Monochromatic Clique Decompositions of Graphs

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Abstract: Let G be a graph whose edges are colored with k colors, and $\mathcal{H} = (H_1, \ldots, H_k)$ be a k-tuple of graphs. A *monochromatic $\mathcal{H}$-decomposition* of G is a partition of the edge set of G such that each part is either a single edge or forms a monochromatic copy of H_i in color i, for some $1 \le i \le k$. Let $\phi_k(n, \mathcal{H})$ be the smallest number ϕ, such that, for every order-n graph and every k-edge-coloring, there is a monochromatic $\mathcal{H}$-decomposition with at most ϕ elements. Extending the previous results of Liu and Sousa [Monochromatic K_r-decompositions of graphs, *J Graph Theory* 76 (2014), 89–100], we solve this problem when each graph in $\mathcal{H}$ is a clique and $n \ge n_0(\mathcal{H})$ is sufficiently large. J. Graph Theory 80: 287–298, 2015

Keywords: *monochromatic graph decomposition; Turán number; Ramsey number*

1. INTRODUCTION

All graphs in this article are finite, undirected, and simple. For standard graph-theoretic terminology the reader is referred to [3].

Given two graphs G and H, an *H-decomposition* of G is a partition of the edge set of G such that each part is either a single edge or forms a subgraph isomorphic to H. Let $\phi(G, H)$ be the smallest possible number of parts in an H-decomposition of G. It is easy to see that, if H is nonempty, we have $\phi(G, H) = e(G) - \nu_H(G)(e(H) - 1)$, where $\nu_H(G)$ is the maximum number of pairwise edge-disjoint copies of H that can be packed into G. Dor and Tarsi [4] showed that if H has a component with at least three edges then it is NP-complete to determine if a graph G admits a partition into copies of H. Thus, it is NP-hard to compute the function $\phi(G, H)$ for such H. Nonetheless, many exact results were proved about the extremal function

$$\phi(n, H) = \max\{\phi(G, H) \mid v(G) = n\},$$

which is the smallest number such that any graph G of order n admits an H-decomposition with at most $\phi(n, H)$ elements.

This function was first studied, in 1966, by Erdős et al. [6], who proved that $\phi(n, K_3) = t_2(n)$, where K_s denotes the complete graph (clique) of order s, and $t_{r-1}(n)$ denotes the number of edges in the *Turán graph* $T_{r-1}(n)$, which is the unique $(r - 1)$-partite graph on n vertices that has the maximum number of edges. A decade later, Bollobás [2] proved that $\phi(n, K_r) = t_{r-1}(n)$, for all $n \geq r \geq 3$.

Recently, Pikhurko and Sousa [13] studied $\phi(n, H)$ for arbitrary graphs H. Their result is the following.

Theorem 1.1 [13]. *Let H be any fixed graph of chromatic number $r \geq 3$. Then,*

$$\phi(n, H) = t_{r-1}(n) + o(n^2).$$

Let $\mathrm{ex}(n, H)$ denote the maximum number of edges in a graph on n vertices not containing H as a subgraph. The result of Turán [20] states that $T_{r-1}(n)$ is the unique extremal graph for $\mathrm{ex}(n, K_r)$. The function $\mathrm{ex}(n, H)$ is usually called the *Turán function* for H. Pikhurko and Sousa [13] also made the following conjecture.

Conjecture 1.2 [13]. *For any graph H of chromatic number $r \geq 3$, there exists $n_0 = n_0(H)$ such that $\phi(n, H) = \mathrm{ex}(n, H)$ for all $n \geq n_0$.*

A graph H is *edge-critical* if there exists an edge $e \in E(H)$ such that $\chi(H) > \chi(H - e)$, where $\chi(H)$ denotes the *chromatic number* of H. For $r \geq 4$, a *clique-extension of order r* is a connected graph that consists of a K_{r-1} plus another vertex, say v, adjacent to at most $r - 2$ vertices of K_{r-1}. Conjecture 1.2 has been verified by Sousa for some edge-critical graphs, namely, clique-extensions of order $r \geq 4$ ($n \geq r$) [18] and the cycles of length 5 ($n \geq 6$) and 7 ($n \geq 10$) [17, 19]. Later, Özkahya and Person [12] verified the conjecture for all edge-critical graphs with chromatic number $r \geq 3$. Their result is the following.

