} = \{F\}$ consists of a single forbidden k-graph, we write $\mathrm{ex}(n, F)$ and $\pi(F)$ for $\mathrm{ex}(n, \{F\})$ and $\pi(\{F\})$.

Mubayi and Rödl [18] conjectured that there is a family $\mathscr{F}$ of 3-graphs such that

$$\pi(\mathscr{F}) < \min\{\pi(F) : F \in \mathscr{F}\}, \tag{1}$$

and commented that the result should even hold for a family $\mathscr{F}$ of size two. Balogh [1] proved the conjecture, calling this phenomenon the *non-principality* of the Turán function. This is in sharp contrast with the case of graphs ($k = 2$) where the Erdős–Stone–Simonovits Theorem [7,3] applies, giving that

$$\pi(\mathscr{F}) = \min\left\{1 - \frac{1}{\chi(F) - 1} : F \in \mathscr{F}\right\} = \min\{\pi(F) : F \in \mathscr{F}\}.$$

Unfortunately, the family in [1] has many members, many more than two. In Section 2 we present a new approach which shows how the so-called stability results lead to families $\mathscr{F}$ satisfying (1) and consisting of two k-graphs only. Combined with the authors' recent results [16,19] this approach allows us to prove that for every $k \geqslant 3$, there is a non-principal k-graph family $\mathscr{F}$ with $|\mathscr{F}| = 2$, thus completely answering the question in [18] (see also Balogh [1, p. 177]).

In Section 3 we show how to extend the ideas of Balogh [1] to arbitrary k-graphs, $k \geqslant 3$. Although this seems to give non-principal families having many elements (a result weaker than that in Section 2), this method is very simple and self-contained. So we include it too.

Many of the (conjectured) extremal examples for (hyper)graph Turán problems have large independent sets. Motivated by this observation, Erdős and Sós [5] restricted the underlying k-graphs in this problem, by requiring that they have no large independent sets. This new class of problems has become known as the *Ramsey–Turán problems*. More precisely, for $0 < \delta \leqslant 1$,

$$\mathrm{ex}(n, \mathscr{F}, \delta) = \max\left\{|G| : G \subset \binom{[n]}{k} \text{ s.t. } G \text{ is } \mathscr{F}\text{-free and } \alpha(G) < \delta n\right\},$$

or zero if no such hypergraph exists. The *Ramsey–Turán density* $\rho(\mathscr{F})$ is defined as

$$\sup_{\delta(n)}\left\{\limsup_{n\to\infty} \frac{\mathrm{ex}(n, \mathscr{F}, \delta(n))}{\binom{n}{k}} : \ \delta(n) \to 0 \text{ as } n \to \infty\right\}.$$

Very little is known about the parameter ρ for k-graphs. Indeed, computing $\rho(\mathscr{F})$ seems even more difficult than computing $\pi(\mathscr{F})$. For example, it is unknown whether $\rho(K_{2,2,2})$ is 0, where $K_{2,2,2}$ is the complete 3-partite graph with two vertices in each part.

Based on the complexity of determining ρ, one would suspect that ρ is also non-principal in the sense of this paper. Indeed, this is the case. Let $\mathscr{F}$ be any k-graph family satisfying (1). Let $\mathscr{F}' = \{F(2) : F \in \mathscr{F}\}$ be obtained from $\mathscr{F}$ by blowing-up each member with factor 2 (that is, each vertex is cloned once). Erdős and Sós [6] proved that if for any edge $D \in F$ there is another edge $D' \in F$ with $|D \cap D'| \geqslant 2$, then $\rho(F) = \pi(F)$. This result extends easily to families of k-graphs, yielding that $\rho(\mathscr{F}') = \pi(\mathscr{F}')$ and $\rho(F) = \pi(F)$ for any $F \in \mathscr{F}'$. By the supersaturation result of Erdős and Simonovits [4], blow-ups preserve the Turán density: $\pi(\mathscr{F}) = \pi(\mathscr{F}')$. Hence, $\mathscr{F}'$ is non-principal with respect to both the Turán and Ramsey–Turán densities.

