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Notes

Hypergraph Turán densities can have arbitrarily large algebraic degree

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ABSTRACT

Grosu (2016) [11] asked if there exist an integer $r \geq 3$ and a finite family of r-graphs whose Turán density, as a real number, has (algebraic) degree greater than $r-1$. In this note we show that, for all integers $r \geq 3$ and d, there exists a finite family of r-graphs whose Turán density has degree at least d, thus answering Grosu's question in a strong form.

1. Introduction

For an integer $r \geq 2$, an *r-uniform hypergraph* (henceforth, an *r-graph*) H is a collection of r-subsets of some finite set V. Given a family $\mathcal{F}$ of r-graphs, we say H is *$\mathcal{F}$-free* if it does not contain any member of $\mathcal{F}$ as a subgraph. The *Turán number* $\mathrm{ex}(n, \mathcal{F})$ of $\mathcal{F}$ is the maximum number of edges in an $\mathcal{F}$-free r-graph on n vertices. The *Turán density* $\pi(\mathcal{F})$ of $\mathcal{F}$ is defined as $\pi(\mathcal{F}) := \lim_{n\to\infty} \mathrm{ex}(n, \mathcal{F})/\binom{n}{r}$; the existence of the limit was established in [12]. The study of $\mathrm{ex}(n, \mathcal{F})$ is one of the central topics in extremal graph

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and hypergraph theory. For the hypergraph Turán problem (i.e. the case $r \geq 3$), we refer the reader to the surveys by Keevash [13] and Sidorenko [18].

For $r \geq 3$, determining the value of $\pi(\mathcal{F})$ for a given r-graph family $\mathcal{F}$ is very difficult in general, and there are only a few known results. For example, the problem of determining $\pi(K_\ell^r)$ raised by Turán [19] in 1941, where K_ℓ^r is the complete r-graph on ℓ vertices, is wide open and the \$500 prize of Erdős for solving it for at least one pair $\ell > r \geq 3$ is still unclaimed.

For every integer $r \geq 2$, define

$$\Pi_{\text{fin}}^{(r)} := \{\pi(\mathcal{F})\colon \mathcal{F} \text{ is a finite family of } r\text{-graphs}\}, \quad \text{and}$$
$$\Pi_{\infty}^{(r)} := \{\pi(\mathcal{F})\colon \mathcal{F} \text{ is a (possibly infinite) family of } r\text{-graphs}\}.$$

For $r = 2$ the celebrated Erdős–Stone–Simonovits theorem [6,7] determines the Turán density for every family $\mathcal{F}$ of graphs; in particular, it holds that

$$\Pi_{\infty}^{(2)} = \Pi_{\text{fin}}^{(r)} = \{1\} \cup \{1 - 1/k\colon \text{integer } k \geq 1\}.$$

The problem of understanding the sets $\Pi_{\text{fin}}^{(r)}$ and $\Pi_{\infty}^{(r)}$ of possible r-graph Turán densities for $r \geq 3$ has attracted a lot of attention. One of the earliest results here is the theorem of Erdős [5] from the 1960s that $\Pi_{\infty}^{(r)} \cap (0, r!/r^r) = \emptyset$ for every integer $r \geq 3$. However, our understanding of the locations and the lengths of other maximal intervals avoiding r-graph Turán densities and the right accumulation points of $\Pi_{\infty}^{(r)}$ (the so-called *jump problem*) is very limited; for some results in this direction see e.g. [1,8,9,17,21].

It is known that the set $\Pi_{\infty}^{(r)}$ is the topological closure of $\Pi_{\text{fin}}^{(r)}$ (and thus the former set is easier to understand) and that $\Pi_{\infty}^{(r)}$ has cardinality of continuum (and thus is strictly larger than the countable set $\Pi_{\text{fin}}^{(r)}$), see respectively Proposition 1 and Theorem 2 in [16].

For a while it was open whether $\Pi_{\text{fin}}^{(r)}$ can contain an irrational number (see the conjecture of Chung and Graham in [3, Page 95]), until such examples were independently found by Baber and Talbot [2] and by the second author [16]. However, the question of Jacob Fox ([16, Question 27]) whether $\Pi_{\text{fin}}^{(r)}$ can contain a transcendental number remains open.

