Supersaturation problem for the bowtie

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A R T I C L E   I N F O

A B S T R A C T

The Turán function $\mathrm{ex}(n, F)$ denotes the maximal number of edges in an F-free graph on n vertices. We consider the function $h_F(n, q)$, the minimal number of copies of F in a graph on n vertices with $\mathrm{ex}(n, F) + q$ edges. The value of $h_F(n, q)$ has been extensively studied when F is bipartite or colour-critical. In this paper we investigate the simplest remaining graph F, namely, two triangles sharing a vertex, and establish the asymptotic value of $h_F(n, q)$ for $q = o(n^2)$.

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1. Introduction

The *Turán function* $\mathrm{ex}(n, F)$ of a graph F is the maximum number of edges in an F-free graph on n vertices. In 1907, Mantel [22] proved that $\mathrm{ex}(n, K_3) = \lfloor n^2/4 \rfloor$, where K_r denotes the complete graph on r vertices. The fundamental paper of Turán [35] solved this extremal problem for cliques: the *Turán graph* $T_r(n)$, the complete r-partite graph on n vertices with parts of size $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$, is the unique maximum K_{r+1}-free graph on n vertices. Thus the Turán function satisfies $\mathrm{ex}(n, K_{r+1}) = |E(T_r(n))|$.

Stated in the contrapositive, this implies that a graph with $\mathrm{ex}(n, K_{r+1}) + 1$ edges (where, by default, n denotes the number of vertices) contains at least one copy of K_{r+1}. Rademacher (1941, unpublished) showed that a graph with $\lfloor n^2/4 \rfloor + 1$ edges contains not just one but at least $\lfloor n/2 \rfloor$ copies of a triangle. This is perhaps the first result in the so-called "theory of supersaturated graphs" that focuses on the *supersaturation function*

$$h_F(n, q) = \min\{\#F(H) : |V(H)| = n, |E(H)| = \mathrm{ex}(n, F) + q\},$$

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the minimum number of F-subgraphs in a graph H on n vertices and $\mathrm{ex}(n, F) + q$ edges. (We say that G is a *subgraph* of H if $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$; we call G an *F-subgraph* if it is isomorphic to F.)

One possible construction for graphs with the minimal number of copies of F is to add some q edges to a maximum F-free graph. Denote by $t_F(n, q)$ the smallest number of F-subgraphs that can be achieved this way. Clearly, $h_F(n, q) \leq t_F(n, q)$. In fact, this bound is sharp for cliques, when q is small. Erdős [3] extended Rademacher's result by showing that $h_{K_3}(n, q) = t_{K_3}(n, q) = q\lfloor n/2 \rfloor$ for $q \leq 3$. Later, he [4] showed that there exists some small constant $\varepsilon_{K_r} > 0$ such that $h_{K_r}(n, q) = t_{K_r}(n, q)$ for all $q \leq \varepsilon_{K_r} n$. Lovász and Simonovits [20,21] found the best possible value of ε_{K_r} as $n \to \infty$, settling a long-standing conjecture of Erdős [3]. In fact, the second paper [21] completely solved the $h_{K_r}(n, q)$-problem when $q = o(n^2)$. The case $q = \Omega(n^2)$ of the supersaturation problem for cliques has been actively studied and proved notoriously difficult. Only recently was an asymptotic solution found: by Razborov [28] for K_3 (see also Fisher [9]), by Nikiforov [25] for K_4, and by Reiher [29] for general K_r.

The supersaturation problem was also considered for general graphs F. If F is bipartite, then there is a beautiful (and still open) conjecture of Erdős–Simonovits (see [33]) and Sidorenko [31] whose positive solution would determine $h_F(n, q)$ asymptotically for $q = \Omega(n^2)$. We refer the reader to some recent papers on the topic, [2,13,15,18,34], that contain many references.

For non-bipartite F, the value of $h_F(n, q)$ has also been considered for general colour-critical graphs. A graph is called r-(*colour*)-*critical* if its chromatic number is $r + 1$ while the removal of some edge from the graph reduces its chromatic number. Simonovits [32] established that if F is r-critical, then for large enough n the unique maximal F-free graph is $T_r(n)$. Pikhurko and Yilma [27] extending the results of Mubayi [24] established that, similarly to cliques, for every colour-critical graph F there exists $\varepsilon_F > 0$ such that when $q \leq \varepsilon_F n$, we have $h_F(n, q) = t_F(n, q)$. In addition, they established the asymptotic size of $h_F(n, q)$ when $q = o(n^2)$.

As far as we know, the supersaturation problem has not been considered for graphs which have chromatic number at least 3 and are not colour-critical, apart from some general (and rather weak) bounds by Erdős and Simonovits [7]. In this paper, we investigate a 'simplest' such graph, namely the *bowtie* which consists of two copies of K_3 merged at a vertex. We refer to this vertex as the *central vertex* of the bowtie.

Despite the simple nature of the bowtie, bowtie-free graphs have been crucial in several areas, such as in countable universal graphs [1,16], in Ramsey theory [14] and Hrushovski property [8], etc.

From this point on F denotes the bowtie and we will assume that n is sufficiently large. The main contribution of this paper is twofold. First we establish that when $q = o(n^2)$ any graph on n vertices and $\mathrm{ex}(n, F) + q$ edges with minimal number of copies of F contains the Turán graph $T_2(n)$. Based on this we establish the asymptotic value of $h_F(n, q)$ when $q = o(n^2)$.

2. Main results

The Turán function of the bowtie F has been established by Erdős, Füredi, Gould and Gunderson [6], namely $\mathrm{ex}(n, F) = \lfloor n^2/4 \rfloor + 1$ for $n \geq 5$. In addition, the extremal graphs are also known when n is large enough, each being $T_2(n)$ with an arbitrary edge added; this follows from more general results in e.g. Glebov [11] or Liu [19]. When n is odd, this leads to two non-isomorphic versions of the extremal graph. Our first result shows that, when n is large enough and $q = o(n^2)$, the only way to construct a graph containing as few bowties as possible is to add edges to $T_2(n)$.

Theorem 1. *There is $\delta > 0$ such that every graph H with $n \geq 1/\delta$ vertices, $\mathrm{ex}(n, F) + q \leq (1/4 + \delta)n^2$ edges and $h_F(n, q)$ copies of the bowtie F contains $T_2(n)$ as a subgraph.*

Theorem 1 implies that $t_F(n, q) = h_F(n, q)$ when $q = o(n^2)$. Once this structural property of the graph has been established, we deduce the asymptotic number of bowties present.

