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The minimum number of triangles in graphs of given order and size

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Abstract

In the 1940s and 50s, Erdős and Rademacher raised the quantitative question of determining the number of triangles one can guarantee in a graph of given order and size. This problem has garnered much attention and, in a major breakthrough, was solved asymptotically by Razborov in 2008, whose results were extended by Nikiforov and Reiher. In this paper, we provide an exact solution for all large graphs whose edge density is bounded away from one. This proves almost every case of a conjecture of Lovász and Simonovits from 1975.

Keywords: Extremal graph theory, Erdős-Rademacher problem, supersaturation.

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1 History of the problem

Let $g_r(n,e)$ be the minimum number of r-cliques in a graph with n vertices and e edges. Perhaps the first theorem of extremal combinatorics is Mantel's theorem from 1907 [11], which asserts that the unique largest triangle-free graph on n vertices is the complete balanced bipartite graph $T_2(n) := K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$. Turán's [19] famous generalisation from 1940 states that the largest K_r-free graph is the complete balanced $(r-1)$-partite graph $T_r(n)$. In other words, $g_r(n,e) = 0$ if and only if $e \le t_r(n)$, where $t_r(n)$ is the number of edges in $T_r(n)$.

In this paper we are interested in determining $g_3(n,e)$, the minimum number of triangles in an n-vertex e-edge graph. Rademacher (unpublished) showed in 1941 that in fact $g_3(n,t_2(n)+1) = \lfloor n/2\rfloor$. The unique extremal graph is obtained by adding an edge to the larger class of $T_2(n)$. Erdős [3,4] conjectured the generalisation that $g_3(n,t_2(n)+\ell) = \ell\lfloor n/2\rfloor$ for $\ell < \lfloor n/2\rfloor$. This bound is attained by adding ℓ edges to the larger class of $T_2(n)$ so that these new edges do not span a triangle. The conjecture is readily seen to be false for $\ell \ge \lfloor n/2\rfloor$. Erdős was able to prove his conjecture when $\ell < cn$ for some positive absolute constant c. The conjecture was eventually proved in totality for large n by Lovász and Simonovits in 1975 [9]. This was extended significantly by the same authors who determined $g_3(n,e)$ whenever $t_k(n) \le e \le t_k(n) + \alpha n^2$, where $\alpha = \alpha(k) > 0$. We state this result precisely below.

Let us now turn to the asymptotic problem of determining

$$g_3(\lambda) := \lim_{n\to\infty} \frac{g_3(n, \lfloor \lambda\binom{n}{2}\rfloor)}{\binom{n}{3}} \tag{1}$$

for all $\lambda \in [0,1]$. Goodman [7] and also Nordhaus and Stewart [14] proved the convex lower bound $g_3(\lambda) \ge \lambda(2\lambda - 1)$. An elegant argument of Bollobás [1] (see also Chapter VI in [2]) shows that a lower bound on $g_3(\lambda)$ is obtained by taking the piecewise linear function connecting the points $\left(1-\frac{1}{k}, (1-\frac{1}{k})(1-\frac{2}{k})\right)$ for all positive integers k. In 2008, a major breakthrough of Razborov [17] used his newly developed theory of *flag algebras* to determine $g_3(\lambda)$ for all $\lambda \in [0,1]$. (Fisher [6] had previously used a different method to determine $g_3(\lambda)$ in the range $1/2 \le \lambda \le 2/3$, a result reproved by Razborov [16].) One can show (see [15]) that, for all n and e,

$$\frac{n^3}{6} g_3\left(\frac{2e}{n^2}\right) \le g_3(n,e) \le \frac{n^3}{6} g_3\left(\frac{2e}{n^2}\right) + O\left(\frac{n^4}{n^2-2e}\right).$$

2 Related problems: supersaturation and graph densities

Before stating some of the results mentioned above in more detail, let us discuss the more general *supersaturation problem* of determining $g_F(n,e)$, the minimum number of copies of F in an n-vertex e-edge graph (so $g_{K_r}(n,e) = g_r(n,e)$). The range of e for which $g_F(n,e) = 0$ is well understood. Indeed, given a fixed graph F, let $\mathrm{ex}(n,F)$ denote the minimum number of edges in an F-free n-vertex graph, i.e. the maximum e for which $g_F(n,e) = 0$. Erdős and Stone [5] proved that $\mathrm{ex}(n,F) = t_{\chi(F)-1}(n) + o(n^2)$, where $\chi(F)$ is the chromatic number of F. The supersaturation phenomenon observed by Erdős and Simonovits [21] asserts that every n-vertex e-edge graph G with $e \geq \mathrm{ex}(n,F) + \Omega(n^2)$ contains not just one copy of F, but in fact a positive proportion of all $|V(F)|$-sized vertex subsets in $V(G)$ span a copy of F. (This also extends to hypergraphs.)