Grosu [11] initiated a systematic study of algebraic properties of the sets $\Pi_{\text{fin}}^{(r)}$ and $\Pi_{\infty}^{(r)}$. He proved a number of general results that, in particular, directly give further examples of irrational Turán densities.

Recall that the *(algebraic) degree* of a real number α is the minimum degree of a non-zero polynomial p with integer coefficients that vanishes on α; it is defined to be ∞ if no such p exists (that is, if the real α is transcendental). In the same paper, Grosu [11, Problem 3] posed the following question.

Problem 1.1 *(Grosu).* Does there exist an integer $r \geq 3$ such that $\Pi_{\text{fin}}^{(r)}$ contains an algebraic number α of degree strictly larger than $r - 1$?

Apparently, all r-graph Turán densities that Grosu knew or could produce with his machinery had degree at most $r-1$, explaining this expression in his question. His motivation for asking this question was that if, on input $\mathcal{F}$, we can compute an upper bound on the degree of $\pi(\mathcal{F})$ as well as on the absolute values of the coefficients of its minimal polynomial, then we can compute $\pi(\mathcal{F})$ exactly, see the discussion in [11, Page 140].

In this short note we answer Grosu's question in the following stronger form.

Theorem 1.2. *For every integer $r \geq 3$ and for every integer d there exists an algebraic number in $\Pi_{\text{fin}}^{(r)}$ whose minimal polynomial has degree at least d.*

Our proof for Theorem 1.2 is constructive; in particular, for $r = 3$ we will show that the following infinite sequence is contained in $\Pi_{\text{fin}}^{(3)}$:

$$\frac{1}{\sqrt{3}}, \quad \frac{1}{\sqrt{3-\frac{2}{\sqrt{3}}}}, \quad \frac{1}{\sqrt{3-\frac{2}{\sqrt{3-\frac{2}{\sqrt{3}}}}}}, \quad \frac{1}{\sqrt{3-\frac{2}{\sqrt{3-\frac{2}{\sqrt{3-\frac{2}{\sqrt{3}}}}}}}}, \quad \ldots \quad . \tag{1}$$

2. Preliminaries

In this section, we introduce some preliminary definitions and results that will be used later.

For an integer $r \geq 2$, an (r-uniform) *pattern* is a pair $P = (m, \mathcal{E})$, where m is a positive integer, $\mathcal{E}$ is a collection of r-multisets on $[m] := \{1, \ldots, m\}$, where by an *r-multiset* we mean an unordered collection of r elements with repetitions allowed. Let $V_1, \ldots, V_m$ be disjoint sets and let $V = V_1 \cup \cdots \cup V_m$. The *profile* of an r-set $R \subseteq V$ (with respect to $V_1, \ldots, V_m$) is the r-multiset on $[m]$ that contains element i with multiplicity $|R \cap V_i|$ for every $i \in [m]$. For an r-multiset $S \subseteq [m]$, let $S(\!(V_1, \ldots, V_m)\!)$ consist of all r-subsets of V whose profile is S. We call this r-graph the *blowup* of S and the r-graph

$$\mathcal{E}(\!(V_1, \ldots, V_m)\!) := \bigcup_{S \in \mathcal{E}} S(\!(V_1, \ldots, V_m)\!)$$

is called the *blowup* of $\mathcal{E}$ (with respect to $V_1, \ldots, V_m$). We say that an r-graph H is a *P-construction* if it is a blowup of $\mathcal{E}$. Note that these are special cases of the more general definitions from [16].

It is easy to see that the notion of a pattern is a generalization of a hypergraph, since every r-graph is a pattern in which $\mathcal{E}$ is a collection of (ordinary) r-sets. For most families $\mathcal{F}$ whose Turán problem was resolved, the extremal $\mathcal{F}$-free constructions are blowups of some simple pattern. For example, let $P_B := (2, \{\{\!\{1,2,2\}\!\}, \{\!\{1,1,2\}\!\}\})$, where we use $\{\!\{\ \}\!\}$ to distinguish multisets from ordinary sets. Then a P_B-construction is a 3-graph H whose vertex set can be partitioned into two parts V_1 and V_2 such that H consists of all triples that have nonempty intersections with both V_1 and V_2. A famous result in the hypergraph Turán theory is that the pattern P_B characterizes the structure of all maximum 3-graphs of sufficiently large order that do not contain a Fano plane (see [4,10,14]).

