Untangling planar graphs from a specified vertex position—Hard cases

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Given a planar graph G, we consider drawings of G in the plane where edges are represented by straight line segments (which possibly intersect). Such a drawing is specified by an injective embedding π of the vertex set of G into the plane. Let $fix(G, \pi)$ be the maximum integer k such that there exists a crossing-free redrawing π' of G which keeps k vertices fixed, i.e., there exist k vertices $v_1, \ldots, v_k$ of G such that $\pi(v_i) = \pi'(v_i)$ for $i = 1, \ldots, k$. Given a set of points X, let $fix^X(G)$ denote the value of $fix(G, \pi)$ minimized over π locating the vertices of G on X. The absolute minimum of $fix(G, \pi)$ is denoted by $fix(G)$.

For the wheel graph W_n, we prove that $fix^X(W_n) \leq (2 + o(1))\sqrt{n}$ for every X. With a somewhat worse constant factor this is also true for the fan graph F_n. We inspect also other graphs for which it is known that $fix(G) = O(\sqrt{n})$.

We also show that the minimum value $fix(G)$ of the parameter $fix^X(G)$ is always attainable by a collinear X.

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1. Introduction

1.1. The problem of untangling a planar graph

In a *plane graph*, each vertex v is a point in $\mathbb{R}^2$ and each edge uv is represented as a continuous plane curve with endpoints u and v. All such curves are supposed to be non-self-crossing and any two of them either have no common point or share a common endvertex. An underlying abstract graph of a plane graph is called *planar*. A planar graph can be drawn as a plane graph in many ways, and the Wagner–Fáry–Stein theorem (see, e.g., [11]) states that there always exists a *straight line drawing* in which every edge is represented by a straight line segment.

Let $V(G)$ denote the vertex set of a planar graph G. In this paper, by a *drawing* of G we mean an arbitrary injective map $\pi : V(G) \to \mathbb{R}^2$. We suppose that each edge uv of G is drawn as the straight line segment with endpoints $\pi(u)$ and $\pi(v)$. Due to possible edge crossings and even overlaps, π may not be a plane drawing of G. Hence it is natural to ask:

How many vertices have to be moved to obtain from π a plane (i.e., crossing-free) straight line drawing of G?

Alternatively, we could allow in π curved edges without their exact specification; such a drawing could be always assumed to be a plane graph. Then our task would be to *straighten* π rather than to eliminate edge crossings.

More formally, for a planar graph G and a drawing π, let

$$fix(G, \pi) = \max_{\pi'} \left| \left\{ v \in V(G) : \pi'(v) = \pi(v) \right\} \right|$$

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where the maximum is taken over all plane straight line drawings π' of G. Furthermore, let

$$fix(G) = \min_\pi fix(G, \pi). \tag{1}$$

In other words, $fix(G)$ is the maximum number of vertices which can be fixed in any drawing of G while *untangling* it.

No efficient algorithm determining the parameter $fix(G)$ is known. Moreover, computing $fix(G, \pi)$ is known to be NP-hard [8,18].

Improving a result of Goaoc et al. [8], Bose et al. [5] showed that

$$fix(G) \geq (n/3)^{1/4}$$

for every planar graph G, where here and in the rest of this paper n denotes the number of vertices in the graph under consideration. Better bounds on $fix(G)$ are known for cycles [12], trees [8,5] and, more generally, outerplanar graphs [8,14]. In all these cases it was shown that $fix(G) = \Omega(n^{1/2})$. For cycles Cibulka [6] proves a better lower bound of $\Omega(n^{2/3})$.

Here we are interested in upper bounds on $fix(G)$, that is, in examples of graphs with small $fix(G)$. Moreover, let X be an arbitrary set of n points in the plane and define

$$fix^X(G) = \min_\pi \{fix(G, \pi) : \pi(V(G)) = X\}.$$

Note that $fix(G) = \min_X fix^X(G)$. This notation allows us to formalize another natural question. Can untangling of a graph become easier if the set X of vertex positions has some special properties (say, if it is known that X is *collinear*, i.e., lies on a line, or is in *convex position*, i.e., no $x \in X$ lies in the convex hull of $X \setminus \{x\}$)? This question admits several variations:

- For which X can one attain equality $fix^X(G) = fix(G)$?
- Are there graphs with $fix^X(G)$ small for *all* X?
- Are there graphs such that $fix^X(G)$ is for some X considerably larger than $fix(G)$?

1.2. Prior results

The *cycle* (resp. *path*; *empty graph*) on n vertices will be denoted by C_n (resp. P_n; E_n). Recall that the *join* of vertex-disjoint graphs G and H is the graph $G * H$ consisting of the union of G and H and all edges between $V(G)$ and $V(H)$. The graphs $W_n = C_{n-1} * E_1$ (resp. $F_n = P_{n-1} * E_1$; $S_n = E_{n-1} * E_1$) are known as *wheels* (resp. *fans*; *stars*). By kG we denote the disjoint union of k copies of a graph G.

Pach and Tardos [12] were the first to establish a principal fact: some graphs can be drawn so that, in order to untangle them, one has to shift almost all their vertices. In fact, this is already true for cycles. More precisely, Pach and Tardos [12] proved that

$$fix^X(C_n) = O((n \log n)^{2/3}) \quad \text{for any } X \text{ in convex position.} \tag{2}$$

The best known upper bounds are of the form $fix(G) = O(\sqrt{n})$. Goaoc et al. [9][1] showed it for certain triangulations. More specifically, they proved that

$$fix^X(P_{n-2} * P_2) < \sqrt{n} + 2 \quad \text{for any collinear } X. \tag{3}$$

Shortly after [9] and independently of it, there appeared our manuscript [10], which was actually a starting point of the current paper. For infinitely many n, we constructed a family $\mathcal{H}_n$ of 3-connected planar graphs on n vertices with $\max_{H \in \mathcal{H}_n} fix(H) = o(n)$. Though no explicit bound was specified in [10], a simple analysis of our construction reveals that

$$fix^X(H_n) \leq 2\sqrt{n} + 1 \quad \text{for any } X \text{ in convex position,} \tag{4}$$

where H_n denotes an arbitrary member of $\mathcal{H}_n$. While the graphs in $\mathcal{H}_n$ are not as simple as $P_{n-2} * P_2$ and the subsequent examples in the literature, the construction of $\mathcal{H}_n$ has the advantage that this class contains graphs with certain special properties, such as bounded vertex degrees. By a later result of Cibulka [6], we have $fix(G) = O(\sqrt{n}(\log n)^{3/2})$ for every G with maximum degree and diameter bounded by a logarithmic function. Note in this respect that $\mathcal{H}_n$ contains graphs with bounded maximum degree that have diameter $\Omega(\sqrt{n})$.