Theorem 1.3 [12]. *For any edge-critical graph H with chromatic number $r \geq 3$, there exists $n_0 = n_0(H)$ such that $\phi(n, H) = \mathrm{ex}(n, H)$, for all $n \geq n_0$. Moreover, the only graph attaining $\mathrm{ex}(n, H)$ is the Turán graph $T_{r-1}(n)$.*

Recently, as an extension of Özkahya and Person's work (and as further evidence supporting Conjecture 1.2), Allen et al. [1] improved the error term obtained by Pikhurko and Sousa in Theorem 1.1. In fact, they proved that the error term $o(n^2)$ can be replaced by $O(n^{2-\alpha})$ for some $\alpha > 0$. Furthermore, they also showed that this error term has the correct order of magnitude. Their result is indeed an extension of Theorem 1.3 since the error term $O(n^{2-\alpha})$ that they obtained vanishes for every edge-critical graph H.

Motivated by the recent work about H-decompositions of graphs, a natural problem to consider is the Ramsey (or colored) version of this problem. More precisely, let G be a graph on n vertices whose edges are colored with k colors, for some $k \geq 2$ and let $\mathcal{H} = (H_1, \ldots, H_k)$ be a k-tuple of fixed graphs, where repetition is allowed. A *monochromatic $\mathcal{H}$-decomposition* of G is a partition of its edge set such that each part is either a single edge, or forms a monochromatic copy of H_i in color i, for some $1 \leq i \leq k$. Let $\phi_k(G, \mathcal{H})$ be the smallest number, such that, for any k-edge-coloring of G, there exists a monochromatic $\mathcal{H}$-decomposition of G with at most $\phi_k(G, \mathcal{H})$ elements. Our goal is to study the function

$$\phi_k(n, \mathcal{H}) = \max\{\phi_k(G, \mathcal{H}) \mid v(G) = n\},$$

which is the smallest number ϕ such that, any k-edge-colored graph of order n admits a monochromatic $\mathcal{H}$-decomposition with at most ϕ elements. In the case when $H_i \cong H$ for every $1 \leq i \leq k$, we simply write $\phi_k(G, H) = \phi_k(G, \mathcal{H})$ and $\phi_k(n, H) = \phi_k(n, \mathcal{H})$.

The function $\phi_k(n, K_r)$, for $k \geq 2$ and $r \geq 3$, has been studied by Liu and Sousa [11], who obtained results involving the Ramsey numbers and the Turán numbers. Recall that for $k \geq 2$ and integers $r_1, \ldots, r_k \geq 3$, the *Ramsey number for $K_{r_1}, \ldots, K_{r_k}$*, denoted by $R(r_1, \ldots, r_k)$, is the smallest value of s, such that, for every k-edge-coloring of K_s, there exists a monochromatic K_{r_i} in color i, for some $1 \leq i \leq k$. For the case when $r_1 = \cdots = r_k = r$, for some $r \geq 3$, we simply write $R_k(r) = R(r_1, \ldots, r_k)$. Since $R(r_1, \ldots, r_k)$ does not change under any permutation of $r_1, \ldots, r_k$, without loss of generality, we assume throughout that $3 \leq r_1 \leq \cdots \leq r_k$. The Ramsey numbers are notoriously difficult to calculate, even though, it is known that their values are finite [15]. To this date, the values of $R(3, r_2)$ have been determined exactly only for $3 \leq r_2 \leq 9$, and these are shown in the following table [14].

r_2	3	4	5	6	7	8	9
$R(3, r_2)$	6	9	14	18	23	28	36

The remaining Ramsey numbers that are known exactly are $R(4, 4) = 18$, $R(4, 5) = 25$, and $R(3, 3, 3) = 17$. The gap between the lower bound and the upper bound for other Ramsey numbers is generally quite large.

For the case $R(3, 3) = 6$, it is easy to see that the only 2-edge-coloring of K_5 not containing a monochromatic K_3 is the one where each color induces a cycle of length 5. From this 2-edge-coloring, observe that we may take a "blow-up" to obtain a 2-edge-coloring of the Turán graph $T_5(n)$, and easily deduce that $\phi_2(n, K_3) \geq t_5(n)$. See Figure 1.

This example was the motivation for Liu and Sousa [11] to study K_r-monochromatic decompositions of graphs, for $r \geq 3$ and $k \geq 2$. They have recently proved the following result.