The problem of determining $g_r(\lambda)$ for integers $r \geq 4$ (defined analogously) has also received a great deal of attention. Lovász and Simonovits [10] (following Moon and Moser [12]), extended Goodman's bound to larger cliques. Nikiforov [13] reproved Razborov's result and also determined $g_4(\lambda)$ for all $\lambda \in [0,1]$. Recently, Reiher [18] determined $g_r(\lambda)$ for all $\lambda \in [0,1]$ and all $r \geq 3$.

The case of bipartite F is very different. A famous conjecture of Sidorenko [20] asserts that $g_F(\lambda)$ is attained by a random graph of density λ (we do not give a precise statement of the conjecture here). It is known to be true in some cases.

3 The solution to the asymptotic problem

Let us return to the problem of determining $g_3(n,e)$. To state Razborov's result on $g_3(\lambda)$ in an enlightening manner, we will require the following crucial definitions. Given a positive integer n, e such that $e \leq \binom{n}{2}$, let

$$k = k(n,e) := \min\{\ell \in \mathbb{N} : e \leq t_\ell(n)\}. \tag{2}$$

Given $\ell \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_\ell \in \mathbb{R}$, for convenience we write $e(K^\ell_{\alpha_1,\ldots,\alpha_\ell}) := \sum_{ij \in \binom{[\ell]}{2}} \alpha_i\alpha_j$ and $K_3(K^\ell_{\alpha_1\ldots,\alpha_\ell}) := \sum_{hij \in \binom{[\ell]}{3}} \alpha_h\alpha_i\alpha_j$ in analogy with the number of edges and triangles in the complete ℓ-partite graph $K^\ell_{n_1,\ldots,n_\ell}$ which is defined when the n_i are positive integers. Now let $c = c(n,e) \in \mathbb{R}$ be such that $c \geq 1/k$ and

$$e(K^k_{cn,\ldots,cn,n-(k-1)cn}) = e, \quad \text{i.e.} \quad c = \frac{1}{k}\left(1 + \sqrt{1 - \frac{2ke}{(k-1)n^2}}\right). \tag{3}$$

Razborov proved the following:

Theorem 3.1 ([17]) *For all positive integers n, e with $e \leq \binom{n}{2}$ and k, c defined as as above, we have that*

$$g_3(n,e) = K_3(K^k_{cn,\ldots,cn,n-(k-1)cn}) + o(n^3). \tag{4}$$

Informally, this says that asymptotically, the number of triangles is minimised by taking a complete partite graph such that all but the smallest part have the same order (which is roughly cn). Nikiforov [13] strengthened this by achieving a better error bound. However, neither result allows one to extract structural information about the *extremal* graphs, i.e. those n-vertex e-edge graphs which contain precisely $g_3(n,e)$ triangles. But we can obtain a family of *almost* extremal graphs as follows. Let $\mathcal{H}_{\text{approx}}(n,c)$ be the set of n-vertex graphs H with vertex partition $V_1, \ldots, V_k$ such that

- $H[V_i]$ is empty for all $i \in [k-2]$; $H[V_i, V_j]$ is complete bipartite whenever $i \neq j$ and $\{i,j\} \neq \{k-1,k\}$; and $H[V_{k-1} \cup V_k]$ is triangle-free and $e(H[V_{k-1} \cup V_k]) = |V_{k-1}||V_k|$.
- $|V_1| = \ldots = |V_{k-1}| = \lfloor cn \rfloor$.

So one graph in $\mathcal{H}_{\text{approx}}(n,c)$ is the complete k-partite graph $K^k_{\lfloor cn \rfloor,\ldots,\lfloor cn \rfloor,n-(k-1)\lfloor cn \rfloor}$. Observe that every graph in $\mathcal{H}_{\text{approx}}(n,c)$ contains the same number of triangles, but not necessarily exactly e edges. Pikhurko and Razborov proved a 'stability' version of Theorem 3.1: that every extremal graph has the approximate structure of a graph in $\mathcal{H}_{\text{approx}}(n,c)$.