In subsequent papers [16,5], examples of graphs with small $fix(G)$ were found in special classes of planar graphs, such as outerplanar and even acyclic graphs. Spillner and Wolff [16] showed for the fan graph that

$$fix^X(F_n) < 2\sqrt{n} + 1 \quad \text{for any collinear } X \tag{5}$$

and Bose et al. [5] established for the star forest with $n = k^2$ vertices that

$$fix^X(kS_k) \leq 3\sqrt{n} - 3 \quad \text{for any collinear } X. \tag{6}$$

Finally, Cibulka [6] proved that

$$fix^X(G) = O((n \log n)^{2/3}) \quad \text{for any } X \text{ in convex position}$$

for all 3-connected planar graphs.

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[1] The papers [9,16] from conference proceedings were subsequently combined into the journal paper [8].
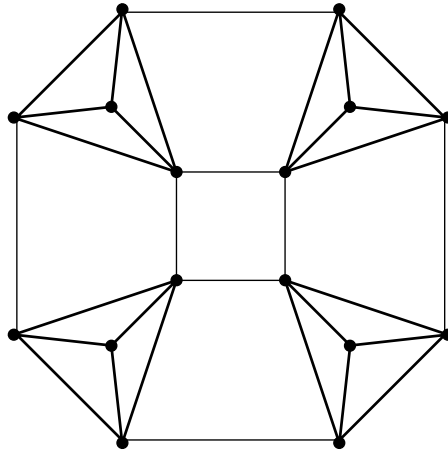
Fig. 1. Example of a graph in $\mathcal{H}_{16}$.

1.3. Our present contribution

In Section 2 we notice that the choice of a collinear vertex position in (3), (5) and (6) is actually optimal for proving upper bounds on $fix(G)$. Specifically, we show that for any G the equality $fix^X(G) = fix(G)$ is attained by some collinear X (see Theorem 2.1).

In Section 3 we extend the bound $fix(G) = O(\sqrt{n})$ in the strongest way with respect to specification of vertex positions. We prove that

$$fix^X(W_n) \leq (2 + o(1))\sqrt{n} \quad \text{for every } X, \tag{7}$$

$$fix^X(F_n) \leq (2\sqrt{2} + o(1))\sqrt{n} \quad \text{for every } X \tag{8}$$

(see Theorem 3.5). Let us define

$$FIX(G) = \max_X fix^X(G)$$

(while $fix(G) = \min_X fix^X(G)$). With this notation, (7) and (8) read

$$FIX(W_n) \leq (2 + o(1))\sqrt{n} \quad \text{and} \quad FIX(F_n) \leq (2\sqrt{2} + o(1))\sqrt{n}.$$

In Section 4 we discuss an approach attempting to give an analog of (7) for the aforementioned family of graphs $\mathcal{H}_n$. A member of this family is defined as a plane graph of the following kind. Let $k \geq 3$ and $n = k^2$. Draw k triangulations, each having k vertices, such that none of them lies inside an inner face of any other triangulation. Connect these triangulations by some more edges making the whole graph 3-connected. $\mathcal{H}_n$ is the set of all 3-connected planar graphs obtainable in this way. This set is not empty. Indeed, we can allocate the k triangulations in a cyclic order and connect each neighboring pair by two vertex-disjoint edges as shown in Fig. 1. Note that k new edges form a cycle C_k and the other k new edges participate in a cycle C_{2k}. If we remove any two vertices from the graph obtained, each triangulation and the whole "cycle" stay connected (since the aforementioned cycles C_k and C_{2k} are vertex-disjoint, at most one of them can get disconnected).

Note that, if we start with triangulations with bounded vertex degrees, the above construction gives us a graph with bounded maximum degree. In this situation our argument for (7) does not work. We hence undertake a different approach.

Given a set of colored points in the plane, we call it *clustered* if its monochromatic parts have pairwise disjoint convex hulls. Given a set X of $n = k^2$ points, let $C(X)$ denote the maximum cardinality of a clustered subset existing in X under any balanced coloring of X in k colors (see Definition 4.1). It is not hard to show (see Lemma 4.2) that

$$fix^X(H_n) \leq C(X) + k, \tag{9}$$

where H_n denotes an arbitrary graph in $\mathcal{H}_n$. We prove that $C(X) = O(n/\log n)$ for every X, which implies that $FIX(H_n) = O(n/\log n)$ (Theorem 4.4).

Better upper bounds for $C(X)$ would give us better upper bounds for $FIX(H_n)$. Note that $C(X)$ has relevance also to the star forest kS_k, namely

$$fix^X(kS_k) \geq C(X) - k \tag{10}$$

(see part 2 of Lemma 4.2). Thus, if there were a set X with $C(X) \gg k$, the parameter $FIX(kS_k)$ would be far apart from $fix(kS_k)$.

As we do not know how close or far away the parameters $fix(G)$ and $FIX(G)$ are for $G = H_n$ and $G = kS_k$, the two graph families deserve further attention. Section 5 is devoted to estimation of $fix^X(G)$ for X in *weakly convex position*, which means

that the points in X lie on the boundary of a convex body (including the cases where X is in convex position and where X is a collinear set). Since $C(X) < 2k$ for any X in weakly convex position, by (9) we obtain $fix^X(H_n) < 3\sqrt{n}$ for such X (Theorem 5.2).

This result for $\mathcal{H}_n$ together with the stronger results obtained for W_n and F_n in Section 3 might suggest that $fix^X(G) = O(fix(G))$ should hold for any G whenever X is in weakly convex position. The simplest case where we are not able to confirm this conjecture is $G = kS_k$. By (9) and (10) we have $fix^X(H_n) \leq fix^X(kS_k) + 2k$ for any k and $n = k^2$, and bounding $fix^X(kS_k)$ from above seems harder. Nevertheless, even here we have a rather tight bound: if X is in weakly convex position, then $fix^X(kS_k) = O(\sqrt{n}\, 2^{\alpha(\sqrt{n})})$, where $\alpha(\cdot)$ denotes the inverse Ackermann function (Theorem 5.4).

We conclude with a list of open questions in Section 6.

2. Hardness of untangling from a collinear position

Theorem 2.1. *For every planar graph G we have $fix(G) = fix^X(G)$ for some collinear X.*

Theorem 2.1 can be deduced from [5, Lemma 1]. For the reader's convenience, we give a self-contained proof.

Proof. Let $fix^-(G)$ denote the minimum value of $fix^X(G)$ over collinear X. We have $fix(G) \leq fix^-(G)$ by definition. The theorem actually states the converse inequality $fix(G) \geq fix^-(G)$. That is, given an arbitrary drawing $\pi : V(G) \to \mathbb{R}^2$, we have to show that it can be untangled while keeping at least $fix^-(G)$ vertices fixed.