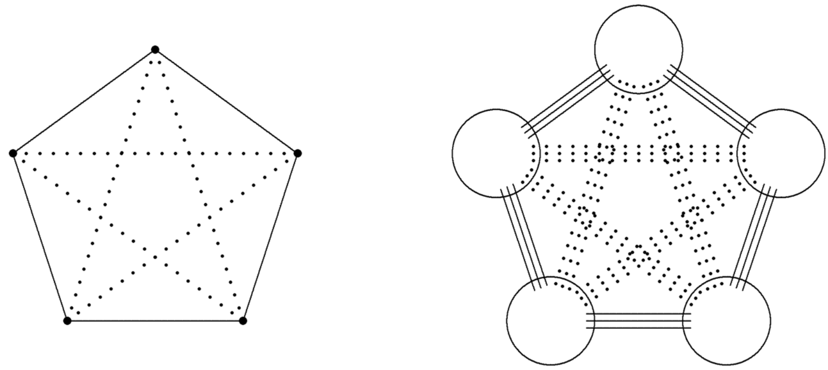

FIGURE 1. The 2-edge-coloring of K_5, and its blow-up

Theorem 1.4 [11].

(a) $\phi_k(n, K_3) = t_{R_k(3)-1}(n) + o(n^2)$;
(b) $\phi_k(n, K_3) = t_{R_k(3)-1}(n)$ *for* $k = 2, 3$ *and* n *sufficiently large;*
(c) $\phi_k(n, K_r) = t_{R_k(r)-1}(n)$, *for* $k \geq 2$, $r \geq 4$ *and* n *sufficiently large.*

Moreover, the only graph attaining $\phi_k(n, K_r)$ *in cases (b) and (c) is the Turán graph* $T_{R_k(r)-1}(n)$.

They also made the following conjecture.

Conjecture 1.5 [11]. *Let* $k \geq 4$. *Then* $\phi_k(n, K_3) = t_{R_k(3)-1}(n)$ *for* $n \geq R_k(3)$.

Here, we will study an extension of the monochromatic K_r-decomposition problem when the clique K_r is replaced by a fixed k-tuple of cliques $\mathcal{C} = (K_{r_1}, \ldots, K_{r_k})$. Our main result, stated in Theorem 1.6, is clearly an extension of Theorem 1.4. Also, it verifies Conjecture 1.5 for sufficiently large n.

Theorem 1.6. *Let* $k \geq 2$, $3 \leq r_1 \leq \cdots \leq r_k$, *and* $R = R(r_1, \ldots, r_k)$. *Let* $\mathcal{C} = (K_{r_1}, \ldots, K_{r_k})$. *Then, there is an* $n_0 = n_0(r_1, \ldots, r_k)$ *such that, for all* $n \geq n_0$, *we have*

$$\phi_k(n, \mathcal{C}) = t_{R-1}(n).$$

Moreover, the only order-n graph attaining $\phi_k(n, \mathcal{C})$ *is the Turán graph* $T_{R-1}(n)$ *(with a k-edge-coloring that does not contain a color-i copy of* K_{r_i} *for any* $1 \leq i \leq k$*).*

The upper bound of Theorem 1.6 is proved in Section 2. The lower bound follows easily by the definition of the Ramsey number. Indeed, take a k-edge-coloring f' of the complete graph K_{R-1} without a monochromatic K_{r_i} in color i, for all $1 \leq i \leq k$. Note that f' exists by definition of the Ramsey number $R = R(r_1, \ldots, r_k)$. Let $u_1, \ldots, u_{R-1}$ be the vertices of the K_{R-1}. Now, consider the Turán graph $T_{R-1}(n)$ with a k-edge-coloring f that is a "blow-up" of f'. That is, if $T_{R-1}(n)$ has partition classes $V_1, \ldots, V_{R-1}$, then for $v \in V_j$ and $w \in V_\ell$ with $j \neq \ell$, we define $f(vw) = f'(u_j u_\ell)$. Then, $T_{R-1}(n)$ with this k-edge-coloring has no monochromatic K_{r_i} in color i, for every $1 \leq i \leq k$. Therefore, $\phi_k(n, \mathcal{C}) \geq \phi_k(T_{R-1}(n), \mathcal{C}) = t_{R-1}(n)$ and the lower bound in Theorem 1.6 follows.