Curiously, the situation with graphs remains open.

Problem 1. *Do there exist graphs G_1, G_2 for which*

$$\rho(\{G_1, G_2\}) < \min\{\rho(G_1), \rho(G_2)\}?$$

What about if we require $\rho(\{G_1, G_2\}) > 0$ as well?

2. Non-principal families of size 2

We need some preliminary definitions before we can start proving the claimed results.

To obtain the *cone* $\mathrm{cn}(F)$ of a k-graph F, enlarge each edge of F by a new common vertex x:

$$\mathrm{cn}(F) := \{\{x\} \cup D : D \in F\}.$$

Let $\mathrm{cn}^{(i)}(F)$ be obtained from F by iterating the cone-operation i times. Define $\mathrm{cn}(\mathscr{F}) := \{\mathrm{cn}(F) : F \in \mathscr{F}\}$. Let K_m^k be the complete k-graph on m vertices.

We call two order-n k-graphs F and G *ε-close* if there is a bijection $f : V(F) \to V(G)$ between their vertex sets such that the number of k-subsets $D \subset V(F)$ for which $D \in F \not\Leftrightarrow f(D) \in G$ is at most $\varepsilon\binom{n}{k}$. In other words, we can make F isomorphic to G by adding and removing at most $\varepsilon\binom{n}{k}$ edges.

A k-graph G is *F-extremal* if it is a maximum F-free k-graph of order $v(G)$ (that is, $|G| = \mathrm{ex}(v(G), F)$). Let us call a k-graph F *stable* if for any $\delta > 0$ there are $\varepsilon > 0$ and n_0 such that for $n \geqslant n_0$ any F-free k-graph G of order n with at least $(\pi(F) - \varepsilon)\binom{n}{k}$ edges is δ-close to an F-extremal k-graph. Erdős [2] and Simonovits [20] independently proved that every 2-graph is stable.

The following lemma gives us a new approach to generating non-principal families.

Lemma 2. *Let F be a stable k-graph. Suppose that we can find a k-graph H of order h and a constant $c = c(H)$ such that for some n_1:*

1. *$\pi(H) \geqslant \pi(F)$ and*
2. *any F-extremal k-graph of order $n \geqslant n_1$ contains at least cn^h copies of H.*

Then

$$\pi(\{F, H\}) < \min(\pi(F), \pi(H)). \tag{2}$$

Proof. Let $\varepsilon > 0$ and $n_0 \geqslant n_1$ be constants satisfying the stability assumption for F with $\delta = c$.

Suppose on the contrary that $\pi(\{F, H\}) \geqslant \pi(F)$. Then there is a k-graph G_n on $n \geqslant n_0$ vertices such that

1. G_n is $\{F, H\}$-free and
2. $|G_n| > (\pi(F) - \varepsilon)\binom{n}{k}$.

By the definition of ε and n_0, this k-graph G_n is c-close to an F-extremal k-graph G'_n on n vertices. By hypothesis, G'_n contains at least cn^h copies of H. Each edge of G'_n lies in at most $k!\,n^{h-k}$ copies of H. Consequently, in order to delete all copies of H from G'_n, we need to delete at least $(cn^h/k!\,n^{h-k}) > c\binom{n}{k}$ edges from G'_n. This contradicts the fact that G_n is H-free and is c-close to G'_n. □

Theorem 3. *For every $k \geqslant 3$ there are k-graphs F and H satisfying* (2).

Proof. Let the k-graph H_l^k be obtained from the complete 2-graph K_l^2 by enlarging each edge by a set of $k-2$ new vertices. Let $T_k(n, l)$, $k \leqslant l$, be the k-graph obtained by partitioning $[n] = V_1 \cup \cdots \cup V_l$ into l almost equal parts and taking those k-sets which intersect every part in at most one vertex.

Mubayi [16] proved that for arbitrary $l \geqslant k \geqslant 3$ we have

$$\pi(H_{l+1}^k) = \frac{l(l-1)\ldots(l-k+1)}{l^k}. \tag{3}$$

Pikhurko [19], building upon the results in [16], proved that H^k_{l+1} is stable with $T_k(n,l)$ being the unique maximum H^k_{l+1}-free graph of order n for $l \geqslant k \geqslant 3$ and $n \geqslant n_0(k,l)$.