Theorem 3.2 ([15]) *For every $\varepsilon > 0$, there are $\delta, n_0 > 0$ such that for every graph G on $n \geq n_0$ vertices with at most $g_3(n,e) + \delta\binom{n}{3}$ triangles, there exists $H \in \mathcal{H}_{\text{approx}}(n,c)$ such that one can obtain H from G by changing at most $\varepsilon\binom{n}{2}$ adjacencies.*

The sizes of the parts of graphs in $\mathcal{H}_{\text{approx}}(n,c)$ are chosen somewhat arbitrarily, for concreteness. Indeed, though the theorem states that every near-extremal graph is close to some $H \in \mathcal{H}_{\text{approx}}(n,c)$, this family is *not* conjectured to be extremal.

4 The Lovász-Simonovits conjecture and new results

As we stated earlier, Lovász and Simonovits [10] exactly determined $g_3(n,e)$ for those pairs n,e with $0 \le e - t_k(n) \le \alpha n^2$ where $\alpha = \alpha(k) > 0$. Let us say that an (n,e)-*graph* is a simple graph with n vertices and e edges. Lovász and Simonovits went further by characterising extremal graphs for such pairs n,e. To state their result, we need to define a family of graphs. Let $n,e \in \mathbb{N}$ be such that $e \le \binom{n}{2}$. Define $\mathcal{H}_1(n,e)$ to be the family of (n,e)-graphs H with the property that $V(H)$ has partition $A \cup B$ such that $H[A]$ is a complete $(k-2)$-partite graph, $H[A,B]$ is complete bipartite and $H[B]$ is triangle-free. Let $h(n,e) := \min\{K_3(H) : H \in \mathcal{H}_1(n,e)\}$. Lovász and Simonovits proved the following:

Theorem 4.1 ([10]) *For all integers $k \ge 2$ there exists $\alpha = \alpha(k) > 0$ such that, for all positive integers (n,e) with $t_k(n) \le e \le t_k(n) + \alpha n^2$, we have that $g_3(n,e) = h(n,e)$.*

In fact they were able to characterise the extremal graphs in this range, and indeed there are extremal graphs which do *not* lie in $\mathcal{H}_1(n,e)$. The constant $\alpha(k)$ in the proof of Theorem 4.1 is so small that Lovász and Simonovits 'did not even dare to estimate $\alpha(2)$'. This result encompasses the entire set of non-trivial known values of $g_3(n,\cdot)$. Lovász and Simonovits made the bold conjecture that in fact $g_3(n,e) = h(n,e)$ for all valid pairs n,e. Our main result proves almost every remaining case of this conjecture for almost the entire range of e.

Theorem 4.2 *For all $\varepsilon > 0$, there exists $n_0 > 0$ such that for all positive integers $n \ge n_0$ and $e \le \binom{n}{2} - \varepsilon n^2$ edges, we have $g_3(n,e) = h(n,e)$.*

We regard this as a complete solution to the problem of minimising the number of triangles in the range under consideration. Indeed, some simple algebra, which we do not include here, allows one to evaluate $h(n,e)$ and thus $g_3(n,e)$. Currently we are working on an extension of our result which characterises the extremal graphs.

5 Some remarks on the proof of Theorem 4.2

The asymptotic results of Fisher, Razborov, Nikiforov, Pikhurko-Razborov and Reiher all use analytic methods. Such techniques do not seem to be helpful for the exact problem, and indeed our proof of Theorem 4.2 uses purely combinatorial methods. At its heart, our proof uses the well-known stability method: Theorem 3.2 implies that any extremal graph G is structurally close to *some* H in $\mathcal{H}_{\text{approx}}(n, c)$ and hence some graph in $\mathcal{H}_1(n, e)$. Then the goal would be to analyse G and show that it cannot contain any imperfections and must in fact lie in $\mathcal{H}_1(n, e)$. The stability approach stems from work of Erdős and Simonovits [21] and has been used to solve many major problems in extremal combinatorics.

However, a significant obstacle is the fact that there is a large family of conjectured extremal graphs. Given any $H \in \mathcal{H}_1(n, e)$ with vertex partition $A \cup B$ as in the definition, one can obtain a different $H' \in \mathcal{H}_1(n, e)$ such that $K_3(H') = K_3(H)$ simply by replacing $H[B]$ with another triangle-free graph containing the same number of edges. In general, there are many choices for this triangle-free graph. Indeed, some simple algebra determines the subfamily $\mathcal{H}_1^{\min}(n, e)$ of $\mathcal{H}_1(n, e)$ which minimises the number of triangles.

An additional difficulty is that $\mathcal{H}_1(n, e)$ does not in fact contain every extremal graph, as in Theorem 4.1. So our goal as stated above must be modified. Full details of the proof may be found in [8].

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