In particular, when all the cliques in $\mathcal{C}$ are equal, Theorem 1.6 completes the results obtained previously by Liu and Sousa in Theorem 1.4. In fact, we get the following direct corollary from Theorem 1.6.

Corollary 1.7. *Let $k \geq 2$, $r \geq 3$ and n be sufficiently large. Then,*

$$\phi_k(n, K_r) = t_{R_k(r)-1}(n).$$

Moreover, the only order-n graph attaining $\phi_k(n, K_r)$ is the Turán graph $T_{R_k(r)-1}(n)$ (with a k-edge-coloring that does not contain a monochromatic copy of K_r).

2. PROOF OF THEOREM 1.6

In this section, we will prove the upper bound in Theorem 1.6. Before presenting the proof we need to introduce the tools. Throughout this section, let $k \geq 2$, $3 \leq r_1 \leq \cdots \leq r_k$ be an increasing sequence of integers, $R = R(r_1, \ldots, r_k)$ be the Ramsey number for $K_{r_1}, \ldots, K_{r_k}$, and $\mathcal{C} = (K_{r_1}, \ldots, K_{r_k})$ be a fixed k-tuple of cliques.

We first recall the following stability theorem of Erdős and Simonovits [5, 16].

Theorem 2.1 (Stability Theorem [5,16]). *Let $r \geq 3$, and G be a graph on n vertices with $e(G) \geq t_{r-1}(n) + o(n^2)$ and not containing K_r as a subgraph. Then, there exists an $(r-1)$-partite graph G' on n vertices with partition classes $V_1, \ldots, V_{r-1}$, where $|V_i| = \frac{n}{r-1} + o(n)$ for $1 \leq i \leq r-1$, that can be obtained from G by adding and subtracting $o(n^2)$ edges.*

Next, we recall the following result of Győri [7, 8] about the existence of edge-disjoint copies of K_r in graphs on n vertices with more than $t_{r-1}(n)$ edges.

Theorem 2.2 [7,8]. *For every $r \geq 3$ there is C such that every graph G with $n \geq C$ vertices and $e(G) = t_{r-1}(n) + m$ edges, where $m \leq \binom{n}{2}/C$, contains at least $m - Cm^2/n^2$ edge-disjoint copies of K_r.*

Now, we will consider coverings and packings of cliques in graphs. Let $r \geq 3$ and G be a graph. Let $\mathcal{K}$ be the set of all K_r-subgraphs of G. A *K_r-cover* is a set of edges of G meeting all elements in $\mathcal{K}$, that is, the removal of a K_r-cover results in a K_r-free graph. A *K_r-packing* in G is a set of pairwise edge-disjoint copies of K_r. The *K_r-covering number* of G, denoted by $\tau_r(G)$, is the minimum size of a K_r-cover of G, and the *K_r-packing number* of G, denoted by $\nu_r(G)$, is the maximum size of a K_r-packing of G. Next, a *fractional K_r-cover* of G is a function $f : E(G) \to \mathbb{R}_+$, such that $\sum_{e \in E(H)} f(e) \geq 1$ for every $H \in \mathcal{K}$, that is, for every copy of K_r in G the sum of the values of f on its edges is at least 1. A *fractional K_r-packing* of G is a function $p : \mathcal{K} \to \mathbb{R}_+$ such that $\sum_{H \in \mathcal{K}: e \in E(H)} p(H) \leq 1$ for every $e \in E(G)$, that is, the total weight of K_r's that cover any edge is at most 1. Here, $\mathbb{R}_+$ denotes the set of nonnegative real numbers. The *fractional K_r-covering number* of G, denoted by $\tau_r^*(G)$, is the minimum of $\sum_{e \in E(G)} f(e)$ over all fractional K_r-covers f, and the *fractional K_r-packing number* of G, denoted by $\nu_r^*(G)$, is the maximum of $\sum_{H \in \mathcal{K}} p(H)$ over all fractional K_r-packings p.

One can easily observe that

$$\nu_r(G) \leq \tau_r(G) \leq \binom{r}{2} \nu_r(G).$$

For $r = 3$, we have $\tau_3(G) \leq 3\nu_3(G)$. A long-standing conjecture of Tuza [21] from 1981 states that this inequality can be improved as follows.