Observe that $\pi(K^k_{k+1}) \geqslant \frac{3}{8} \geqslant \pi(H^k_{k+2})$ for all $k \geqslant 3$. The lower bound on $\pi(K^k_{k+1})$ follows from the following construction. Partition $[n] = A \cup B$ into two almost equal parts. Split the family of all k-sets X intersecting both A and B into two k-graphs G_0 and G_1 according to the parity of $|X \cap A|$. Both G_0 and G_1 are K^k_{k+1}-free: For any $(k+1)$-set $Y \subset [n]$ intersecting A in $s \in [1,k]$ elements, the k-sets $Y \setminus \{a\}$ and $Y \setminus \{b\}$, where $a \in Y \cap A$ and $b \in Y \cap B$, have the intersections with A of sizes $s-1$ and s, respectively, that is, of different parities. Now, $|G_0| + |G_1| = (1 - 2^{-k+1} + o(1))\binom{n}{k}$, so one of these has size at least $\frac{1}{2}(1 - 2^{-k+1} + o(1))\binom{n}{k} \geqslant (\frac{3}{8} + o(1))\binom{n}{k}$. Consequently, $\pi(K^k_{k+1}) \geqslant \frac{3}{8}$.On the other hand, the Turán density $\pi(H^k_{k+2}) = (k+1)!/(k+1)^k$ is a decreasing function of k which equals $\frac{3}{8}$ for $k=3$.

Finally, note that $T_k(n, k+1)$ contains at least $(\lfloor n/(k+1) \rfloor)^{k+1}$ copies of K^k_{k+1}. Lemma 2, whose all assumptions are satisfied, implies that the pair $\{H^k_{k+2}, K^k_{k+1}\}$ is non-principal for all $k \geqslant 3$. □

2.1. *Using other stability results*

Although the paper [19] was accepted by the *Journal of Combinatorial Theory, Series B*, its publication is suspended because of a disagreement between the author and the publisher over the copyright terms.[1] So, it might be useful (and of independent interest) to see for which k we can infer the conclusion of Theorem 3 without referring to [19].

In a recent manuscript [17] we present a self-contained solution of the Turán problem for the *generalized fan* F^k_l which is a k-graph closely related to H^k_l. More precisely, the edge set of F^k_l comprises $[k]$ together with $E_{ij} \cup \{i,j\}$ over all pairs $\{i,j\} \in \binom{[l]}{2} \setminus \binom{[k]}{2}$, where E_{ij} are pairwise disjoint $(k-2)$-sets consisting of vertices outside $[l]$. It was proved that, for $l \geqslant k \geqslant 3$ and all large n, the k-graph F^k_{l+1} is stable and $T_k(n,l)$ is the unique extremal F^k_{l+1}-free graph. Since the same holds if we forbid H^k_{l+1}, the proof of Theorem 3, with obvious modifications, shows that for any $k \geqslant 3$, we have

$$\pi(\{F^k_{k+2}, K^k_{k+1}\}) < \min(\pi(F^k_{k+2}),\ \pi(K^k_{k+1})).$$

The k-graph F^k_{k+2} is in a sense simpler than H^k_{k+2}, having only $2k+2$ edges when compared to $|H^k_{k+2}| = \binom{k+2}{2}$.

Let $k = 2l \geqslant 4$ be even. Let $F = \{A \cup B, A \cup C, B \cup C\}$, where A, B, C are disjoint l-sets. Frankl [8] showed that $\pi(F) = \frac{1}{2}$. Keevash and Sudakov [15, Theorem 3.4] showed that F is stable. Every extremal k-graph G' for F on $n \geqslant n_0$ vertices has vertex partition $X \cup Y$, $|X| \approx |Y| \approx n/2$, and consists of all edges intersecting X (and also Y) in an odd number of vertices. Let us take $H = \mathrm{cn}(K^{k-1}_m)$ where $m = m(k)$ is a sufficiently large integer to satisfy $(k!/m^k)\binom{m}{k} > \frac{1}{2}$. The latter implies that $\pi(H) > \frac{1}{2}$, because $T_k(n,m)$ does not contain H. As G' contains $(2 + o(1))(n/2)\binom{n/2}{m}$ copies of H, Lemma 2 implies that the family $\{F, H\}$ is non-principal.