For a pattern $P = (m, \mathcal{E})$, let the *Lagrange polynomial* of $\mathcal{E}$ be

$$\lambda_{\mathcal{E}}(x_1, \ldots, x_m) := r! \sum_{E \in \mathcal{E}} \prod_{i=1}^{m} \frac{x_i^{E(i)}}{E(i)!},$$

where $E(i)$ is the multiplicity of i in the r-multiset E. In other words, $\lambda_{\mathcal{E}}$ gives the asymptotic edge density of a large blowup of $\mathcal{E}$, given its relative part sizes x_i.

The *Lagrangian* of P is defined as follows:

$$\lambda(P) := \sup \{\lambda_{\mathcal{E}}(x_1, \ldots, x_m) \colon (x_1, \ldots, x_m) \in \Delta_{m-1}\},$$

where $\Delta_{m-1} := \{(x_1, \ldots, x_m) \in [0,1]^m \colon x_1 + \ldots + x_m = 1\}$ is the standard $(m-1)$-dimensional simplex in $\mathbb{R}^m$. Since we maximise a polynomial (a continuous function) on a compact space, the supremum is in fact the maximum and we call the vectors in Δ_{m-1} attaining it *P-optimal*. Note that the Lagrangian of a pattern is a generalization of the well-known *hypergraph Lagrangian* that has been successfully applied to Turán-type problems (see e.g. [1,9,20]), with the basic idea going back to Motzkin and Straus [15].

For $i \in [m]$ let $P - i$ be the pattern obtained from P by *removing index* i, that is, we remove i from $[m]$ and delete all multisets containing i from E (and relabel the remaining indices to form the set $[m-1]$). We call P *minimal* if $\lambda(P - i)$ is strictly smaller than $\lambda(P)$ for every $i \in [m]$, or equivalently if no P-optimal vector has a zero entry. For example, the 2-graph pattern $P := (3, \{ \{\!\{1, 2\}\!\}, \{\!\{1, 3\}\!\} \})$ is not minimal as $\lambda(P) = \lambda(P - 3) = 1/2$.

In [16], the second author studied the relations between possible hypergraph Turán densities and patterns. One of the main results from [16] is as follows.

Theorem 2.1 *([16]). For every minimal pattern P there exists a finite family $\mathcal{F}$ of r-graphs such that $\pi(\mathcal{F}) = \lambda(P)$, and moreover, every maximum $\mathcal{F}$-free r-graph is a P-construction.*

Let $r \geq 3$ and $s \geq 1$ be two integers. Given an r-uniform pattern $P = (m, \mathcal{E})$, one can create an $(r+s)$-uniform pattern $P + s := (m + s, \hat{\mathcal{E}})$ in the following way: for every $E \in \mathcal{E}$ we insert the s-set $\{m+1, \ldots, m+s\}$ into E, and let $\hat{\mathcal{E}}$ denote the resulting family of $(r+s)$-multisets. For example, if $P = (3, \{\{\!\{1, 2, 3\}\!\}, \{\!\{1, 3, 3\}\!\}, \{\!\{2, 3, 3\}\!\}\})$, then $P + 1 = (4, \{\{\!\{1, 2, 3, 4\}\!\}, \{\!\{1, 3, 3, 4\}\!\}, \{\!\{2, 3, 3, 4\}\!\}\})$.

The following observation follows easily from the definitions.

Observation 2.2. *If P is a minimal pattern, then $P + s$ is a minimal pattern for every integer $s \geq 1$.*

For the Lagrangian of $P+s$ we have the following result.

Proposition 2.3. *Suppose that $r \geq 2$ is an integer and P is an r-uniform pattern. Then for every integer $s \geq 1$ we have*

$$\lambda(P+s) = \frac{r^r(s+r)!}{(r+s)^{r+s}r!}\,\lambda(P).$$

In particular, the real numbers $\lambda(P+s)$ and $\lambda(P)$ have the same degree.