Conjecture 2.3 [21]. *For every graph G, we have $\tau_3(G) \leq 2\nu_3(G)$.*

Conjecture 2.3 remains open although many partial results have been proved. By using the earlier results of Krivelevich [10], and Haxell and Rödl [9], Yuster [22] proved the following theorem which will be crucial to the proof of Theorem 1.6. In the case $r = 3$, it is an asymptotic solution of Tuza's conjecture.

Theorem 2.4 [22]. *Let $r \geq 3$ and G be a graph on n vertices. Then*

$$\tau_r(G) \leq \left\lfloor \frac{r^2}{4} \right\rfloor \nu_r(G) + o(n^2). \tag{1}$$

We now prove the following lemma that states that a graph G with n vertices and at least $t_{R-1}(n) + \Omega(n^2)$ edges falls quite short of being optimal.

Lemma 2.5. *For every $k \geq 2$ and $c_0 > 0$ there are $c_1 > 0$ and n_0 such that for every graph G of order $n \geq n_0$ with at least $t_{R-1}(n) + c_0 n^2$ edges, we have $\phi_k(G, \mathcal{C}) \leq t_{R-1}(n) - c_1 n^2$.*

Proof. Suppose that the lemma is false, that is, there is $c_0 > 0$ such that for some increasing sequence of n there is a graph G on n vertices with $e(G) \geq t_{R-1}(n) + c_0 n^2$ and $\phi_k(G, \mathcal{C}) \geq t_{R-1}(n) + o(n^2)$. Fix a k-edge-coloring of G and, for $1 \leq i \leq k$, let G_i be the subgraph of G on n vertices that contains all edges with color i.

Let $m = e(G) - t_{R-1}(n)$, and let $s \in \{0, \ldots, k\}$ be the maximum such that

$$r_1 = \ldots = r_s = 3.$$

Let us very briefly recall the argument from [11] that shows $\phi_k(G, \mathcal{C}) \leq t_{R-1}(n) + o(n^2)$, adopted to our purposes. If we remove a K_{r_i}-cover from G_i for every $1 \leq i \leq k$, then we destroy all copies of K_R in G. By Turán's theorem, at most $t_{R-1}(n)$ edges remain. Thus,

$$\sum_{i=1}^{k} \tau_{r_i}(G_i) \geq m. \tag{2}$$

By Theorem 2.4, if we decompose G into a maximum K_{r_i}-packing in each G_i and the remaining edges, we obtain that

$$\begin{aligned}
\phi_k(G, \mathcal{C}) &\leq e(G) - \sum_{i=1}^{k} \left(\binom{r_i}{2} - 1 \right) \nu_{r_i}(G_i) \\
&\leq t_{R-1}(n) + m - \sum_{i=1}^{k} \frac{\binom{r_i}{2} - 1}{\lfloor r_i^2/4 \rfloor} \tau_{r_i}(G_i) + o(n^2) \\
&\leq t_{R-1}(n) + m - \sum_{i=1}^{k} \tau_{r_i}(G_i) - \frac{1}{4} \sum_{i=s+1}^{k} \tau_{r_i}(G_i) + o(n^2) \ \leq \ t_{R-1}(n) + o(n^2).
\end{aligned} \tag{3}$$

The third inequality holds since $(\binom{r}{2} - 1)/\lfloor r^2/4 \rfloor \geq 5/4$ for $r \geq 4$ and is equal to 1 for $r = 3$.

Let us derive a contradiction from this by looking at the properties of our hypothetical counterexample G. First, all inequalities that we saw have to be equalities within an additive term $o(n^2)$. In particular, the slack in (2) is $o(n^2)$, that is,

$$\sum_{i=1}^{k} \tau_{r_i}(G_i) = m + o(n^2). \tag{4}$$

Also, $\sum_{i=s+1}^{k} \tau_{r_i}(G_i) = o(n^2)$. In particular, we have that $s \geq 1$. To simplify the later calculations, let us redefine G by removing a maximum K_{r_i}-packing from G_i for each $i \geq s+1$. The new graph is still a counterexample to the lemma if we decrease c_0 slightly, since the number of edges removed is at most $\sum_{i=s+1}^{k} \binom{r_i}{2}\tau_{r_i}(G_i) = o(n^2)$.