For $k=3$ we can use the stability result either for the Fano plane, established independently by Füredi and Simonovits [12] and by Keevash and Sudakov [14], or for

$$F_{3,2} := \{\{1,2,3\}, \{1,4,5\}, \{2,4,5\}, \{3,4,5\}\}$$

established by Füredi, Pikhurko, and Simonovits [11] (cf. also [10]). In both cases we can take $H = \mathrm{cn}(K^2_m)$ for some sufficiently large m.

3. Balogh's construction for general $k \geqslant 3$

A partition $V(F) = \cup_{i=1}^m A_i$ of the vertex set of a k-graph F is called a $(k_1, \ldots, k_m)$*-partition* if every edge of F intersects A_i in precisely k_i vertices, $i \in [m]$. Let $\mathcal{N}_{k,l}$ be the (infinite) family consisting of all k-graphs not admitting a $(k-l, l)$-partition. Let the k-graph $T_k := \mathrm{cn}^{(k-2)}(K^2_3)$, where the operation $\mathrm{cn}(F)$ is defined in Section 2.

Theorem 4. *For every $k \geqslant 3$ there exists a finite family $\mathcal{F}$ of k-graphs which satisfies* (1).

[1] See http://www.math.cmu.edu/~pikhurko/Copyright.html for more details.

Proof. Our construction generalizes that of Balogh [1]. We consider first the following (infinite) family:

$$\mathscr{H} = \mathscr{H}_k := \{T_k\} \cup \bigcup_{i=0}^{k-3} \mathrm{cn}^{(i)}(\mathscr{N}_{k-i,1}).$$

We show that $\mathscr{H}$ satisfies (1) (with min replaced by inf) and then explain how to obtain the required finite $\mathscr{F}$ from it.

Let G be any $\mathscr{H}$-free k-graph of order n. We prove by induction on $k \geqslant 2$ that $|G| \leqslant p_k(n)$, where $p_k(n) := \prod_{i=1}^{k} \lfloor (n+i-1)/k \rfloor$ is the maximum size of a k-partite k-graph on n vertices.

The claim is true for $k = 2$, since in this case $\mathscr{H} = \{K_3^2\}$ and (a special case of) Turán's theorem applies. Let $k \geqslant 3$. As $\mathscr{N}_{k,1} \subset \mathscr{H}$, G admits a $(1, k-1)$-partition $A \cup B$. For any $x \in A$ the *link graph*

$$G_x := \{D \not\ni x : D \cup \{x\} \in G\}$$

is $\mathscr{H}_{k-1}$-free. Moreover, all edges of G_x lie inside B by the definition of $A \cup B$. By the induction assumption, $|G_x| \leqslant p_{k-1}(b)$, where $b := |B|$. As each edge intersects A in precisely one vertex, we have

$$|G| = \sum_{x \in A} |G_x| \leqslant (n-b)\, p_{k-1}(b) \leqslant p_k(n),$$

proving the claim.

Thus $\pi(\mathscr{H}) \leqslant k!/k^k$. In fact, we have equality here as k-partite k-graphs demonstrate.

On the other hand, a maximum k-graph G of order n with a $(2, 1, \ldots, 1)$-partition has about $(n/k)^{k-2}\binom{2n/k}{2} \approx 2(k!/k^k)\binom{n}{k}$ edges and is $\mathscr{H} \setminus \{T_k\}$-free. Also, by taking a maximum k-partite k-graph and replacing the last three parts by the T_3-free 3-graph of density $\frac{2}{7}$ constructed by Frankl and Füredi [9] we add $\Omega(n^k)$ edges (note that $p_3(n)/\binom{n}{3} \approx \frac{2}{9} < \frac{2}{7}$). A routine analysis shows that the constructed k-graph is T_k-free.