Choose Cartesian coordinates in the plane such that $\pi(V(G))$ is located between the lines $y = 0$ and $y = 1$. Let $p_x, p_y : \mathbb{R}^2 \to \mathbb{R}$ denote the projections onto the x-axis and the y-axis, respectively. We also suppose that the axes are chosen such that the map $\lambda = p_x\pi$ is injective. Let us view λ as a drawing of G, aligning all the vertices on the line $y = 0$. By definition, there is a plane drawing λ' of G such that the set of fixed vertices $F = \{v \in V(G) : \lambda'(v) = \lambda(v)\}$ has cardinality at least $fix^-(G)$.

Given a set $A \subset \mathbb{R}^2$ and a real $\varepsilon > 0$, let $N_\varepsilon(A)$ denote the ε-neighborhood of A in the Euclidean metric. For each pair of disjoint edges e, e' in λ', there is an ε such that $N_\varepsilon(e) \cap N_\varepsilon(e') = \emptyset$. Since G is finite, we can assume that the latter is true with the same ε for all disjoint pairs e, e'.

We now define a drawing $\pi' : V(G) \to \mathbb{R}^2$ by setting

$$\pi'(v) = \begin{cases} (p_x\pi(v), \varepsilon p_y\pi(v)) & \text{if } v \in F, \\ \lambda'(v) & \text{otherwise.} \end{cases}$$

Note that $\pi'(v) \in N_\varepsilon(\lambda'(v))$ for every $v \in V(G)$. Since λ' is crossing-free, so is π'.

Finally, define a linear transformation of the plane by $a(x, y) = (x, \varepsilon^{-1}y)$ and consider $\pi'' = a\pi'$. Clearly, π'' is a plane drawing of G and all vertices in F stay fixed under the transition from π to π''. $\square$

3. Hardness of untangling from every vertex position

In Section 3.1 we state known results on the longest monotone subsequences in a random permutation. These results are used in Section 3.2 for proving upper bounds on $FIX(W_n)$ and $FIX(F_n)$.

3.1. Monotone subsequences in a random permutation

By a *permutation* of $[N] = \{1, 2, \ldots, N\}$ we will mean a sequence $S = s_1 s_2 \ldots s_N$ where each positive integer $i \leq N$ occurs once (that is, S determines a one-to-one map $S : [N] \to [N]$ by $S(i) = s_i$). A subsequence $s_{i_1} s_{i_2} \ldots s_{i_k}$, where $i_1 < i_2 < \cdots < i_k$, is *increasing* if $s_{i_1} < s_{i_2} < \cdots < s_{i_k}$. The length of a longest increasing subsequence of S will be denoted by $\ell(S)$.

Lemma 3.1. *Let S_N be a uniformly random permutation of $\{1, 2, \ldots, N\}$.*

1. *(Pilpel [13])* $\mathbb{E}[\ell(S_N)] \leq \sum_{i=1}^{N} 1/\sqrt{i} \leq 2\sqrt{N} - 1$.
2. *(Frieze [7]; Bollobás and Brightwell [4]) For any real $\epsilon > 0$ there is a $\beta = \beta(\epsilon) > 0$ such that for all $N \geq N(\epsilon)$ we have*

$$\mathbb{P}\left[\ell(S_N) \geq \mathbb{E}[\ell(S_N)] + N^{1/4+\epsilon}\right] \leq \exp\left(-N^\beta\right).$$

Further concentration results for $\ell(S_N)$ are obtained in [17,3].

Lemma 3.1 shows that $\ell(S_N) \leq 2N^{1/2}(1 + N^{-1/4+\epsilon})$ with probability at least $1 - \exp\left(-N^\beta\right)$. We will also need a bound for another parameter of S_N, roughly speaking, for the maximum total length of two non-interweaving monotone subsequences of S_N. Let us define this parameter more precisely. A subsequence of a permutation S will be called *monotone* if it can be made increasing by shifting and/or reversing (for example, 21543). This notion is rather natural if we regard S as a *circular permutation*, i.e., S is considered up to shifts. Call two subsequences S' and S'' of S *non-interweaving* if they have no common element and S has no subsequence $s_{i_1} s_{i_2} s_{i_3} s_{i_4}$ with s_{i_1}, s_{i_3} occurring in S' and s_{i_2}, s_{i_4} in S'' or vice versa. Define $\ell_2(S)$ to be the sum of the lengths of S' and S'' maximized over non-interweaving monotone subsequences of S.

Lemma 3.2. *Let S_N be a uniformly random permutation of $\{1, 2, \ldots, N\}$. For any real $\epsilon > 0$ there is a $\gamma = \gamma(\epsilon) > 0$ such that for all $N \geq N(\epsilon)$ we have*

$$\mathbb{P}\left[\ell_2(S_N) \geq 2\sqrt{2}N^{1/2} + 2N^{1/4+\epsilon}\right] \leq \exp\left(-N^\gamma\right). \tag{11}$$

Proof. Given a sequence $S_N = s_1 s_2 \ldots s_N$ and a pair of indices $1 \leq i < j \leq N$, consider the splitting of the circular version of S_N into two parts $P_1 = s_i \ldots s_{j-1}$ and $P_2 = s_j \ldots s_N s_1 \ldots s_{i-1}$. Let $P_1' = s_{j-1} \ldots s_i$ and $P_2' = s_{i-1} \ldots s_1 s_N \ldots s_j$ be the reverses of P_1 and P_2. Define

$$\lambda_{ij} = \max\{\ell(P_1), \ell(P_1')\} + \max\{\ell(P_2), \ell(P_2')\}.$$

Note that $\ell_2(S_N) = \lambda_{ij}$ for some pair i, j. Since there are only polynomially many such pairs, it suffices to show for each i, j that the inequality

$$\lambda_{ij} \geq 2\sqrt{2}N^{1/2} + 2N^{1/4+\epsilon} \tag{12}$$

holds with an exponentially small probability. Denote the length of P_k by N_k, so we have $N_1 + N_2 = N$. For each $k = 1, 2$, note that both $\ell(P_k)$ and $\ell(P_k')$ are distributed identically to $\ell(S_{N_k})$.

Suppose first that N_1 or N_2 is relatively small, say, $N_1 \leq 2(\sqrt{2} - 1)\sqrt{N}$. Then (12) implies that

$$\ell(P_2) \geq 2N_2^{1/2} + 2N_2^{1/4+\epsilon}$$

or this estimate is true for P_2'. Provided N and, hence, N_2 are large enough, we conclude by Lemma 3.1 that (12) happens with probability at most $2\exp(-N_2^\beta) \leq 2\exp\left(-\frac{1}{2}N^\beta\right)$.