Suppose that we remove, for each $i \leq s$, an arbitrary (not necessarily minimum) K_3-cover F_i from G_i such that

$$\sum_{i=1}^{s} |F_i| \leq m + o(n^2). \tag{5}$$

Let $G' \subseteq G$ be the obtained K_R-free graph. (Recall that we assumed that G_i is K_{r_i}-free for all $i \geq s+1$.) Let $G_i' \subseteq G_i$ be the color classes of G'. We know by (5) that $e(G') \geq t_{R-1}(n) + o(n^2)$. Since G' is K_R-free, we conclude by the Stability Theorem (Theorem 2.1) that there is a partition $V(G) = V(G') = V_1 \dot\cup \ldots \dot\cup V_{R-1}$ such that

$$\forall\, i \in \{1, \ldots, R-1\}, \quad |V_i| = \frac{n}{R-1} + o(n) \qquad \text{and} \qquad |E(T) \setminus E(G')| = o(n^2), \tag{6}$$

where T is the complete $(R-1)$-partite graph with parts $V_1, \ldots, V_{R-1}$.

Next, we essentially expand the proof of (1) for $r = 3$ and transform it into an algorithm that produces K_3-coverings F_i of G_i, with $1 \leq i \leq s$, in such a way that (5) holds but (6) is impossible whatever $V_1, \ldots, V_{R-1}$ we take, giving the desired contradiction.

Let H be an arbitrary graph of order n. By the LP duality, we have that

$$\tau_r^*(H) = \nu_r^*(H). \tag{7}$$

By the result of Haxell and Rödl [9] we have that

$$\nu_r^*(H) = \nu_r(H) + o(n^2). \tag{8}$$

Krivelevich [10] showed that

$$\tau_3(H) \leq 2\tau_3^*(H). \tag{9}$$

Thus, $\tau_3(H) \leq 2\nu_3(H) + o(n^2)$ giving (1) for $r = 3$.

The proof of Krivelevich [10] of (9) is based on the following result.

Lemma 2.6. *Let H be an arbitrary graph and $f : E(H) \to \mathbb{R}_+$ be a minimum fractional K_3-cover. Then $\tau_3(H) \leq \frac{3}{2}\,\tau_3^*(H)$ or there is $xy \in E(H)$ with $f(xy) = 0$ that belongs to at least one triangle of H.*

Proof. If there is an edge $xy \in E(H)$ that does not belong to a triangle, then necessarily $f(xy) = 0$ and xy does not belong to any optimal fractional or integer K_3-cover. We can remove xy from $E(H)$ without changing the validity of the lemma. Thus, we can assume that every edge of H belongs to a triangle.

Suppose that $f(xy) > 0$ for every edge xy of H, for otherwise we are done. Take a maximum fractional K_3-packing p. Recall that it is a function that assigns a weight

$p(xyz) \in \mathbb{R}_+$ to each triangle xyz of H such that for every edge xy the sum of weights over all K_3's of H containing xy is at most 1, that is,

$$\sum_{z \in \Gamma(x) \cap \Gamma(y)} p(xyz) \leq 1, \tag{10}$$

where $\Gamma(v)$ denotes the set of neighbors of the vertex v in H.

This is the dual LP to the minimum fractional K_3-cover problem. By the complementary slackness condition (since f and p are optimal solutions), we have equality in (10) for every $xy \in E(H)$. This and the LP duality imply that

$$\tau_3^*(H) = \nu_3^*(H) = \sum_{\text{triangle } xyz} p(xyz) = \frac{1}{3} \sum_{xy \in E(H)} \sum_{z \in \Gamma(x) \cap \Gamma(y)} p(xyz) = \frac{1}{3} e(H).$$

On the other hand $\tau_3(H) \leq \frac{1}{2}\, e(H)$: take a bipartite subgraph of H with at least half of the edges; then the remaining edges form a K_3-cover. Putting the last two inequalities together, we obtain the required result. ■

Let $1 \leq i \leq s$. We now describe an algorithm for finding a K_3-cover F_i in G_i. Initially, let $H = G_i$ and $F_i = \emptyset$. Repeat the following.