Theorem 2. *For every $c > 0$ there is $\delta > 0$ such that for all natural numbers $n \geq 1/\delta$ and $q \leq \delta n^2$ the following holds. If we write $2(q + 1) = dn + m$ for $d, m \in \mathbb{N}$ with $0 \leq m < n$ and set*

$$e_1 = \left\lfloor \frac{dn}{4} + \frac{\min\{m, n/2\}}{2} \right\rfloor \quad and \quad e_2 = q + 1 - e_1,$$

then

$$h_F(n, q) = (1 \pm c)\frac{n}{2}\left[\binom{e_1}{2} + \binom{e_2}{2} + m\binom{d+1}{2}\frac{n}{2} + (n - m)\binom{d}{2}\frac{n}{2} + 4e_1e_2\right].$$

Note that if $n \to \infty$ and $q/n \to \infty$ in Theorem 2, then $d + 1 = (1 + o(1))d = (1 + o(1))2q/n$ and $e_1, e_2 = (1 + o(1))q/2$, resulting in a simpler formula

$$h_F(n, q) = (1 + o(1))\frac{9}{8}q^2n.$$

Proof outline of Theorems 1 and 2. We start by showing an upper bound on $h_F(n, q)$. This can be achieved by counting the number of bowties in an arbitrary graph on n vertices with $ex(n, F) + q$ edges (Lemma 4). Therefore the number of bowties in any extremal graph H is small and the Graph Removal Lemma (Theorem 5) allows us to conclude that H can be made bowtie-free by removing a small number of edges. In addition, since the chromatic number of the bowtie is 3, the Erdős–Simonovits Stability Theorem (Theorem 6) implies that the vertex set of the obtained bowtie-free graph can be partitioned into two sets, V_1 and V_2, such that almost $|E(T_2(n))|$ edges are present between the two parts. Therefore the original graph H also has almost all edges between V_1 and V_2 (Lemma 7).

The key step is to establish that every edge between V_1 and V_2 is present (Proposition 3). In order to achieve this we need to determine the number of bowties containing a given edge and the number of bowties created by inserting an edge. In particular we need to compare the number of bowties containing an edge in V_1 or V_2 and the number of bowties created by inserting an edge between V_1 and V_2. When the number of edges spanned by V_1 and V_2 is small, the number of bowties has to be counted very precisely in order to determine which is smaller (Lemma 11). On the other hand, when the number of edges spanned by V_1 and V_2 is large, a simpler and less accurate estimate suffices (Lemma 15). In both cases there is an edge in V_1 or V_2 which is contained in more bowties than the number of bowties the insertion of any edge between V_1 and V_2 would create. Since H is minimal, with respect to the number of copies of bowties, every edge between V_1 and V_2 must be present.

Let us ignore the triangles spanned by V_1 and V_2 for the moment, later we will see that no such triangles exist in any extremal graph (Lemma 16). We shall express the number of bowties using an explicit formula. A bowtie can be formed in three different ways (see Fig. 1). Since we ignore triangles spanned by V_1 and V_2, any triangle in the graph contains exactly one edge spanned by V_1 or V_2. As a bowtie consists of two triangles, it must have exactly two edges spanned by V_1 or V_2. We distinguish the following cases, depending on whether both edges are spanned by the same set or not, in addition if they are spanned by the same set we consider the cases when the two edges are adjacent or non-adjacent.

Recall that every edge between V_1 and V_2 is present. Therefore, any pair of disjoint edges in V_1 is contained in $|V_2|$ bowties and similarly any pair of disjoint edges in V_2 is contained in $|V_1|$ bowties. If the two edges are adjacent, then they are contained in $|V_2|(|V_2| - 1)$ and $|V_1|(|V_1| - 1)$ bowties respectively. Finally any two edges, where one edge is spanned by V_1 and the other edge is spanned by V_2, are contained in $2(n - 4)$ bowties.

The previous argument implies that the number of bowties depends only on the size of V_1 and V_2 and the degree sequence of the graphs spanned by these sets. In fact, in the optimal degree sequence the degrees of any two vertices in the same part V_i may differ by at most one. Due to the small number of edges spanned by V_1 and V_2 one can rearrange the edges inside these parts so that each part spans an almost regular and triangle-free graph (Lemma 16). Destroying triangles inside a part decreases the number of bowties, justifying our earlier decision to ignore them.

In order to finish the proof of Theorem 1 we only need to show that $|V_1| = \lceil n/2 \rceil$ or $|V_1| = \lfloor n/2 \rfloor$. Note that if $|V_1| = \lceil n/2 \rceil$ and $|V_2| = \lfloor n/2 \rfloor$, the number of edges spanned by V_1 and V_2 is exactly
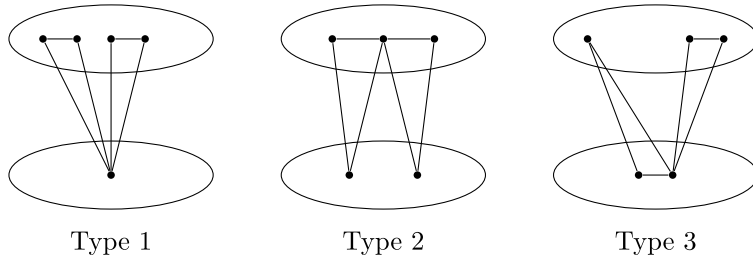
Fig. 1. Types of bowties.

$q + 1$. If we increase the number of vertices in V_1 by a and, in order to keep the total number of vertices unchanged, at the same time we decrease the number of vertices in V_2 by the same amount, then the number of edges spanned by V_1 and V_2 increases by at least a^2. In addition, the number of bowties containing edge pairs in V_1 decreases (as any such bowtie is either Type 1 or Type 2), while for edge pairs in V_2 the number of bowties increases. On the other hand, for a fixed pair of edges, one in V_1 and the other in V_2, the number of bowties containing both these edges remains unchanged. Roughly speaking, the number of bowties created by adding the a^2 new edges has to be counterbalanced by the decrease in the number of Type 1 and Type 2 bowties. This would be possible only if there was a large difference between the number of edges in V_1 and V_2. However, we show that this is not the case (Lemma 18) and thus a must be 0.

Bounding the difference between the number of edges spanned by V_1 and V_2 has an additional advantage. Together with the exact size of V_1 and V_2 it also implies that, disregarding a couple of vertices, when looking at the graphs spanned by V_1 and V_2 the difference in the degree of any pair of vertices is at most 1 (Lemma 19). In fact, in one of the partitions almost every vertex has the same degree (Lemma 21). These provide us with a good approximation on the degree sequence of the graphs spanned by V_1 and V_2 and thus also on the total number of bowties.

Organisation of the paper. The remainder of the paper is divided into four parts. In Section 3 we show that any extremal graph H contains a complete spanning bipartite subgraph (Proposition 3). Section 4 states any additional lemmas needed for Theorems 1 and 2 and gives the proofs of these results. The proof of the remaining technical lemmas can be found in Section 5. Section 6 contains some concluding remarks.

3. Complete bipartite subgraph: proof of Proposition 3

Throughout this section let H be a graph on n vertices and $\mathrm{ex}(n, F) + q$ edges containing the minimal number of bowties. Let $V := V(H)$ denote the vertex set of H and $E := E(H)$ its edge set. The following proposition is vital in proving Theorem 1.