Proof. Assume that $P = (m, \mathcal{E})$. Let $\hat{P} := P + s = (m+s, \widehat{\mathcal{E}})$. Let $(x_1, \ldots, x_{m+s}) \in \Delta_{m+s-1}$ be a $\hat{P}$-optimal vector. Note from the definition of Lagrange polynomial that

$$\lambda(\hat{P}) = \lambda_{\hat{\mathcal{E}}}(x_1, \ldots, x_{m+s}) = \frac{(r+s)!}{r!}\,\lambda_{\mathcal{E}}(x_1, \ldots, x_m) \prod_{i=m+1}^{m+s} x_i.$$

Let $x := \frac{1}{s}\sum_{i=m+1}^{m+s} x_i$ and note that $\sum_{i=1}^{m} x_i = 1 - sx$. Since $\lambda_{\mathcal{E}}$ is a homogeneous polynomial of degree r, we have

$$\lambda_{\mathcal{E}}(x_1, \ldots, x_m) = \lambda_{\mathcal{E}}\left(\frac{x_1}{1-sx}, \ldots, \frac{x_m}{1-sx}\right)(1-sx)^r \leq \lambda(P)(1-sx)^r.$$

This and the AM-GM inequality give that

$$\lambda(\hat{P}) = \frac{(r+s)!}{r!}\,\lambda_{\mathcal{E}}(x_1, \ldots, x_m) \prod_{i=m+1}^{m+s} x_i \leq \frac{(r+s)!}{r!}\,\lambda(P)(1-sx)^r x^s.$$

For $x \in [0, 1/s]$, the function $(1-sx)^r(rx)^s$, as the product of $s+r$ non-negative terms summing to r, is maximized when all terms are equal, that is, at $x = \frac{1}{r+s}$. So

$$\lambda(\hat{P}) \leq \frac{(r+s)!}{r!}\,\lambda(P)(1-sx)^r x^s \leq \frac{r^r(s+r)!}{(r+s)^{r+s}r!}\,\lambda(P).$$

To prove the other direction of this inequality, observe that if we take $(x_1, \ldots, x_m) = \frac{r}{r+s}(y_1, \ldots, y_m)$, where $(y_1, \ldots, y_m) \in \Delta_{m-1}$ is P-optimal, and take $x_{m+1} = \cdots = x_{m+s} = \frac{1}{r+s}$, then all inequalities above hold with equalities. ■

3. Proof of Theorem 1.2

In this section we prove Theorem 1.2. By Theorem 2.1, it suffices to find a sequence of r-uniform minimal patterns $(P_k)_{k=1}^{\infty}$ such that the degree of the real number $\lambda(P_k)$ goes to infinity as k goes to infinity. Furthermore, by Observation 2.2 and Proposition 2.3, it suffices to find such a sequence for $r = 3$. So we will assume that $r = 3$ in the rest of this note.

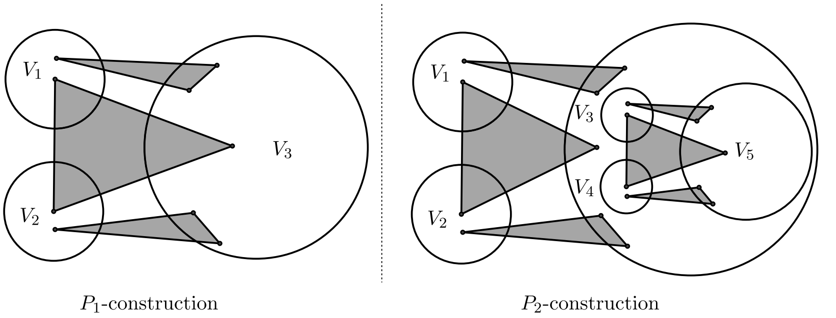

Fig. 1. Constructions with one level and two levels.

To start with, we let $P_1 := (3, \{\{\!\{1,2,3\}\!\}, \{\!\{1,3,3\}\!\}, \{\!\{2,3,3\}\!\}\})$. Recall that a 3-graph H is a P_1-construction (see Fig. 1) if there exists a partition $V(H) = V_1 \cup V_2 \cup V_3$ such that the edge set of H consists of

(a) all triples that have one vertex in each V_i,
(b) all triples that have one vertex in V_1 and two vertices in V_3, and
(c) all triples that have one vertex in V_2 and two vertices in V_3.