Suppose now that $N_k > 2(\sqrt{2}-1)\sqrt{N}$ for both $k = 1, 2$ and that N is large enough. Since $N_1^{1/2} + N_2^{1/2} \leq 2\left(\frac{N_1+N_2}{2}\right)^{1/2} = \sqrt{2}N^{1/2}$, the inequality (12) entails that for $k = 1$ or $k = 2$ we must have

$$\ell(P_k) > 2N_k^{1/2} + N_k^{1/4+\epsilon}$$

or this estimate must be true for P_k'. By Lemma 3.1, the event (12) happens with probability no more than $4\exp\left(-c^\beta N^{\beta/2}\right)$, where $c = 2(\sqrt{2} - 1)$.

We see that, whatever N_1 and N_2 are, (11) holds for any positive $\gamma < \beta/2$ and large enough N. $\quad\square$

3.2. Graphs with small FIX(G)

Recall that $FIX(G) = \max_X fix^X(G)$. If $FIX(G)$ is small, this means that no special properties of the set of vertex locations can make the untangling problem for G easy.

Lemma 3.3. *For any 3-connected planar graph G on n vertices with maximum vertex degree $N = n - o(\sqrt{n})$ we have*

$$FIX(G) \leq (2 + o(1))\sqrt{n}.$$

Proof. We have to prove that $fix^X(G) \leq (2 + o(1))\sqrt{n}$ for any set X of n points. Let $X = \{x_1, \ldots, x_n\}$ and define $X_N = \{x_1, \ldots, x_N\}$. We need to fix the *north* direction in the plane $\mathbb{R}^2$. For definiteness, let it be determined by the vector $(0, 1)$. Given a point p in the plane, we define a permutation S_p describing the order in which the points in X_N are visible from the standpoint of p. If $p = x_s$ with $s \leq N$, we take p as the first visible point, that is, let s be the first index in the sequence S_p. Now, we look around, starting from the north, in a clockwise direction and put i before j in S_p if we see x_i earlier than x_j. If x_i and x_j lie in the same direction from p, we see the nearer point first, that is, i precedes j in S_p whenever $x_i \in [p, x_j]$.

Define an equivalence relation $\equiv$ such that $S \equiv S'$ if S and S' are obtainable from one another by a shift. Let us show that the quotient set $Q = \left\{S_p : p \in \mathbb{R}^2\right\} /_\equiv$ is finite and estimate its cardinality. Suppose first that not all points in X_N are collinear. Let L be the set of lines passing through at least two points in X_N. After removal of all lines in L, the plane is split into connected components that will be called *L-faces*. Any intersection point of two lines will be called an *L-vertex*. The L-vertices lying on a line in L split this line into *L-edges*. Exactly two L-edges for each line are unbounded. It is easy to see that $S_p \equiv S_{p'}$ whenever p and p' belong to the same L-face or the same L-edge. It follows that $|Q|$ does not exceed the total amount of L-faces, L-edges, and L-vertices.

Let us express this bound in terms of $l = |L| \leq \binom{N}{2}$. If we erase all the unbounded L-edges, we obtain a crossing-free straight line drawing of a planar graph with at most $\binom{l}{2}$ vertices. It has less than $\frac{3}{2}l^2 - \frac{3}{2}l$ edges and less than $l^2 - l$ faces. Restoring the unbounded L-edges, we see that the total number of L-edges is less than $\frac{3}{2}l^2 + \frac{1}{2}l$ and the number of L-faces is

less than $l^2 + l$. Therefore,

$$|Q| < (l^2 + l) + \left(\frac{3}{2}l^2 + \frac{1}{2}l\right) + \left(\frac{1}{2}l^2 - \frac{1}{2}l\right) < \frac{3}{4}N^4.$$

In the much simpler case of a collinear X_N, we have $|Q| \leq N$.

Let c be a vertex of G with maximum vertex degree. By the Whitney theorem on embeddability of 3-connected graphs, the neighbors of c appear around c in the same circular order $v_1, \ldots, v_N$ in any plane drawing of G. Pick up a random permutation σ of $\{1, \ldots, N\}$ and consider a drawing $\pi : V(G) \to X$ such that $\pi(v_i) = x_{\sigma(i)}$. Let π' be an unentanglement of π. Let $p = \pi'(c)$ and denote the set of all shifts and reverses of the permutation S_p by $\mathcal{S}_p$.

We have to estimate the number of vertices remaining fixed under the transition from π to π', that is, the cardinality of the set $F = \left\{\pi(v) : v \in V(G), \pi(v) = \pi'(v)\right\}$. Let $F^* = \{\pi(v_i) \in F : i \leq N\}$, which is the subset of F corresponding to the fixed neighbors of c. Note that $|F \setminus F^*| \leq n - N$ and recall that $n - N = o(\sqrt{n})$ by our assumption. It follows that $|F| \leq |F^*| + o(\sqrt{n})$, and we have to estimate $|F^*|$.

The points in F^* go around p in the canonical Whitney order. This means that the indices of the corresponding vertices form an increasing subsequence in $\sigma^{-1}S$ for some $S \in \mathcal{S}_p$. For each S, the composition $\sigma^{-1}S$ is a random permutation of $\{1, \ldots, N\}$. Recall that, irrespectively of the choice of $p = \pi'(c)$, there are at most $2N|Q| < \frac{3}{2}N^5$ possibilities for S. By Lemma 3.1, every increasing subsequence of $\sigma^{-1}S$ has length at most $2N^{1/2} + N^{1/4+\epsilon}$ with probability at least $1 - O\left(N^5 \exp\left(-N^\beta\right)\right)$. Thus, if N is sufficiently large, we have $|F^*| \leq (2 + o(1))\sqrt{n}$ for all unentanglements π' of some drawing π (in fact, this is true for almost all π). This implies the required bound $|F| \leq (2 + o(1))\sqrt{n}$.   □

While Lemma 3.3 immediately gives us a bound on $FIX(W_n)$ for the wheel graph, this lemma does not apply directly to the fan graph F_n because it is not 3-connected and has a number of essentially different plane drawings. Nevertheless, all these drawings are still rather structured, which makes analysis of the fan graph only a bit more complicated. Indeed, denote the central vertex of F_n by c and let $v_1 \ldots v_{n-1}$ be the path of the other vertices. Let α be a plane drawing of F_n. Label each edge $\alpha(c)\alpha(v_i)$ with number i and denote the circular sequence in which the labels follow each other around $\alpha(c)$ by R_α. Split R_α into two pieces. Let R'_α be the sequence of labels starting with 1, ending with $n - 1$, and containing all intermediate labels if we go around $\alpha(c)$ clockwise. Let R''_α be the counter-clockwise analog of R'_α. Note that R'_α and R''_α overlap in $\{1, n-1\}$.

Lemma 3.4. *Both R'_α and R''_α are monotone.*

Proof. We proceed by induction on n. The base case of $n = 3$ is obvious. Suppose that the claim is true for all plane drawings of F_n and consider an arbitrary plane drawing α of F_{n+1}. Let β be obtained from α by erasing $\alpha(v_n)$ along with the incident edges. Obviously, β is a plane drawing of F_n.