Proposition 3. *For an arbitrary graph H on n vertices and $\mathrm{ex}(n, F) + q$ edges containing $h_F(n, q)$ bowties, where $q = o(n^2)$ and n is large enough, admits a partition $V = V_1 \cup V_2$ such that $E(K(V_1, V_2)) \subseteq E$, $|V_1|, |V_2| = (1 + o(1))n/2$ and the number of edges spanned by each of V_1 and V_2 is at most $4q + 4$.*

The remainder of this section is devoted to proving Proposition 3. Let $\{V_1, V_2\}$ be a max-cut of H. The crucial part of the proof is to show that both parts have size $(1 + o(1))n/2$ and every edge between the parts is present, i.e. $E(K(V_1, V_2)) \subseteq E$.

The key tools used for the proof are the Graph Removal Lemma (Theorem 5) and the Stability Theorem (Theorem 6). Roughly speaking, the graph removal lemma states that if a graph contains only a few copies of bowties, which holds for H (Lemma 4), then removing a few edges makes it bowtie-free. Together with the Stability Theorem this implies that V_1 and V_2 both have size roughly $n/2$ and most edges between the parts are present (Lemma 7).

For $i = 1, 2$ let $v_i = |V_i|$. Denote the edges spanned by V_1 or V_2 by B and set $b = |B|$. We call the edges in B *bad*. The degree sequence of several graphs on V plays a crucial role in counting the

number of bowties. For any $v \in V$ and $E' \subseteq \binom{V}{2}$ let $d_{E'}(v)$ be the degree of vertex v in (V, E') (the graph with vertex set V and edge set E'). In the special case when $E' = B$ we call $d_B(v)$ the *bad degree*.

Our proof strategy is to show that there exists a bad edge which is contained in more bowties then inserting an edge between the parts would create, should such an edge be missing. Therefore removing the bad edge destroys more bowties than inserting the edge between the parts creates leading to a graph with fewer bowties and implying that every edge between V_1 and V_2 is present. Showing this requires a very precise analysis, when the number of bad edges is small (Lemma 11). However this fails once the number of bad edges becomes large and an alternate proof is required (Lemma 15).

In order to prove the above lemmas we need a lower bound on the maximal number of bowties a bad edge is contained in (Lemma 12) and an upper bound on the number of bowties created when an edge between the parts is inserted (Lemma 13). The latter is more difficult. Roughly speaking an upper bound on the bad degree leads to an upper bound on the number of triangles containing the inserted edge, which leads to an upper bound on the number of bowties. At first we can only prove a weak upper bound on the bad degree (Lemma 9). However if the vertex has many neighbours in the other part, which is true for the majority of the vertices, a significantly better bound exists (Lemma 10). In fact, once we show that every vertex has many neighbours in the other part (Lemma 14) the tighter bound applies to every vertex.

As mentioned earlier we need to show that the number of bowties in H, i.e. $h_F(n, q)$, is small.

Lemma 4. *For $q \leq n^2/20$ we have that*

$$h_F(n, q) \leq (q + 1)^2 \left(13n/4 + 13\right).$$

Proof. In order to prove the statement we construct a graph G satisfying $|V(G)| = n$, $|E(G)| = \mathrm{ex}(n, F) + q$ and $\#F(G) \leq (q + 1)^2(13n/4 + 13)$. Partition the vertex set of G into two parts U_1 and U_2 such that $|U_1| = \lceil n/2 \rceil$ and $|U_2| = \lfloor n/2 \rfloor$. Every edge between U_1 and U_2 will be included in the graph, i.e. $K(U_1, U_2) \subseteq G$. This determines $\lceil n/2 \rceil \lfloor n/2 \rfloor$ edges in G and thus we only need to establish the position of the remaining $q + 1$ edges.

Denote by $n' = \lfloor n/4 \rfloor$. Let W_1, W_2 be disjoint subsets of U_1 of size n'. The remaining $q + 1$ edges are placed between W_1 and W_2 such that the degrees are as equal as possible. In particular, for $i = 1, 2$, $(q + 1) \bmod n'$ vertices in W_i have $\lceil (q + 1)/n' \rceil$ neighbours in W_{3-i}, while the remaining vertices have $\lfloor (q+1)/n' \rfloor$ neighbours in W_{3-i}. Note that such a construction is possible by Lemma 22, as $\lfloor (q + 1)/n' \rfloor < n'$ for sufficiently large n.

In order to determine the number of bowties we count for every vertex v the number of bowties in G where the central vertex is v. For $v \in V(G)$ any two edge disjoint triangles containing v form a bowtie. Since no edges are spanned by U_2, every triangle must contain an edge in U_1 and any such edges must be between W_1 and W_2. Therefore for $v \in U_2$ there are at most $\binom{q+1}{2}$ bowties centred at v.

Now consider bowties centred at $v \in W_i$ for $i = 1, 2$. Recall that any triangle must contain an edge in U_1 and a vertex in U_2. Therefore any bowtie centred at v must contain two different vertices in $N(v) \cap U_1$ and two vertices in U_2. These can be selected in at most

$$\binom{|N(v) \cap U_1|}{2} \frac{n^2}{4}$$

ways, giving an upper bound on the number of bowties. Let $k \equiv q + 1 \pmod{n'}$. Then we have

$$
\begin{aligned}
\#F(G) &\leq 2k \binom{\lceil (q+1)/n' \rceil}{2} \frac{n^2}{4} + 2(n'-k) \binom{\lfloor (q+1)/n' \rfloor}{2} \frac{n^2}{4} + \frac{n}{2} \binom{q+1}{2} \\
&\leq \frac{n}{2} \left[k \binom{\lceil (q+1)/n' \rceil}{2} n + (n'-k) \binom{\lfloor (q+1)/n' \rfloor}{2} n + \frac{(q+1)^2}{2} \right] \\
&\leq \frac{n}{4} \left[n' \left(\frac{q+1}{n'} \right)^2 n + kn \frac{2(q+1)}{n'} + (q+1)^2 \right].
\end{aligned}
$$

Note that $k \leq q + 1$ and $n' \geq n/4 - 1$, therefore

$$
\#F(G) \leq \frac{n}{4} (q+1)^2 \left[\frac{3n}{n/4 - 1} + 1 \right] \leq (q+1)^2 \left(13n/4 + 13 \right)
$$

for large enough n. $\quad\square$

Next we state the Graph Removal Lemma and the Stability Theorem. Using them we show that the vertex set of the extremal graph H can be partitioned into two sets V_1, V_2 such that both sets contain roughly $n/2$ vertices and most edges between V_1 and V_2 are present.