Hence, $\pi(F) \geqslant \frac{9}{7}\pi(\mathscr{H})$ for every $F \in \mathscr{H}$. There is an n_0 such that, for example, $\mathrm{ex}(n_0, \mathscr{H})/\binom{n}{k} \leqslant \frac{8}{7}\pi(\mathscr{H})$. As $\mathrm{ex}(n, \mathscr{H})/\binom{n}{k}$ is non-increasing, see [13], $\pi(\mathscr{F}) \leqslant \frac{8}{7}\pi(\mathscr{H})$, where $\mathscr{F}$ consists of all k-graphs from $\mathscr{H}$ with at most n_0 vertices. The obtained (finite) family $\mathscr{F}$ has clearly all the required properties. □

References

[1] J. Balogh, The Turán density of triple systems is not principal, J. Combin. Theory A 100 (2002) 176–180.
[2] P. Erdős, Some recent results on extremal problems in graph theory. Results, in: Theory of Graphs (International Symposium, Rome, 1966), Gordon and Breach, New York, 1967, pp. 117–123 (English), 124–130 (French).
[3] P. Erdős, M. Simonovits, A limit theorem in graph theory, Stud. Sci. Math. Hungar. 51–57 (1966).
[4] P. Erdős, M. Simonovits, Supersaturated graphs and hypergraphs, Combinatorica 3 (1983) 181–192.
[5] P. Erdős, V.T. Sós, Problems and results on Ramsey–Turán type theorems (preliminaryy report), Congr. Numer. 26 (1980) 17–23.
[6] P. Erdős, V.T. Sós, On Ramsey–Turán type theorems for hypergraphs, Combinatorica 2 (1982) 289–295.
[7] P. Erdős, A.H. Stone, On the structure of linear graphs, Bull. Amer. Math. Soc. 52 (1946) 1087–1091.
[8] P. Frankl, Asymptotic solution of a Turán-type problem, Graphs Combin. 6 (1990) 223–227.
[9] P. Frankl, Z. Füredi, An exact result for 3-graphs, Discrete Math. 50 (1984) 323–328.
[10] Z. Füredi, O. Pikhurko, M. Simonovits, The Turán density of the hypergraph $\{abc, ade, bde, cde\}$, Electron. J. Combin. 10 (2003) 7.
[11] Z. Füredi, O. Pikhurko, M. Simonovits, Triple systems with empty neighbourhoods, Combin. Probab. Comput. 14 (2005) 795–813.
[12] Z. Füredi, M. Simonovits, Triple systems not containing a Fano configuration, Combin. Probab. Comput. 14 (2005) 467–488.
[13] G.O.H. Katona, T. Nemetz, M. Simonovits, On a graph problem of Turán, Mat. Fiz. Lapok 15 (1964) 228–238 (in Hungarian).
[14] P. Keevash, B. Sudakov, The Turán number of the Fano plane, Combinatorica 25 (2005) 561–574.
[15] P. Keevash, B. Sudakov, On a hypergraph Turán problem of Frankl, Combinatorica 25 (2006) 673–706.
[16] D. Mubayi, A hypergraph extension of Turán's theorem, J. Combin Theory B 96 (2006) 122–134.
[17] D. Mubayi and O. Pikhurko, A new generalization of Mantel's theorem to k-graphs, J. Combin. Theory B 97 (2007) 669–678.
[18] D. Mubayi, V. Rödl, On the Turán number of triple systems, J. Combin. Theory A 100 (2002) 135–152.
[19] O. Pikhurko, Exact computation of the hypergraph Turán function for expanded complete 2-graphs, J. Combin Theory B, accepted for publication, E-print arXiv:math.CO/0510227. The publication is suspended, see ⟨http://www.math.cmu.edu/~pikhurko/Copyright.html⟩.
[20] M. Simonovits, A method for solving extremal problems in graph theory, stability problems, in: Theory of Graphs (Proceedings of Colloqium, Tihany, 1966), Academic Press, New York, 1968, pp. 279–319.