In the drawing α of F_{n+1}, we consider the triangle T with vertices $\alpha(c)$, $\alpha(v_{n-1})$, and $\alpha(v_n)$. Clearly, all points $\alpha(v_i)$ for $i \leq n - 2$ are inside T or all of them are outside. In both cases, $n - 1$ and n are neighbors in R_α. Therefore, R_α is obtainable from R_β by inserting n, on the one side or the other, next to $n - 1$. It follows that R'_α is obtained from R'_β either by appending n after $n - 1$ or by replacing $n - 1$ with n (the same is true for R''_α and R''_β). It remains to note that both operations preserve monotonicity.   □

We are now prepared for obtaining upper bounds on $FIX(G)$ for the wheel graph W_n and the fan graph F_n. Note that, up to a small constant factor, these bounds match the lower bound $fix(F_n) \geq fix(W_n) \geq \sqrt{n-2}$ (which follows, e.g., from [14, Theorem 4.1]).

Theorem 3.5.
1. $FIX(W_n) \leq (2 + o(1))\sqrt{n}.$
2. $FIX(F_n) \leq (2\sqrt{2} + o(1))\sqrt{n}.$

Proof. The bound for W_n follows directly from Lemma 3.3 as observed before.

As for F_n, notice that the argument of Lemma 3.3 becomes applicable if, in place of the Whitney theorem, we use Lemma 3.4. Let π be a random location of $V(F_n)$ on X, as in the proof of Lemma 3.3. More precisely, let $v_1 \ldots v_{n-1}$ denote the path of non-central vertices in F_n. We pick a random permutation σ of $\{1, \ldots, n-1\}$ and set $\pi(v_i) = x_{\sigma(i)}$. As established in the proof of Lemma 3.3, the set X determines a set of permutations $\mathcal{S}_X$ with $|\mathcal{S}_X| = O(n^4)$ such that, from any standpoint p in the plane, the vertices $v_1, \ldots, v_{n-1}$ are visible in the circular order $\tau_p = \sigma^{-1}S$ for some $S \in \mathcal{S}_X$.

Let α be any untangling of π and R_α be the associated order on the neighborhood of the central vertex $\alpha(c)$. By Lemma 3.4, R_α consists of two monotone parts R'_α and R''_α. The set F of fixed vertices is correspondingly split into F' and F''. Since R'_α and R''_α overlap in two elements, F' and F'' can have one or two common vertices. If this happens, we remove those from F''. Notice that the indices of the vertices in F' and in F'' form non-interweaving monotone subsequences of $\tau_{\alpha(c)}$. Therefore, $|F'| + |F''| \leq \ell_2(\tau_{\alpha(c)})$ and part 2 of the theorem follows from Lemma 3.2.   □

4. Making convex hulls disjoint

In Section 1.2 we listed the few graphs for which an upper bound $fix(G) = O(\sqrt{n})$ is known, namely $P_{n-2} * P_2$, F_n, any $H_n \in \mathcal{H}_n$, and kS_k. By Theorem 3.5 in the former two cases we have a stronger result $FIX(G) = O(\sqrt{n})$ (note that $P_{n-2} * P_2$ contains W_n as a subgraph). We now consider a problem related to estimating the parameters $FIX(H_n)$ and $FIX(kS_k)$.

Definition 4.1. Let $n = k^2$ and X be an n-point set in the plane. Given a partition $X = X_1 \cup \cdots \cup X_k$, we regard $\mathcal{X} = \{X_1, \ldots, X_k\}$ as a coloring of X in k colors. We will consider only *balanced* $\mathcal{X}$ with each $|X_i| = k$. Call a set $Y \subseteq X$ *clustered* if the monochromatic classes $Y_i = Y \cap X_i$ have pairwise disjoint convex hulls. Let $C(X, \mathcal{X})$ denote the largest size of a clustered subset of X. Finally, define $C(X) = \min_{\mathcal{X}} C(X, \mathcal{X})$.

Lemma 4.2. *Let H_n denote an arbitrary graph in $\mathcal{H}_n$, where $n = k^2$ with $k \geq 3$.*

1. $\mathrm{fix}^X(H_n) \leq C(X) + k$.
2. $\mathrm{fix}^X(kS_k) \geq C(X) - k$.

Proof. 1. Recall that H_n is defined as a plane graph whose vertex set $V(H_n) = V_1 \cup \cdots \cup V_k$ is partitioned so that each V_i spans a triangulation and these k triangulations are in the outer faces of each other. Take $\mathcal{X}$ such that $C(X, \mathcal{X}) = C(X)$ and $\pi : V(H_n) \to X$ such that $\pi(V_i) = X_i$. Consider an unentanglement π' of π and denote the set of fixed vertex locations by Y. By the Whitney theorem, π' is obtainable from the plane graph H_n by a homeomorphism of the plane, possibly after turning some inner face of H_n into the outer face. Since V_i spans a triangulation in H_n, the convex hull of $\pi'(V_i)$ is a triangle T_i. Since the corresponding triangulations are pairwise disjoint in H_n, the triangles T_i are pairwise disjoint possibly with a single exception for some T_s containing all the other triangles. Let $Y_i = Y \cap X_i$. It follows that the convex hulls of the Y_i's do not intersect, perhaps with an exception for a single set Y_s. The exception may occur if π' is homeomorphic to a version of H_n with different outer face. Therefore, $|Y| \leq C(X) + k$, where the term k corresponds to the exceptional Y_s.

2. Given an arbitrary drawing $\pi : V(kS_k) \to X$ of the star forest, we have to untangle it while keeping at least $C(X) - k$ vertices fixed. Let $V(kS_k) = V_1 \cup \cdots \cup V_k$ where each V_i is the vertex set of a star component. Define a coloring $\mathcal{X}$ of X by $X_i = \pi(V_i)$. Let Y be a largest clustered subset of X. Choose pairwise disjoint open convex sets $C_1, \ldots, C_k$ such that C_i contains $Y_i = Y \cap X_i$ for all i. Redraw kS_k so that, for each i, the ith star component is contained in C_i. It is clear that, doing this, we can leave all non-central vertices in Y fixed. Thus, we have at least $|Y| - k \geq C(X) - k$ fixed vertices. $\square$

Lemma 4.3. *For any set X of $n = k^2$ points in the plane, we have $C(X) = O(n/\log n)$.*

Proof. Let $B(X)$ denote the set of all balanced k-colorings of X, i.e., the set of partitions $X = X_1 \cup \cdots \cup X_k$ with each $|X_i| = k$. We have $|B(X)| = n!/(k!)^k$.

Call a k-tuple of subsets $Z_1, \ldots, Z_k \subset X$ a *crossing-free coloring* of X if the Z_i's have pairwise disjoint convex hulls. We do not exclude the possibility that some Z_i's are empty and the coloring is partial, i.e., $\bigcup_{i=1}^{k} Z_i \subsetneq X$. Denote the set of all crossing-free colorings of X by $F(X)$.