Theorem 5 (*Graph Removal Lemma, see e.g. [17]*)**.** *Let Γ be a graph with f vertices. Then for every $\varepsilon_1 > 0$ there exists $\varepsilon_2 > 0$ such that every graph H with $n \geq 1/\varepsilon_2$ vertices and at most $\varepsilon_2 n^f$ copies of Γ can be made Γ-free by removing at most $\varepsilon_1 n^2$ edges.*

Theorem 6 (*Stability Theorem [5,32]*)**.** *Let $r \geq 2$ and Γ be a graph with chromatic number $r + 1$. Then for every $\varepsilon_1 > 0$ there exists $\varepsilon_2 > 0$ such that every Γ-free graph on $n \geq 1/\varepsilon_2$ vertices with at least $|E(T_r(n))| - \varepsilon_2 n^2$ edges contains an r-partite subgraph with at least $|E(T_r(n))| - \varepsilon_1 n^2$ edges.*

Lemma 7. *Let $\{V_1, V_2\}$ be a max-cut of H. For every $\delta_1 > 0$ there exist $\delta_2 > 0$ and n_0 such that for every $n > n_0$ and $q + 1 < \delta_2 n^2$ we have*

$$
|E(H) \triangle E(K(V_1, V_2))| \leq \delta_1 n^2 + q + 1
$$

and

$$
n/2 - 2\sqrt{\delta_1} n \leq |V_1|, |V_2| \leq n/2 + 2\sqrt{\delta_1} n.
$$

Proof. First we show that the number of edges between V_1 and V_2 is at least $\lceil n/2 \rceil \lfloor n/2 \rfloor - \delta_1 n^2$. Since $\{V_1, V_2\}$ is a max-cut, this follows if H contains a bipartite graph with at least $\lceil n/2 \rceil \lfloor n/2 \rfloor - \delta_1 n^2$ edges. Note that the chromatic number of the bowtie is 3. Therefore, Theorem 6 implies that there exists a constant $\varepsilon_1 > 0$ such that every bowtie-free graph with at least $\lceil n/2 \rceil \lfloor n/2 \rfloor - \varepsilon_1 n^2$ edges contains a bipartite subgraph with at least $\lceil n/2 \rceil \lfloor n/2 \rfloor - \delta_1 n^2$ edges.

Therefore, H contains such a bipartite graph if it has a bowtie-free subgraph with at least $\lceil n/2 \rceil \lfloor n/2 \rfloor - \varepsilon_1 n^2$ edges. Theorem 5 implies that there exists an $\varepsilon_2 > 0$ such that every graph with at most $\varepsilon_2 n^5$ copies of bowties can be made bowtie-free by removing at most $\varepsilon_1 n^2$ edges. This holds for every q satisfying $q + 1 < \sqrt{\varepsilon_2} n^2 / 2$ because by Lemma 4 the number of bowties in H is at most

$$
(q+1)^2 \left(13n/4 + 13 \right) \leq \varepsilon_2 n^5.
$$

In fact, after the removal of these edges the graph still contains at least

$$
\lceil n/2 \rceil \lfloor n/2 \rfloor + q + 1 - \varepsilon_1 n^2 \geq \lceil n/2 \rceil \lfloor n/2 \rfloor - \varepsilon_1 n^2
$$

edges.

Thus we have a bipartite subgraph with at least $\lceil n/2 \rceil \lfloor n/2 \rfloor - \delta_1 n^2$ edges and the first statement follows. Note that a complete bipartite graph on n vertices where one part has size at least

$n/2 + 2\sqrt{\delta_1} n$ contains at most

$$\frac{n^2}{4} - 4\delta_1 n^2 \overset{\delta_1 n^2 > 1/8}{\leq} \lceil n/2 \rceil \lfloor n/2 \rfloor - 2\delta_1 n^2$$

edges, implying the required bounds on the size of the individual parts. □

Let $\varepsilon > 0$ be sufficiently small. Apply Lemma 7 for $\delta_1 = \varepsilon^2/4$ which gives some $\delta_2 > 0$. Since $q = o(n^2)$, we have that $q + 1 \leq \delta_2 n^2$, for large enough n. Thus $\{V_1, V_2\}$, a max-cut of H, satisfies

$$n/2 - \varepsilon n \leq |V_1|, |V_2| \leq n/2 + \varepsilon n$$

and

$$|E(H) \triangle E(K(V_1, V_2))| \leq \varepsilon^2 n^2/4 + q + 1 \leq \varepsilon^2 n^2.$$

Note that

$$b \leq |E(H) \triangle E(K(V_1, V_2))| \leq \varepsilon^2 n^2. \tag{1}$$

In order to establish bounds on $d_B(v)$ we need to determine the number of bowties any edge is contained in.

Lemma 8. *Every edge in H is contained in less than* 13bn *bowties.*

Proof. Assume for contradiction that there exists an edge f contained in at least $13bn$ bowties. We will show that there exists a pair of non-adjacent vertices $w_1, w_2 \in V$ such that inserting the edge $\{w_1, w_2\}$ creates less than $13bn$ bowties. Thus the graph created by removing f and inserting $\{w_1, w_2\}$ has less bowties and contradicts the minimality of H.

Note that the number of vertices $w \in V_1$ with $d_B(w) \geq 6b/n$ is at most $2b/(6b/n) = n/3$. This implies that the number of vertices in V_1 with $d_B(w) < 6b/n$ is at least $(1 - 2\varepsilon)\frac{n}{2} - \frac{n}{3} \geq n/8$. Since for each of these vertices $d_B(w) < 6b/n \leq 6\varepsilon^2 n$, two of them must be non-adjacent.

Let $w_1, w_2 \in V_1$ be a pair of non-adjacent vertices with $d_B(w_1), d_B(w_2) \leq 6b/n$ and we count the bowties which would be created if we would insert this edge. Note that any such bowtie must contain at least one more edge in B. We start with the bowties which contain an edge in B adjacent to $\{w_1, w_2\}$. In order to count the number of these bowties we first count the number of triangles containing an edge in B adjacent to w_1 or w_2. Note that for w_1 there are $d_B(w_1)$ such edges and each of these edges creates a triangle with at most $v_2 + d_B(w_1)$ vertices. Therefore there are at most $((1 + 2\varepsilon)n/2 + 6b/n)6b/n$ such triangles, and similarly for w_2. Thus the number of triangles containing an edge in B adjacent to w_1 or w_2 is at most $((1 + 2\varepsilon)n/2 + 6b/n)12b/n$. Note that the codegree of w_1, w_2 is at most n. Consequently the total number of bowties created in this fashion is at most

$$\left((1 + 2\varepsilon)\frac{n}{2} + \frac{6b}{n} \right) \frac{12b}{n} n \overset{(1)}{<} 8bn.$$

Now consider the case when the bowtie contains an edge in B which is not adjacent to $\{w_1, w_2\}$. Select an edge $e \in B$, this can be done in b ways. The final vertex of the bowtie can be picked in at most n ways. Note that once the central vertex of the bowtie has been selected the structure of the bowtie has been determined, as it has to contain the edges e and $\{w_1, w_2\}$ and these two edges have to be disjoint. Therefore we create at most $5bn$ bowties this way.

In total we created less than $13bn$ bowties, leading to a contradiction. □

Using Lemma 8 we obtain an upper bound on $d_B(v)$.

Lemma 9. *For every* $v \in V$ *we have* $d_B(v) < 5\varepsilon^{2/3}n$.