The pattern P_1 was studied by Yan and Peng in [20], where they proved that there exists a single 3-graph whose Turán density is given by P_1-constructions which, by $\lambda(P_1) = 1/\sqrt{3}$, is an irrational number. It seems that some other patterns could be used to prove Theorem 1.2; however, the obtained sequence of Turán densities (i.e. the sequence in (1)) produced by using P_1 is nicer than those produced by the other patterns that we tried.

Next, we define the pattern $P_{k+1} = (2k+3, \mathcal{E}_{k+1})$ for every $k \geq 1$ inductively. It is easier to define what a P_{k+1}-construction is rather than to write down the definition of P_{k+1}: for every integer $k \geq 1$ a 3-graph H is a P_{k+1}-*construction* if there exists a partition $V(H) = V_1 \cup V_2 \cup V_3$ such that

(a) the induced subgraph $H[V_3]$ is a P_k-construction, and
(b) $H \setminus H[V_3]$ consists of all triples whose profile is in $\{\{\!\{1,2,3\}\!\}, \{\!\{1,3,3\}\!\}, \{\!\{2,3,3\}\!\}\}$.

The pattern P_k can be written down explicitly, although this is not necessary for our proof later. For example, $P_2 = (5, \mathcal{E}_2)$ (see Fig. 1), where

$$\begin{aligned}\mathcal{E}_2 = \{&\{\!\{1,2,3\}\!\}, \{\!\{1,2,4\}\!\}, \{\!\{1,2,5\}\!\}, \{\!\{1,3,3\}\!\}, \{\!\{1,3,4\}\!\}, \{\!\{1,3,5\}\!\},\\ &\{\!\{1,4,4\}\!\}, \{\!\{1,4,5\}\!\}, \{\!\{1,5,5\}\!\}, \{\!\{2,3,3\}\!\}, \{\!\{2,3,4\}\!\}, \{\!\{2,3,5\}\!\},\\ &\{\!\{2,4,4\}\!\}, \{\!\{2,4,5\}\!\}, \{\!\{2,5,5\}\!\}, \{\!\{3,4,5\}\!\}, \{\!\{3,5,5\}\!\}, \{\!\{4,5,5\}\!\}\}.\end{aligned}$$

Our first result determines the Lagrangian of P_k for every $k \geq 1$. For convenience, we set $P_0 := (1, \{\emptyset\})$ and $\lambda_0 := 0$.

Proposition 3.1. *For every integer $k \geq 0$, we have $\lambda(P_{k+1}) = 1/\sqrt{3 - 2\lambda(P_k)}$ and the pattern P_{k+1} is minimal. In particular, $(\lambda(P_k))_{k=1}^{\infty}$ is the sequence in (1).*

Proof. We use induction on k where the base $k = 0$ is easy to check directly (or can be derived by adapting the forthcoming induction step to work for $k = 0$). Let $k \geq 1$.

Let us prove that $\lambda(P_{k+1}) = 1/\sqrt{3 - 2\lambda(P_k)}$. Recall that $P_k = (2k + 1, \mathcal{E}_k)$ and $P_{k+1} = (2k + 3, \mathcal{E}_{k+1})$. Let $(x_1, \ldots, x_{2k+3}) \in \Delta_{2k+2}$ be a P_{k+1}-optimal vector. Let $x := \sum_{i=3}^{2k+3} x_i = 1 - x_1 - x_2$. It follows from the definitions of P_{k+1} and the Lagrange polynomial that

$$\lambda(P_{k+1}) = \lambda_{\mathcal{E}_{k+1}}(x_1, \ldots, x_{2k+3}) = 6\left(x_1x_2x + (x_1 + x_2)\frac{x^2}{2}\right) + \lambda_{\mathcal{E}_k}(x_3, \ldots, x_{2k+3}). \quad (2)$$

Since $\lambda_{\mathcal{E}_k}(x_3, \ldots, x_{2k+3})$ is a homogeneous polynomial of degree 3, we have

$$\lambda_{\mathcal{E}_k}(x_3, \ldots, x_{2k+3}) = \lambda_{\mathcal{E}_k}\left(\frac{x_3}{x}, \ldots, \frac{x_{2k+3}}{x}\right)x^3 \leq \lambda(P_k)x^3.$$