Let $\mathcal{X} \in B(X)$. An estimate $C(X, \mathcal{X}) \geq a$ means that

$$\sum_{i=1}^{k} |X_i \cap Z_i| \geq a \tag{13}$$

for some $\mathcal{Z} \in F(X)$. Regard $\mathcal{X}$ and $\mathcal{Z}$ as elements of the space $\{1, \ldots, k, k+1\}^X$ of $(k+1)$-colorings of X, where the new color $k+1$ is assigned to the points that are uncolored in $\mathcal{Z}$. Then (13) means that the Hamming distance between $\mathcal{X}$ and $\mathcal{Z}$ does not exceed $n - a$. Note that the $(n-a)$-neighborhood of $\mathcal{Z}$ can contain no more than $\binom{n}{n-a} k^{n-a}$ elements of $B(X)$. Therefore, an estimate $C(X) < a$ would follow from the inequality

$$|F(X)| \binom{n}{a} k^{n-a} < |B(X)|. \tag{14}$$

Given a partition $Z = P_1 \cup \cdots \cup P_m$ of a point set Z, we call it *crossing-free* if the convex hulls of the P_i's are nonempty and pairwise disjoint. According to Sharir and Welzl [15, Theorem 5.2], the overall number of crossing-free partitions of any l-point set Z is at most $O(12.24^l)$. In order to derive from here a bound for the number of crossing-free *colorings*, with each coloring $(Z_1, \ldots, Z_k)$ we associate a partition $(P_1, \ldots, P_m)$ of the union $Z = \bigcup_{i=1}^{k} Z_i$ such that $(P_1, \ldots, P_m)$ is the result of removing all empty sets from the sequence $(Z_1, \ldots, Z_k)$. Since $(P_1, \ldots, P_m)$ is the crossing-free partition of a subset of X, the Sharir–Welzl bound implies that the number of all possible partitions $(P_1, \ldots, P_m)$ obtainable in this way does not exceed $O(24.48^n)$. Since $(Z_1, \ldots, Z_k)$ can be restored from $(P_1, \ldots, P_m)$ in $\binom{k}{m}$ ways, we obtain $|F(X)| < c2^k 24.48^n$ for a constant c. Thus, we would have (14) provided

$$c2^k 24.48^n \frac{n^a}{a!} k^{n-a} \leq \frac{n!}{(k!)^k}.$$

Taking the logarithm of both sides, we see that the latter inequality holds for all sufficiently large n if we set $a = 6.4n/\ln n$. $\square$

Part 1 of Lemma 4.2 and Lemma 4.3 immediately give us the main result of this section.

Theorem 4.4. $FIX(H_n) = O(n/\log n)$ *for an arbitrary $H_n \in \mathcal{H}_n$.*

Note that the bound of Theorem 4.4 is the best upper bound on $FIX(G)$ that we know for graphs with bounded vertex degrees.

5. Hardness of untangling from a weakly convex position

Despite the observations made in Section 4, we do not know whether or not $fix^X(H_n)$ and $fix^X(kS_k)$ are close to, respectively, $fix(H_n)$ and $fix(kS_k)$ for every location X of the vertex set. We now restrict our attention to point sets X in weakly convex position, i.e., on the boundary of a convex plane body.

We will use Davenport–Schinzel sequences defined as follows (see, e.g., [1] for more details). An integer sequence $S = s_1 \ldots s_n$ is called a *(k, p)-Davenport–Schinzel sequence* if the following conditions are met:

- $1 \le s_i \le k$ for each $i \le n$;
- $s_i \ne s_{i+1}$ for each $i < n$;
- S contains no subsequence *xyxyxy* . . . of length $p + 2$ for any $x \ne y$.

By a *subsequence* of S we mean any sequence $s_{i_1} s_{i_2} \ldots s_{i_m}$ with $i_1 < i_2 < \cdots < i_m$. The maximum length of a (k, p)-Davenport–Schinzel sequence will be denoted by $\lambda_p(k)$. We are interested in the particular case of $p = 4$.

We inductively define a family of functions over positive integers:

$$
\begin{array}{ll}
A_1(n) = 2n & n \ge 1, \\
A_k(1) = 2 & k \ge 1, \\
A_k(n) = A_{k-1}(A_k(n-1)) & n \ge 2, \ k \ge 2.
\end{array}
$$

Ackermann's function is defined by $A(n) = A_n(n)$. This function grows faster than any primitive recursive function. The inverse of Ackermann's function is defined by $\alpha(n) = \min \{t \ge 1 : A(t) \ge n\}$.

Agarwal et al. [2] proved that $\lambda_4(k) = O(k2^{\alpha(k)})$. Note that $\alpha(n)$ grows very slowly, e.g., $\alpha(n) \le 4$ for all n up to $A(4)$, which is the exponential tower of twos of height 65 536. Thus, the bound for $\lambda_4(k)$ is nearly linear in k.

Sometimes it will be convenient to identify a sequence $S = s_1 \ldots s_n$ with all its cyclic shifts. This way, $s_j s_n s_1 s_i$, where $i < j$, is a subsequence of S. In such circumstances we will call a sequence *circular*. Subsequences of S will be regarded also as circular sequences. Note that the set of all circular subsequences is the same for S and any of its shifts. The length of S will be denoted by $|S|$.

Lemma 5.1. *Let $k, s \ge 1$ and $S^{k,s}$ be the circular sequence consisting of s successive blocks of the form $12 \ldots k$.*

1. *Suppose that S is a subsequence of $S^{k,s}$ with no 4-subsubsequence of the form xyxy, where $x \ne y$. Then $|S| < k + s$.*
2. *Suppose that S is a subsequence of $S^{k,s}$ with no 6-subsubsequence of the form xyxyxy, where $x \ne y$. Then $|S| < \lambda_4(k) + s \le O(k2^{\alpha(k)}) + s$.*

Proof. 1. We proceed by double induction on k and s. The base case where $k = 1$ and s is arbitrary is trivial. Let $k \ge 2$ and consider a subsequence S with no forbidden subsubsequence. If each of the k elements occurs in S at most once, then $|S| \le k$ and the claimed bound is true. Otherwise, without loss of generality we suppose that S contains $\ell \ge 2$ occurrences of k. Let $A_1, \ldots, A_\ell$ (resp. $B_1, \ldots, B_\ell$) denote the parts of S (resp. $S^{k,s}$) between these ℓ elements. Thus, $|S| = \ell + \sum_{i=1}^{\ell} |A_i|$.