Proof. Assume for contradiction that there exists a vertex $v \in V$ such that $d_B(v) \geq 5\varepsilon^{2/3}n$. Without loss of generality assume that $v \in V_1$. Because $\{V_1, V_2\}$ is a max-cut of H we have that

$|N(v) \cap V_2| \geq 5\varepsilon^{2/3}n$. Let $N_1 \subseteq N(v) \cap V_1$ and $N_2 \subseteq N(v) \cap V_2$ be sets of size exactly $5\varepsilon^{2/3}n$. Since at most $\varepsilon^2 n^2$ edges are missing between V_1 and V_2, there can be only $\varepsilon^2 n^2$ edges missing between N_1 and N_2. Therefore, there are at least $(25\varepsilon^{4/3} - \varepsilon^2)n^2 \geq 24\varepsilon^{4/3}n^2$ edges between N_1 and N_2 and each of these edges is adjacent to at most $10\varepsilon^{2/3}n$ other edges in this set. In addition, by the pigeonhole principle there must exist a vertex $u \in N_1$ such that $|N(u) \cap N_2| \geq 2\varepsilon^{2/3}n$.

Next we count the number of bowties containing the edge $\{u, v\}$. Note that any pair of disjoint edges between N_1 and N_2 creates a bowtie, where the central vertex is v. Recall that u is contained in at least $2\varepsilon^{2/3}n$ edges, where the other end is in N_2. In addition, each of these edges is disjoint from at least $24\varepsilon^{4/3}n^2 - 10\varepsilon^{2/3}n$ edges between N_1 and N_2. Therefore, $\{u, v\}$ is contained in at least

$$2\varepsilon^{2/3}n(24\varepsilon^{4/3}n^2 - 10\varepsilon^{2/3}n) > 25\varepsilon^2 n^3$$

bowties, when n is large enough.

On the other hand, Lemma 8 implies that any edge can be contained in at most $13bn$ bowties and by the bound (1) giving $b \leq \varepsilon^2 n^2$, we have $13bn \leq 13\varepsilon^2 n^3$, leading to a contradiction. □

However, for vertices which have many neighbours in the opposite part, which holds for most vertices, we establish a tighter bound on $d_B(v)$. Let

$$M = E(K(V_1, V_2)) \setminus E(H).$$

We call the edges in M *missing*. Recall that $d_M(v)$ is the degree of vertex v in the graph (V, M). Let $S_i \subseteq V_i$ be the set of vertices $v \in V_i$ satisfying $d_M(v) \geq \varepsilon^{1/4}n$. Since $q + 1 + |M| \leq b$, we have

$$|S_i| \leq \frac{|M|}{\varepsilon^{1/4}n} \leq \frac{b}{\varepsilon^{1/4}n}. \tag{2}$$

Next we will consider the properties of the graph spanned by $V' = V \setminus (S_1 \cup S_2)$, denoted by H'. Let $V'_1 = V_1 \setminus S_1$ and $V'_2 = V_2 \setminus S_2$. The set of edges spanned by V'_1 and V'_2 are denoted by B'_1 and B'_2 respectively. Also let $B' = B'_1 \cup B'_2$. Note that for $i = 1, 2$, for any vertex $v \in V'_i$ we have $d_B(v) \leq d_{B'}(v) + |S_i|$. Since $|S_i|$ is bounded from above by $b/(\varepsilon^{1/4}n)$, any upper bound on $d_{B'}(v)$ also provides an upper bound on $d_B(v)$.

Lemma 10. *For every $v \in V'$ we have $d_{B'}(v) \leq 80b/n + 1$.*

Proof. Assume for contradiction that there exists $v \in V'$ such that $d_{B'}(v) > 80b/n + 1$. Without loss of generality assume $v \in V'_1$. We will show that in this graph there is an edge which is contained in at least $13bn$ bowties which together with Lemma 8 leads to a contradiction.

Note that any vertex in V'_1 has at least $v_2 - \varepsilon^{1/4}n$ neighbours in V_2 and thus any pair of vertices in V'_1 has at least $v_2 - 2\varepsilon^{1/4}n \geq (1 - 2\varepsilon)n/2 - 2\varepsilon^{1/4}n$ common neighbours. Therefore every edge in B'_1 is contained in at least $(1 - 2\varepsilon)n/2 - 2\varepsilon^{1/4}n$ triangles. Any pair of triangles each containing a different edge in B'_1 and a vertex in V_2 is edge disjoint if the vertex in V_2 differs. Therefore, any edge in B' adjacent to v is contained in at least

$$(d_{B'}(v) - 1)\left((1 - 2\varepsilon - 4\varepsilon^{1/4})\frac{n}{2} - 1\right)^2 \geq 16bn$$

bowties, a contradiction. □

Recall that $q + 1 + |M| \leq b$. When b is small, we have $S_1, S_2 = \emptyset$, implying $H = H'$ and $d_{B'}(v) = d_B(v)$. Therefore, the upper bound on the degree in the graph (V, B') in Lemma 10 is actually an upper bound on the degree in (V, B). This enables us to show that $K(V_1, V_2)$ is a subgraph of H.

Lemma 11. *When $b < \varepsilon^{1/4}n$ we have that $M = \emptyset$.*

Proof. Assume for contradiction that $M \neq \emptyset$. Since $b < \varepsilon^{1/4}n$, by (2) we have $|S_1|, |S_2| < 1$, implying that $S_1 = S_2 = \emptyset$. Together with Lemma 10 we have $d_B(w) \leq 80b/n + 1 < 80\varepsilon^{1/4} + 1 < 2$ for every $w \in V$. Therefore, $d_B(w)$ is at most one, which in turn implies that the edges in B are disjoint.

Without loss of generality assume $b_1 \geq b_2$. For every edge $\{w_1, w_2\} \in B_1$ let $t_{\{w_1, w_2\}}$ denote the number of triangles containing either w_1 or w_2 and an edge in B_2.

We show that if $B_2 \neq \emptyset$, then there exists an edge $e \in B_1$ with $t_e \geq b_2 + 1$. Assume for contradiction that there is no such edge. For any $\{w_1, w_2\} \in B_1$ consider the $2b_2$ potential triangles which could contribute to $t_{\{w_1, w_2\}}$. Since the edges in B_2 are disjoint, these triangles are edge disjoint except for the edge in B_2. In order for $t_{\{w_1, w_2\}} \leq b_2$ we must have that b_2 of these triangles are missing at least one edge and thus $d_M(w_1) + d_M(w_2) \geq b_2$. Since the edges in B_1 are also disjoint, we have

$$|M| \geq \sum_{\{w_1, w_2\} \in B_1} (d_M(w_1) + d_M(w_2)) \geq b_1 b_2.$$

Note that because $|M| \leq b - q - 1$ and $q \geq 1$, we have $|M| \leq b - 2 = b_1 + b_2 - 2$. However, no positive integer pair $b_1 \geq b_2 \geq 1$ satisfies $b_1 + b_2 - 2 \geq b_1 b_2$, resulting in a contradiction.