Take a minimum fractional K_3-cover f of H. If the first alternative of Lemma 2.6 is true, pick a K_3-cover of H of size at most $\frac{3}{2}\, \tau_3^*(H)$, add it to F_i and stop. Otherwise, fix some edge $xy \in E(H)$ returned by Lemma 2.6. Let F' consist of all pairs xz and yz over $z \in \Gamma(x) \cap \Gamma(y)$. Add F' to F_i and remove F' from $E(H)$. Repeat the whole step (with the new H and f).

Consider any moment during this algorithm, when we had $f(xy) = 0$ for some edge xy of H. Since f is a fractional K_3-cover, we have that $f(xz) + f(yz) \geq 1$ for every $z \in \Gamma(x) \cap \Gamma(y)$. Thus, if H' is obtained from H by removing 2ℓ such pairs, where $\ell = |\Gamma(x) \cap \Gamma(y)|$, then $\tau_3^*(H') \leq \tau_3^*(H) - \ell$ because f when restricted to $E(H')$ is still a fractional cover (although not necessarily an optimal one). Clearly, $|F_i|$ increases by 2ℓ during this operation. Thus, indeed we obtain, at the end, a K_3-cover F_i of G_i of size at most $2\tau_3^*(G_i)$.

Also, by (7) and (8) we have that

$$\sum_{i=1}^{s} |F_i| \leq 2 \sum_{i=1}^{s} \nu_3(G_i) + o(n^2).$$

Now, since all slacks in (3) are $o(n^2)$, we conclude that

$$\sum_{i=1}^{s} \nu_3(G_i) \leq \frac{m}{2} + o(n^2)$$

and (5) holds. In fact, (5) is equality by (4).

Recall that G_i' is obtained from G_i by removing all edges of F_i and G' is the edge-disjoint union of the graphs G_i'. Suppose that there exist $V_1, \ldots, V_{R-1}$ satisfying (6). Let $M = E(T) \setminus E(G')$ consist of *missing* edges. Thus, $|M| = o(n^2)$.

Let

$$X = \{x \in V(T) \mid \deg_M(x) \geq c_2 n\},$$

where we define $c_2 = (4(R-1))^{-1}$. Clearly,

$$|X| \leq 2|M|/c_2 n = o(n).$$

Observe that, for every $1 \leq i \leq s$, if the first alternative of Lemma 2.6 holds at some point, then the remaining graph H satisfies $\tau_3^*(H) = o(n^2)$. Indeed, otherwise by $\tau_3(G_i) \leq 2\tau_3^*(G_i) - \tau_3^*(H)/2 + o(n^2)$ we get a strictly smaller constant than 2 in (9) and thus a gap of $\Omega(n^2)$ in (3), a contradiction. Therefore, all but $o(n^2)$ edges in F_i come from some *parent edge* xy that had f-weight 0 at some point.

When our algorithm adds pairs xz and yz to F_i with the same parent xy, then it adds the same number of pairs incident to x as those incident to y. Let $\mathcal{P}$ consist of pairs xy that are disjoint from X and were a parent edge during the run of the algorithm. Since the total number of pairs in F_i incident to X is at most $n|X| = o(n^2)$, there are $|F_i| - o(n^2)$ pairs in F_i such that their parent is in $\mathcal{P}$.

Let us show that y_0 and y_1 belong to different parts V_j for every pair $y_0y_1 \in \mathcal{P}$. Suppose on the contrary that, say, $y_0, y_1 \in V_1$. For each $2 \leq j \leq R-1$ pick an arbitrary $y_j \in V_j \setminus (\Gamma_M(y_0) \cup \Gamma_M(y_1))$. Since $y_0, y_1 \notin X$, the possible number of choices for y_j is at least

$$\frac{n}{R-1} - 2c_2 n + o(n) \geq \frac{n}{R-1} - 3c_2 n.$$

Let

$$Y = \{y_0, \ldots, y_{R-1}\}.$$

By the above, we have at least $(\frac{n}{R-1} - 3c_2 n)^{R-2} = \Omega(n^{R-2})$ choices of Y. Note that by the definition, all edges between $\{y_0, y_1\}$ and the rest of Y are present in $E(G')$. Thus, the number of sets Y containing at least one edge of M different from y_0y_1 is at most