Denote the number of elements with at least one occurrence in A_i by k_i. Each element x occurs in at most one of the A_i's because otherwise S would contain a subsequence $xkxk$. It follows that $\sum_{i=1}^{\ell} k_i \le k - 1$. Note that, if we append B_i with an element k, it will consist of blocks $12 \ldots k$. Denote the number of these blocks by s_i and notice the equality $\sum_{i=1}^{\ell} s_i = s$. Since A_i has no forbidden subsequence, we have $|A_i| \le k_i + s_i - 1$. If $k_i \ge 1$, this follows from the induction assumption because A_i can be regarded as a subsequence of S^{k_i, s_i}. If $k_i = 0$, this is also true because then $|A_i| = 0$. Summarizing, we obtain $|S| \le \ell + \sum_{i=1}^{\ell} (k_i + s_i - 1) \le \ell + (k - 1) + s - \ell < k + s$.

2. Let S' be obtained from S by shrinking each block $z \ldots z$ of the same elements to z. Since S' is a $(k, 4)$-Davenport–Schinzel sequence, we have $|S'| \le \lambda_4(k)$. Note now that any two elements neighboring in a shrunken block are at distance at least $k - 1$ in $S^{k,s}$. It readily follows that the total number of elements deleted in S is less than s. □

Theorem 5.2. *Let H_n be an arbitrary graph in $\mathcal{H}_n$. For any X in weakly convex position we have*

$$fix^X(H_n) < 3\sqrt{n}.$$

Proof. By part 1 of Lemma 4.2, it suffices to show that $C(X) < 2k$ for any set X of $n = k^2$ points on the boundary Γ of a convex body. Let $\mathcal{X}$ be the interweaving k-coloring of X where the colors appear along Γ in the circular sequence $S^{k,k}$ as in Lemma 5.1. Suppose that Y is a clustered subset of X. Note that there are no two pairs $\{y_1, y_2\} \subset Y \cap X_i$ and $\{y_1', y_2'\} \subset Y \cap X_j$, $i \ne j$, with intersecting segments $[y_1, y_2]$ and $[y_1', y_2']$. This means that the subsequence of $S^{k,k}$ induced by Y does not contain any pattern *ijij*. By part 1 of Lemma 5.1, we have $|Y| < 2k$ and, hence, $C(X, \mathcal{X}) < 2k$ as required. □

Remark 5.3. With a little more care, we can improve the constant factor in Theorem 5.2 by proving that $fix^X(H_n) \le 2\sqrt{n} + 1$ for any X in weakly convex position.
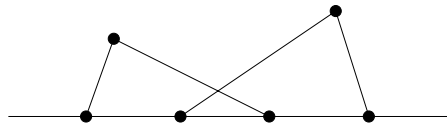
Fig. 2. Proof of part 1 of Theorem 5.4: an *ijij*-subsequence in *A*.

The rest of this section is devoted to the star forest kS_k. This sequence of graphs is of especial interest because this is the only example of graphs for which we know that $fix(G) = O(\sqrt{n})$ but are currently able to prove neither that $FIX(G) = o(n)$ nor that $fix^X(G) = O(\sqrt{n})$ for X in weakly convex position.

The first part of the forthcoming Theorem 5.4 restates [5, Theorem 5] (see (6) in Section 1.2) with a worse factor in front of $\sqrt{n}$; we include it for an expository purpose. The proof of this part is based on part 1 of Lemma 5.1, which we already used to prove Theorem 5.2. The second part, which is our primary interest, requires a more delicate analysis based on part 2 of Lemma 5.1.

Theorem 5.4. *Let kS_k denote the star forest with $n = k^2$ vertices. For every integer $k \geq 2$ we have*

1. *$fix^X(kS_k) < 7\sqrt{n}$ for any collinear X;*
2. *$fix^X(kS_k) = O(\sqrt{n}2^{\alpha(\sqrt{n})})$ for any X in weakly convex position.*

Proof. Define $V = V(kS_k)$. Let $V = \bigcup_{i=1}^{k} V_i \cup C$, where each V_i consists of all $k - 1$ leaves in the same star component and C consists of all k central vertices.

1. Suppose that X consists of points $x_1, \ldots, x_n$ lying on a line ℓ in this order. Consider a drawing $\pi : V \to X$ such that

$$\pi(V_i) = \{x_i, x_{i+k}, x_{i+2k}, \ldots, x_{i+(k-2)k}\} \quad \text{for each } i \leq k, \tag{15}$$
$$\pi(C) = \{x_{n-k+1}, x_{n-k+2}, \ldots, x_n\}.$$

Let π' be a crossing-free straight line redrawing of kS_k. We have to estimate the number of fixed vertices, i.e., those vertices participating in $F = \{\pi(v) : v \in V, \ \pi(v) = \pi'(v)\}$. For this purpose we split F into four parts: $F = A \cup B \cup D \cup E$ where A (resp. B; D) consists of the fixed leaves adjacent to central vertices located in π' above ℓ (resp. below ℓ; on ℓ) and E consists of the fixed central vertices.

Trivially, $|E| \leq k$ and it is easy to see that $|D| \leq 2k$. Let us estimate $|A|$ and $|B|$. Label each x_m by the index i for which $x_m \in \pi(V_i)$ and view $x_1 x_2 \ldots x_{n-k}$ as the sequence $S^{k,k-1}$ defined in Lemma 5.1. Let S be the subsequence induced by the points in A. Note that S does not contain any subsequence $ijij$ because otherwise we would have an edge crossing in π' (see Fig. 2). By part 1 of Lemma 5.1, we have $|A| = |S| < 2k$. The same applies to B. It follows that $|F| = |A| + |B| + |D| + |E| < 7k$, as claimed.

2. Let X be a set of $n = k^2$ points on the boundary Γ of a convex plane body P. It is known that the boundary of a convex plane body is a rectifiable curve and, therefore, we can speak of the length of Γ or its arcs. Clearly, the convex body P plays a nominal role and can be varied once X is fixed. Thus, to avoid unnecessary technical complications in the forthcoming argument, without loss of generality we can suppose that the boundary curve Γ contains only a finite number of (maximal) straight line segments. In particular, we can suppose that Γ contains no straight line segment at all if X is in "strictly" convex position.

We will use the following terminology. A *chord* is a straight line segment whose endpoints lie on Γ. An *arrow* is a directed chord with one endpoint called a *head* and the other called a *tail*. Call an arrow a *median* if its endpoints split Γ into arcs of equal length. Fix the "clockwise" order of motion along Γ and color each non-median arrow in one of two colors, red if the shortest way along Γ from the tail to the head is clockwise and blue if it is counter-clockwise.

Given a point a outside P, we define *quiver Q_a* as follows. For each line going through a and intersecting Γ in exactly two points, h and t, Q_a contains the arrow th directed so that the head is closer to a than the tail.

Given a non-median arrow th, we will denote the shorter component of $\Gamma \setminus \{t, h\}$ by $\Gamma[t, h]$. Our argument will be based on the following elementary fact.