Select $\{w_1, w_2\} \in B_1$ such that $t_{\{w_1, w_2\}}$ is maximal. We will give a lower bound on the number of bowties destroyed when $\{w_1, w_2\}$ is removed. First we consider bowties which have an edge in B_2. Note that a triangle w_1, x, y, where $x, y \in V_2$, forms a bowtie with any triangle containing both w_1 and w_2, but not x and y. Also the codegree of w_1, w_2, not including x and y, is at least $(1 - 2\varepsilon)n/2 - |M| - 2 \geq (1 - 2\varepsilon)n/2 - b$. Since $b_2 + \mathbb{1}_{B_2 \neq \emptyset} = 0$, when $b_2 = 0$ we have $t_{\{w_1, w_2\}} \geq b_2 + \mathbb{1}_{B_2 \neq \emptyset}$. Thus removing $\{w_1, w_2\}$ destroys at least

$$(b_2 + \mathbb{1}_{B_2 \neq \emptyset}) \left((1 - 2\varepsilon)\frac{n}{2} - b \right)$$

such bowties. Now we consider the case when the bowtie contains a second edge in B_1. Note that any four vertices in V_1 have at least $(1 - 2\varepsilon)n/2 - b$ common neighbours. Recall that the edges in B are disjoint. Thus any pair of edges in B_1 are contained in at least $(1 - 2\varepsilon)n/2 - b$ bowties. Therefore, at least

$$(b_1 - 1) \left((1 - 2\varepsilon)\frac{n}{2} - b \right)$$

additional bowties are destroyed when $\{w_1, w_2\}$ is removed. In total, this leads to at least

$$(b - 1 + \mathbb{1}_{B_2 \neq \emptyset}) \left((1 - 2\varepsilon)\frac{n}{2} - b \right) \geq (b - 1 + \mathbb{1}_{B_2 \neq \emptyset}) \left((1 - 2\varepsilon)\frac{n}{2} - \varepsilon^{1/4}n \right)$$

destroyed bowties.

Since $M \neq \emptyset$, there exists $\{u, v\} \in M$. Without loss of generality assume $u \in V_1$ and $v \in V_2$. Now we analyse the number of bowties created when $\{u, v\}$ is added to the graph.

We first consider the case when $B_2 = \emptyset$. Since $B_2 = \emptyset$, v has no neighbours in B_2 and thus inserting $\{u, v\}$ can create at most one triangle, with the third vertex being $z \in V_1$, if it exists. The number of bowties created is equal to the number of triangles containing exactly one of u, v, z as these form a bowtie with the triangle u, v, z. The only neighbour of u in V_1 is z and vice versa. This means that the only way these vertices can be in a triangle is if the triangle contains an edge in B_2. However as $B_2 = \emptyset$ no such triangles exist. Similarly, the only way v can be in a triangle is if the triangle contains an edge in B_1, so there are at most $b_1 = b$ such triangles. Thus removing $\{w_1, w_2\}$ destroys at least

$$(b - 1) \left((1 - 2\varepsilon)\frac{n}{2} - \varepsilon^{1/4}n \right)$$

bowties, while inserting $\{u, v\}$ creates at most b. Since $b \geq |M| + q + 1 \geq 3$ we destroy more triangles than we created, which leads to a contradiction.

On the other hand, if $B_2 \neq \emptyset$, then u can have a neighbour in V_1 and v can have a neighbour in V_2. Denote these vertices by z_1 and z_2 respectively, if it exists. Similarly as before, we are interested in the number of triangles containing exactly one of u, v, z_1 or one of u, v, z_2. We start with the u, v, z_1 triangle. Similarly as before, for each of u and z_1 there are at most b_2 such triangles where the additional vertices of the triangle are in V_2, and for v there are at most b_1 such triangles where the additional vertices are in V_1. Now the only neighbour of u in V_1 is z_1 and vice versa, so we have counted every triangle containing u or z_1. However, v has a neighbour $z_2 \in V_2$ and every vertex in

$V_1 \setminus \{u, z_1\}$ can form a triangle with $\{v, z_2\}$. Therefore, the total number of bowties containing the $\{u, v, z_1\}$ triangle is at most $b_1 + 2b_2 + v_1$. Analogously, one can show for the $\{u, v, z_2\}$ triangle that the number of bowties it is contained in is at most $2b_1 + b_2 + v_2$. So in total at most $3b + n$ bowties have been created, by adding the edge $\{u, v\}$. Now this has to be more than the number of triangles destroyed, so

$$3b + n \geq b\left((1 - 2\varepsilon)\frac{n}{2} - \varepsilon^{1/4} n\right),$$

which leads to a contradiction as $b \geq 3$. $\quad\square$

In the remainder of this section we prove that $M = \emptyset$ when b is large. We start by showing that $S_1 = S_2 = \emptyset$ holds in this case as well. Namely, should there exist a vertex $v \in S_1 \cup S_2$ we show that removing many edges from B and inserting them between V_1 and V_2 such that they are adjacent to v decreases the number of bowties in the graph. As a first step we need to find a suitable set of edges, which when removed from the graph destroy many copies of bowties.

Lemma 12. *For every integer $k \leq |B'|/2$ there exists $D \subseteq B'$ with $|D| = k$ such that removing all edges in D destroys at least*

$$k\frac{bn}{8} - \frac{nk^2}{2}$$

bowties.

Proof. Note that

$$
\sum_{v \in S_1} d_B(v) + \sum_{v \in S_2} d_B(v) \overset{\text{Lemma 9}}{\leq} (|S_1| + |S_2|) 5\varepsilon^{2/3} n
$$

$$
= 10\varepsilon^{5/12}(|S_1| + |S_2|)\varepsilon^{1/4} n/2
$$

$$
\overset{(2)}{\leq} 10\varepsilon^{5/12} b. \tag{3}
$$

Thus

$$
|B'| \geq b - \sum_{v \in S_1} d_B(v) - \sum_{v \in S_2} d_B(v) \overset{(3)}{\geq} (1 - 10\varepsilon^{5/12}) b > 0. \tag{4}
$$

Therefore, B' is non-empty. Without loss of generality we assume that V_1 spans at least as many edges in B' as V_2. Note that any four vertices in V_1' have at least $(1 - 2\varepsilon)n/2 - 4\varepsilon^{1/4} n$ common neighbours. Select an arbitrary pair of edges $\{w_1, w_2\}, \{w_3, w_4\} \in B_1'$. If $\{w_3, w_4\}$ is disjoint from $\{w_1, w_2\}$, then for every shared neighbour of w_1, w_2, w_3, w_4 we have a bowtie. On the other hand, if $\{w_3, w_4\}$ intersects $\{w_1, w_2\}$, then every pair of shared neighbours results in a bowtie. Thus the number of bowties containing $\{w_1, w_2\}$ and $\{w_3, w_4\}$ is at least

$$(1 - 2\varepsilon - 8\varepsilon^{1/4})\frac{n}{2}.$$

Therefore, removing arbitrary k edges from B_1' destroys at least

$$
(1 - 2\varepsilon - 8\varepsilon^{1/4})\frac{n}{2}\left(k\left(\frac{|B'|}{2} - k\right)\right)
$$

$$
\overset{(4)}{\geq} (1 - 2\varepsilon - 8\varepsilon^{1/4})\frac{n}{2}\left(k\left((1 - 10\varepsilon^{5/12})\frac{b}{2} - k\right)\right)
$$

$$
\geq k\frac{bn}{8} - \frac{nk^2}{2}
$$

bowties. $\quad\square$

Next we analyse the number of bowties created if we insert edges between V_1 and V_2.