So it follows from (2) and the 2-variable AM-GM inequality that

$$\begin{aligned}
\lambda(P_{k+1}) &\leq 6\left(\left(\frac{x_1 + x_2}{2}\right)^2 x + (x_1 + x_2)\frac{x^2}{2}\right) + \lambda(P_k)x^3 \\
&= 6\left(\left(\frac{1 - x}{2}\right)^2 x + (1 - x)\frac{x^2}{2}\right) + \lambda(P_k)x^3 = \frac{3x - (3 - 2\lambda(P_k))\,x^3}{2}.
\end{aligned}$$

Since $0 \leq \lambda(P_k) \leq 1$, one can easily show by taking the derivative that the maximum of the function $\left(3x - (3 - 2\lambda(P_k))\,x^3\right)/2$ on $[0, 1]$ is achieved if and only if $x = 1/\sqrt{3 - 2\lambda(P_k)}$, and the maximum value is $1/\sqrt{3 - 2\lambda(P_k)}$. This proves that $\lambda(P_{k+1}) \leq 1/\sqrt{3 - 2\lambda(P_k)}$.

To prove the other direction of this inequality, one just need to observe that when we choose

$$x_1 = x_2 = \frac{1}{2} - \frac{1}{2\sqrt{3 - 2\lambda(P_k)}} \quad \text{and} \quad (x_3, \ldots, x_{2k+3}) = \frac{1}{\sqrt{3 - 2\lambda(P_k)}}\,(y_1, \ldots, y_{2k+1}) \quad (3)$$

where $(y_1, \ldots, y_{2k+1}) \in \Delta_{2k}$ is a P_k-optimal vector, then all inequalities above hold with equality. Therefore, $\lambda(P_{k+1}) = 1/\sqrt{3 - 2\lambda(P_k)}$.

To prove that P_{k+1} is minimal, take any P_{k+1}-optimal vector $(x_1, \ldots, x_{2k+3}) \in \Delta_{2k+2}$; we have to show that it has no zero entries. This vector attains equality in all our inequalities above, which routinely implies that $(x_1, \ldots, x_{2k+3})$ must satisfy (3), for some P_k-optimal vector $(y_1, \ldots, y_{2k+1})$. We see that $x_1 = x_2$ are both non-zero because the sequence $(\lambda(P_0), \ldots, \lambda(P_{k+1}))$ is strictly increasing (since $x < 1/\sqrt{3 - 2x}$ for all $x \in$

$[0, 1))$ and thus $\lambda(P_k) < 1$. The remaining conclusion that $x_3, \dots, x_{2k+3}$ are non-zero follows from the induction hypothesis on $(y_1, \dots, y_{2k+1})$. ■

In order to finish the proof of Theorem 1.2 it suffices to prove that the degree of $\mu_k := \lambda(P_k)$ goes to infinity as $k \to \infty$. This is achieved by the last claim of the following lemma.

Lemma 3.2. *Let $p_1(x) := 3x^2 - 1$ and inductively for $k = 1, 2, \dots$ define*

$$p_{k+1}(x) = (2x^2)^{2^k} p_k\left(\frac{3x^2 - 1}{2x^2}\right), \quad \textit{for } x \in \mathbb{R}.$$

Then the following claims hold for each $k \in \mathbb{N}$:

(a) $p_k(\mu_k) = 0$;
(b) p_k is a polynomial of degree at most 2^k with integer coefficients: $p_k(x) = \sum_{i=0}^{2^k} c_{k,i} x^i$ for some $c_{k,i} \in \mathbb{Z}$;
(c) the integers $b_{k,i} := c_{k,i}$ for even k and $b_{k,i} := c_{k,2^k - i}$ for odd k satisfy the following:

(c.i) for each integer i with $0 \le i \le 2^k$, 3 divides $b_{k,i}$ if and only if $i \ne 2^k$;
(c.ii) 9 does not divide $b_{k,0}$;

(d) the polynomial p_k is irreducible of degree exactly 2^k;
(e) the degree of μ_k is 2^k.

Proof. Let us use induction on k. All stated claims are clearly satisfied for $k = 1$, when $p_1(x) = 3x^2 - 1$ and $\mu_1 = 1/\sqrt{3}$. Let us prove them for $k + 1$ assuming that they hold for some $k \ge 1$.