Claim A. *Let arrows th and $t'h'$ be in the same quiver Q and have the same color. Suppose that $\Gamma[t', h']$ is shorter than $\Gamma[t, h]$. Then both t' and h' lie in $\Gamma[t, h]$.*

Proof of Claim A. Let t^*h^* be the median in Q. Since th and $t'h'$ are of the same color, the four points t, h, t', h' are in the same component of $\Gamma \setminus \{t^*, h^*\}$. The claim readily follows from the fact that the chords th and $t'h'$ do not cross (see Fig. 3). □

After these preliminaries, we begin with the proof. Let $x_1, \ldots, x_n$ be a listing of points in X along Γ. Fix π to be an arbitrary map satisfying (15). Let π' be a crossing-free redrawing of kS_k. Look at the edges in π' with one endpoint $\pi'(v)$ on Γ and the other endpoint elsewhere. Perturbing π' a little at the positions not lying on Γ (and using the regularity assumption made about Γ), we can ensure that

(1) any such edge intersects Γ in at most two points, including $\pi'(v)$ (this is automatically true if Γ contains no straight line segment);

(2) if an edge intersects Γ in two points, it splits Γ into components having different lengths.
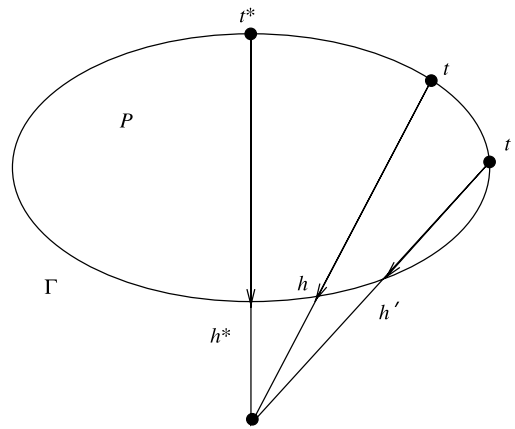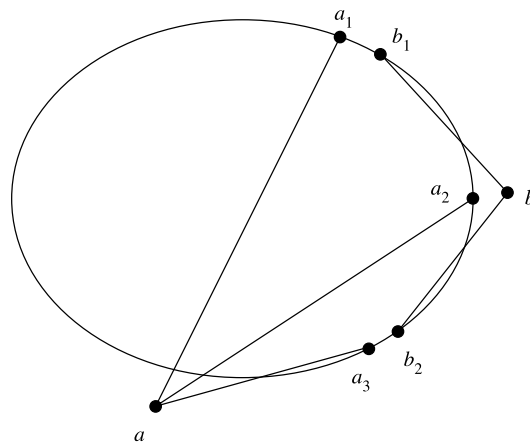
Fig. 3. Proof of Claim A.

Fig. 4. Proof of part 2 of Theorem 5.4: the impossibility of an *ijijij*-subsequence in *R*.

Assume that π' meets these conditions. Let v be a leaf adjacent to a central vertex c. Suppose that $\pi'(v) \in \Gamma$, $\pi'(c) \notin P$, and the segment $\pi'(v)\pi'(c)$ crosses Γ at a point $h \neq \pi'(v)$. By Condition 2, the arrow $\pi'(v)h$ is not a median and hence colored in red or blue. We color each such $\pi'(v)$ in red or blue correspondingly.

Now we split the set of fixed vertices F into five parts. Let E consist of the fixed central vertices, I (resp. O) consist of those fixed leaves such that the edges emanating from them are completely inside (resp. outside) P, and R (resp. B) consist of the red (resp. blue) fixed leaves. By Condition 1, we have $F = E \cup I \cup O \cup R \cup B$.

Trivially, $|E| \leq k$. Like in the proof of the first part of the theorem, notice that the subsequences of $S^{k,k-1}$ corresponding to I and O do not contain *ijij*-subsequences. By part 1 of Lemma 5.1, we have $|I| < 2k$ and $|O| < 2k$.

Finally, consider the subsequence S of $S^{k,k-1}$ corresponding to R and show that it does not contain any *ijijij*-subsubsequence. Assume, to the contrary, that such a subsubsequence exists. This means that $x_1 \ldots x_{n-k}$ contains two interleaving subsequences $a_1 a_2 a_3$ and $b_1 b_2 b_3$ whose elements belong to two different star components of π', with central vertices a and b, respectively. Since a_1, a_2, a_3 are red, Claim A implies that, say, a_2 and a_3 lie on the shorter arc of Γ cut off by the edge aa_1 (see Fig. 4).

Without loss of generality, let b_1 be between a_1 and a_2 and b_2 be between a_2 and a_3. Since b_1 and b_2 are red and π' is crossing-free, it must be the case that bb_1 intersects $\Gamma[a_1, a_2]$ and bb_2 intersects $\Gamma[a_2, a_3]$ (in another point). This makes a contradiction with Claim A.

Thus, S is *ijijij*-free and, by part 2 of Lemma 5.1, we have $|R| = |S| \leq O(k2^{\alpha(k)})$. The same applies to B. Summarizing, we see that $|F| = |E| + |I| + |O| + |R| + |B| \leq O(k2^{\alpha(k)})$, as claimed. $\quad\square$

6. Open problems

1. Can the parameters *fix*(G) and *FIX*(G) be far apart from each other for some planar graphs? Say, is it possible that for infinitely many graphs we have *FIX*(G) $\geq n^{\epsilon}$*fix*(G) with a constant $\epsilon > 0$?

2. Lemma 4.3 states an upper bound $C(X) = O(n/\log n)$ for any set X of $n = k^2$ points in the plane. A trivial lower bound is $C(X) \geq \sqrt{n}$. How do we make the gap closer? By Lemma 4.2, in this way we could show either that $FIX(H_n)$ is close to $fix(H_n)$ or that $FIX(kS_k)$ is far from $fix(kS_k)$.

3. Find upper bounds on $FIX(G)$, at least $FIX(G) = o(n)$, for the cycle C_n, the star forest kS_k, and the uniform binary tree. Recall that upper bounds on $fix(G)$ for these graphs are obtained in [12,5,6], respectively (the uniform binary tree is just a particular instance of the class of graphs with logarithmic vertex degrees and diameter treated in [6]).

4. Let $Fix(G)$ denote the maximum of $fix^X(G)$ over X in weakly convex position. Obviously, $fix(G) \leq Fix(G) \leq FIX(G)$. Note that the first inequality can be strict: for example, $fix(K_4) = 2$ while $Fix(K_4) = 3$ for the tetrahedral graph. Is it true that $Fix(G) = O(fix(G))$? Currently we cannot prove this even for graphs $G = kS_k$; cf. Theorem 5.4.

5. By Theorem 2.1, for every G we have $fix(G) = fix^X(G)$ for some collinear X. Does this equality hold for *every* collinear X? This question is related to the discussion in [14, Section 5.1].

Acknowledgements

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