Lemma 13. *Assume $\{u, v\} \in M$ with $u \in V_1$ and $v \in V_2$. Let G be a graph created from H by removing an arbitrary set of edges $D \subseteq E(H)$ and inserting an arbitrary set of edges from $M \setminus \{u, v\}$. If*

$$|N_G(u) \cap V_1'|, |N_G(v) \cap V_2'| \leq 80b/n + 1, \tag{5}$$

then inserting $\{u, v\}$ into the graph G creates at most $4\varepsilon^{1/4}bn + 6n$ bowties.

Proof. Note that adding the edges in $M \setminus \{u, v\}$ leaves B unchanged and removing the edges in D can only decrease the size of B.

We start by determining an upper bound on the number of triangles in G a given vertex is contained in. Consider an arbitrary vertex $w \in V$. Without loss of generality let $w \in V_1$ and we consider the triangles depending on the number of neighbours it contains in V_2. Should the additional vertices of the triangle both be in V_1 or V_2, they form an edge in B and thus there are at most b such triangles. On the other hand, if the triangle contains two vertices in V_1 and one vertex in V_2, then the second vertex in V_1 has to be a neighbour of w, which can be chosen in $d_{B \setminus D}(w)$ ways and the vertex in V_2 can be chosen in at most n ways. Therefore, the number of triangles containing v is at most $b + d_{B \setminus D}(v)n$. This together with Lemma 9 implies that the number of triangles containing any vertex can be bounded from above by

$$b + 5\varepsilon^{2/3}n^2. \tag{6}$$

However, for $w \in V' \cup \{u, v\}$ a stronger bound holds. Note that for a vertex $w \in V_1$ the value of $d_{B \setminus D}(w)$ is determined by the number of neighbours w has in V_1' and in S_1. For $w \in V_1' \cup \{u\}$, Lemma 10 and (5) imply that the first of these two terms can be bounded from above by $80b/n + 1$, while the second term is at most $\varepsilon^{-1/4}b/n$ by (2). An analogous argument holds for $w \in V_2' \cup \{v\}$. Thus the number of triangles that $w \in V' \cup \{u, v\}$ is contained in is at most

$$b + \left(\frac{80b}{n} + 1 + \frac{b}{\varepsilon^{1/4}n}\right)n = 81b + n + \varepsilon^{-1/4}b \leq 3\varepsilon^{-1/4}b + n. \tag{7}$$

Inserting the edge $\{u, v\}$ creates two types of triangles, depending on whether the third vertex is in V' or not. We first consider the case when the third vertex is in V'. Note that due to our conditions there are at most $|N_G(u) \cap V_1'| + |N_G(v) \cap V_2'| \leq 2(80b/n + 1)$ such triangles. Due to (7) every vertex of these triangles is contained in at most $3\varepsilon^{-1/4}b + n$ triangles, thus the number of bowties is bounded from above by

$$2\left(\frac{80b}{n} + 1\right)3(3\varepsilon^{-1/4}b + n) = 1440\varepsilon^{-1/4}\frac{b^2}{n} + 18\varepsilon^{-1/4}b + 480b + 6n$$

$$\overset{(1)}{\leq} \varepsilon bn + 18\varepsilon^{-1/4}b + 6n$$

$$\leq 2\varepsilon bn + 6n, \tag{8}$$

where the last inequality holds, since n is large enough.

In addition, inserting $\{u, v\}$ also creates at most $|S_1| + |S_2|$ triangles with the third vertex in $S_1 \cup S_2$. Thus, by (2) and (6) an upper bound on these bowties is

$$2\frac{b}{\varepsilon^{1/4}n}3\left(b + 5\varepsilon^{2/3}n^2\right) = \frac{6b^2}{\varepsilon^{1/4}n} + 30\varepsilon^{5/12}bn \overset{(1)}{\leq} 2\varepsilon^{1/4}bn.$$

This together with (8) implies that at most

$$4\varepsilon^{1/4}bn + 6n$$

bowties have been created. □

Now we can show that both S_1 and S_2 are empty and thus $H = H'$.

Lemma 14. *We have $S_1, S_2 = \emptyset$.*

Proof. Assume for contradiction that S_1 or S_2 is non-empty. Without loss of generality assume $v \in S_1$. Then we have that $d_M(v) \geq \varepsilon^{1/4}n$. In addition, by (1) and (2) we have $|S_2| \leq b\varepsilon^{-1/4}n^{-1} \leq \varepsilon^{7/4}n$ and thus there exists $U \subseteq V_2'$ of size $\varepsilon^{1/3}n$ such that no vertex in U is adjacent to v.

Now we will remove $\varepsilon^{1/3}n$ edges from the graph H and insert the edges $\{v, u\}$ for every $u \in U$. In particular, we will first remove every edge in $B \setminus B'$ which is adjacent to v. By Lemma 9 the number of such edges is $d_B(v) \leq 5\varepsilon^{2/3}n < \varepsilon^{1/3}n$. In addition $\varepsilon^{1/3}n \leq \varepsilon^{1/4}n/3 \leq |M|/3 \leq |B|/3$, which is at most $|B'|/2$ by (4). Therefore $0 < \varepsilon^{1/3}n - d_B(v) \leq |B'|/2$ and the remaining $\varepsilon^{1/3}n - d_B(v)$ edges can be removed from B' in accordance to Lemma 12. Denote the set of $\varepsilon^{1/3}n$ removed edges by D. Removing the edges in D destroys at least

$$(\varepsilon^{1/3}n - d_B(v))\frac{bn}{8} - \frac{\varepsilon^{2/3}n^3}{2} \overset{\text{Lemma 9}}{\geq} \frac{\varepsilon^{1/3}bn^2}{30}$$

bowties, where we use $b \geq d_M(v) \geq \varepsilon^{1/4}n$.

Let G be the graph obtained after removing the edges in D. Note that $|N_G(v) \cap V_1| = 0$ and for $u \in V_2'$ we have by Lemma 10 that $|N_G(u) \cap V_2'| \leq 80b/n + 1$. In addition, inserting edges into M keeps these values unchanged. Thus, Lemma 13 is applicable for each of the $\varepsilon^{1/3}n$ edges inserted, and thus at most

$$\varepsilon^{1/3}n(4\varepsilon^{1/4}bn + 6n)$$

bowties are created. Since $b \geq \varepsilon^{1/4}n$

$$\frac{\varepsilon^{1/3}bn^2}{30} > \varepsilon^{1/3}n(4\varepsilon^{1/4}bn + 6n),$$

contradicting the minimality of H. $\square$

Since $H = H'$ we have that $d_B(v) = d_{B'}(v)$. Thus, Lemma 10 implies that the conditions of Lemma 13 are satisfied for any pair $\{u, v\} \in M$, enabling us to prove $M = \emptyset$.