For Part (a), we have by Proposition 3.1 that

$$\frac{3\mu_{k+1}^2 - 1}{2\mu_{k+1}^2} = \frac{3/(3 - 2\mu_k) - 1}{2/(3 - 2\mu_k)} = \mu_k$$

and thus $p_{k+1}(\mu_{k+1}) = (2\mu_{k+1}^2)^{2^k} p_k(\mu_k)$, which is 0 by induction.

Part (b) also follows easily from the induction assumption:

$$p_{k+1}(x) = (2x^2)^{2^k} \sum_{i=0}^{2^k} c_{k,i} \left(\frac{3x^2 - 1}{2x^2}\right)^i = \sum_{i=0}^{2^k} c_{k,i} (3x^2 - 1)^i (2x^2)^{2^k - i}. \tag{4}$$

Let us turn to Part (c). The relation in (4) when taken modulo 3 reads that

$$\sum_{j=0}^{2^{k+1}} c_{k+1,j} x^j \equiv \sum_{i=0}^{2^k} c_{k,i} x^{2^{k+1} - 2i} \pmod 3.$$

Thus, $c_{k+1,j} \equiv c_{k,2^k-j/2} \pmod 3$ for all even j between 0 and 2^{k+1}, while $c_{k+1,j} \equiv 0 \pmod 3$ for odd j (in fact, $c_{k+1,j} = 0$ for all odd j since p_{k+1} is an even function). In terms of the sequences $(b_{\ell,j})_{j=0}^{2^\ell}$, this relation states that

$$b_{k+1,j} \equiv b_{k,j/2} \pmod 3 \quad \text{for all even } j \text{ with } 0 \le j \le 2^k,$$

while $b_{k+1,j} \equiv 0 \pmod 3$ for all odd j. This implies Part (c.i). For Part (c.ii), the relation in (4) when taken modulo 9 gives that $c_{k+1,0} \equiv c_{k,2^k}$ and $c_{k+1,2^{k+1}} \equiv c_{k,0} \cdot 2^{2^k} + c_{k,1} \cdot 3 \cdot 2^{2^k-1}$. Since $c_{k,1}$ is divisible by 3, we have in fact that $c_{k+1,2^{k+1}} \equiv c_{k,0} \cdot 2^{2^k} \equiv c_{k,0} \pmod 9$. By the induction hypothesis, this implies that 9 does not divide $b_{k+1,0}$.

By the argument above, $c_{k+1,2^{k+1}}$ is non-zero module 3 for odd k and non-zero module 9 for even k. Thus, regardless of the parity of k, the degree of the polynomial p_{k+1} is exactly 2^{k+1}. Moreover, p_{k+1} satisfies Eisenstein's criterion for prime $q = 3$ (namely, that q divides all coefficients, except exactly one at the highest power of x or at the constant term while the other of the two is not divisible by q^2). By the criterion (whose proof can be found in e.g. [16, Section 4]), the polynomial p_{k+1} is irreducible, proving Part (d).

By putting the above claims together, we see that μ_{k+1} is a root of an irreducible polynomial of degree 2^{k+1}, establishing Part (e). This completes the proof the lemma (and thus of Theorem 1.2) ■

4. Concluding remarks

Our proof of Theorem 1.2 shows that for every integer d which is a power of 2 there exists a finite family $\mathcal{F}$ of r-graphs such that $\pi(\mathcal{F})$ has algebraic degree d. It seems interesting to know whether this is true for all positive integers.

Problem 4.1. Let $r \ge 3$ be an integer. Is it true that for every positive integer d there exists a finite family $\mathcal{F}$ of r-graphs such that $\pi(\mathcal{F})$ has algebraic degree exactly d?

By considering other patterns, one can get Turán densities in $\Pi_{\text{fin}}^{(r)}$ whose algebraic degrees are not powers of 2. For example, the pattern $([3], \{\{1,2,3\}\}, \{1,2\})$ with *recursive parts* 1 and 2 (where we can take blowups of the single edge $\{\{1,2,3\}\}$ and recursively repeat this step inside the first and the second parts of each added blowup) gives a Turán density in $\Pi_{\text{fin}}^{(3)}$ (by [16, Theorem 3], a generalisation of Theorem 2.1) whose degree can be computed to be 3. However, we did not see any promising way of how to produce a pattern whose Lagrangian has any given degree d.

Data availability

No data was used for the research described in the article.

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