$$|M| \times n^{R-4} = o(n^{R-2}).$$

This is $o(1)$ times the number of choices of Y. Thus, for almost every Y, $H = G'[Y]$ is a clique (except perhaps the pair y_0y_1). In particular, there is at least one such choice of Y; fix it. Let $i \in \{1, \ldots, k\}$ be arbitrary. Adding back the pair y_0y_1 colored i to H (if it is not there already), we obtain a k-edge-coloring of the complete graph H of order R. By the definition of $R = R(r_1, \ldots, r_k)$, there must be a monochromatic triangle on abc of color $h \leq s$. (Recall that we assumed at the beginning that G_j is K_{r_j}-free for each $j > s$.) But abc has to contain an edge from the K_3-cover F_h, say ab. This edge ab is not in G' (it was removed from G). If a, b lie in different parts V_j, then $ab \in M$, a contradiction to the choice of Y. The only possibility is that $ab = y_0y_1$. Then $h = i$. Since both y_0c and y_1c are in G_i', they were never added to the K_3-cover F_i by our algorithm. Therefore, y_0y_1 was never a parent, which is the desired contradiction.

Thus, every $xy \in \mathcal{P}$ connects two different parts V_j. For every such parent xy, the number of its children in M is at least half of all its children. Indeed, for every pair of children xz and yz, at least one connects two different parts; this child necessarily belongs to M. Thus,

$$|F_i \cap M| \geq \frac{1}{2}|F_i| + o(n^2).$$

(Recall that parent edges that intersect X produce at most $2n|X| = o(n^2)$ children.) Therefore,

$$|M| \geq \frac{1}{2}\sum_{i=1}^{s}|F_i| + o(n^2) \geq \frac{m}{2} + o(n^2) = \Omega(n^2),$$

contradicting (6). This contradiction proves Lemma 2.5. ■

We are now able to prove Theorem 1.6.

Proof of the upper bound in Theorem 1.6. Let C be the constant returned by Theorem 2.2 for $r = R$. Let $n_0 = n_0(r_1, \ldots, r_k)$ be sufficiently large to satisfy all the inequalities we will encounter. Let G be a k-edge-colored graph on $n \geq n_0$ vertices. We will show that $\phi_k(G, \mathcal{C}) \leq t_{R-1}(n)$ with equality if and only if $G = T_{R-1}(n)$, and G does not contain a monochromatic copy of K_{r_i} in color i for every $1 \leq i \leq k$.

Let $e(G) = t_{R-1}(n) + m$, where m is an integer. If $m < 0$, we can decompose G into single edges and there is nothing to prove.

Suppose $m = 0$. If G contains a monochromatic copy of K_{r_i} in color i for some $1 \leq i \leq k$, then G admits a monochromatic $\mathcal{C}$-decomposition with at most $t_{R-1}(n) - \binom{r_i}{2} + 1 < t_{R-1}(n)$ parts and we are done. Otherwise, the definition of R implies that G does not contain a copy of K_R. Therefore, $G = T_{R-1}(n)$ by Turán's theorem and $\phi_k(G, \mathcal{C}) = t_{R-1}(n)$ as required.

Now suppose $m > 0$. We can also assume that $m < \binom{n}{2}/C$ for otherwise we are done: $\phi_k(G, \mathcal{C}) < t_{R-1}(n)$ by Lemma 2.5. Thus, by Theorem 2.2, the graph G contains at least $m - Cm^2/n^2 > \frac{m}{2}$ edge-disjoint copies of K_R. Since each K_R contains a monochromatic copy of K_{r_i} in the color-i graph G_i, for some $1 \leq i \leq k$, we conclude that $\sum_{i=1}^{k} \nu_{r_i}(G_i) > \frac{m}{2}$, so that $\sum_{i=1}^{k}(\binom{r_i}{2} - 1)\nu_{r_i}(G_i) \geq \sum_{i=1}^{k} 2\nu_{r_i}(G_i) > m$. We have

$$\phi_k(G, \mathcal{C}) = e(G) - \sum_{i=1}^{k}\binom{r_i}{2}\nu_{r_i}(G_i) + \sum_{i=1}^{k}\nu_{r_i}(G_i) < t_{R-1}(n),$$

giving the required. ■

Remark. By analyzing the above argument, one can also derive the following stability property for every fixed family $\mathcal{C}$ of cliques as $n \to \infty$: every graph G on n vertices with $\phi_k(G, \mathcal{C}) = t_{R-1}(n) + o(n^2)$ is $o(n^2)$-close to the Turán graph $T_{R-1}(n)$ in the edit distance.

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