Lemma 15. *When $b \geq \varepsilon^{1/4}n$ we have $M = \emptyset$.*

Proof. Assume for contradiction $M \neq \emptyset$ and let $\{u, v\} \in M$. Since $|B'| = |B| \geq 2$, by Lemma 12 there exists an edge $\{w_1, w_2\}$ such that removing this edge destroys at least $bn/8 - n/2$ bowties. In addition, Lemma 13 implies that we create at most $4\varepsilon^{1/4}bn + 6n$ bowties when inserting $\{u, v\}$. Since $b \geq \varepsilon^{1/4}n$ we have

$$\frac{bn}{8} - \frac{n}{2} > 4\varepsilon^{1/4}bn + 6n,$$

which is a contradiction. $\square$

We conclude this section with the proof of Proposition 3.

Proof of Proposition 3. The first statement follows trivially from Lemmas 7, 11 and 15.

Next we show that $b_1 \leq 4q + 4$. Assume for contradiction that $b_1 > 4q + 4$. Taking any two disjoint edges e_1, e_2 in B_1 and a vertex in V_2 creates a bowtie. In addition if $e_1, e_2 \in B_1$ are adjacent then any pair of vertices in V_2 creates a bowtie. Thus the number of bowties in the graph is at least

$$\binom{4(q+1)}{2}v_2 \geq \binom{4(q+1)}{2}(1 - 2\varepsilon)\frac{n}{2} > \frac{27}{8}(q+1)^2 n > (q+1)^2(13n/4 + 13)$$

for large enough n. This together with Lemma 4 contradicts the minimality of H. An analogous argument shows $b_2 \leq 4q + 4$.

Let $v_1 = \lceil n/2 \rceil + a$ and $v_2 = \lfloor n/2 \rfloor - a$. Without loss of generality assume $a \geq 0$ i.e. $v_1 \geq v_2$. Then the total number of pairs between V_1 and V_2 is

$$|V_1||V_2| = \lceil n/2 \rceil \lfloor n/2 \rfloor - a^2 + (\lfloor n/2 \rfloor - \lceil n/2 \rceil)a \leq |E(T_2(n))| - a^2$$

implying that $b_1 + b_2 = b \geq q + 1 + a^2$. Together with $b_1 + b_2 \leq 8q + 8 = o(n^2)$ this implies $a = o(n)$ completing the proof. $\square$

4. Proof of Theorems 1 and 2

4.1. Proof of Theorem 1

Throughout this section let H be a graph on n vertices and $\mathrm{ex}(n, F) + q$ edges containing the minimal number of bowties. In the previous section we proved Proposition 3 stating that there exists a partition of the vertex set of H into two sets V_1, V_2, such that every edge between the two sets is present. In addition, both sets contain approximately $n/2$ vertices and the number of edges spanned by each of these sets is small.

Once the partition of Proposition 3 has been established, we only need to examine the structure of the graphs spanned by V_1 and V_2. Denote the edges spanned by V_1 and V_2 by B_1 and B_2, respectively. Set $B = B_1 \cup B_2$ and $b = |B|$. In addition, for $i = 1, 2$ let

$$v_i = |V_i| \quad \text{and} \quad b_i = |B_i|.$$

We start by investigating the number of bowties in such graphs. Let $\widetilde{H}$ be a graph on V containing $\mathrm{ex}(n, F) + q$ edges such that the vertex set of $\widetilde{H}$ can be partitioned into two parts $\widetilde{V}_1, \widetilde{V}_2$ with $E(K(\widetilde{V}_1, \widetilde{V}_2)) \subseteq E(\widetilde{H})$. Denote the set of edges spanned by $\widetilde{V}_1$ and $\widetilde{V}_2$ by $\widetilde{B}_1$ and $\widetilde{B}_2$, respectively, also let $\widetilde{B} = \widetilde{B}_1 \cup \widetilde{B}_2$.

Recall that for any $v \in V$ and $E' \subseteq \binom{V}{2}$ $d_{E'}(v)$ denotes the degree of vertex v in the graph (V, E'). Since $\widetilde{H}$ contains a complete bipartite subgraph we can express a lower bound on the number of bowties found in $\widetilde{H}$ via an explicit formula. Recall that bowties are formed of two triangles, and note that any triangle in $\widetilde{H}$ must contain at least one edge in $\widetilde{B}$. In particular, we restrict ourselves to bowties formed from two triangles which have exactly one edge in $\widetilde{B}$ and thus two edges between $\widetilde{V}_1$ and $\widetilde{V}_2$. Any such bowtie belongs to one of the 3 types of bowties seen in Fig. 1. Any pair of disjoint edges in $\widetilde{V}_1$ is contained in $|\widetilde{V}_2|$ bowties. In addition a pair of adjacent edges in $\widetilde{V}_1$ can be found in $|\widetilde{V}_2|(|\widetilde{V}_2| - 1)$ bowties. Similarly, two disjoint edges in $\widetilde{V}_2$ create $|\widetilde{V}_1|$ bowties, while two adjacent edges create $|\widetilde{V}_1|(|\widetilde{V}_1| - 1)$ bowties. Finally any two edges, where one is spanned by $\widetilde{V}_1$ and the other is spanned by $\widetilde{V}_2$, are contained in $2(n - 4)$ bowties. This implies that the total number of bowties in $\widetilde{H}$ is at least

$$\sum_{i=1}^{2} \left(\sum_{v \in \widetilde{V}_i} \binom{d_{\widetilde{B}}(v)}{2} |\widetilde{V}_{3-i}| (|\widetilde{V}_{3-i}| - 1) + \left(\binom{|\widetilde{B}_i|}{2} - \sum_{v \in \widetilde{V}_i} \binom{d_{\widetilde{B}}(v)}{2} \right) |\widetilde{V}_{3-i}| \right)$$
$$+ 2(n - 4)|\widetilde{B}_1||\widetilde{B}_2|.$$

After trivial simplifications, this lower bound can be rewritten as

$$\#F(\widetilde{H}) \geq f((d_{\widetilde{B}}(v))_{v \in \widetilde{V}_1}, (d_{\widetilde{B}}(v))_{v \in \widetilde{V}_2}), \tag{9}$$

where f is the following function. It takes as input two sequences of non-negative integers, $(d_{1,1}, \ldots, d_{1,v_1^*}) \in \mathbb{N}^{v_1^*}$ and $(d_{2,1}, \ldots, d_{2,v_2^*}) \in \mathbb{N}^{v_2^*}$ such that $v_1^* + v_2^* = n$ and $b_i^* = \frac{1}{2} \sum_{j=1}^{v_i^*} d_{i,j}$ is an integer for $i = 1, 2$. Then the value of the function f is defined as

$$f((d_{1,j})_{j=1}^{v_1^*}, (d_{2,j})_{j=1}^{v_2^*}) = 2(n - 4)b_1^* b_2^*$$
$$+ \sum_{i=1}^{2} \left(\sum_{j=1}^{v_i^*} \binom{d_{i,j}}{2} v_{3-i}^* (v_{3-i}^* - 2) + \binom{b_i^*}{2} v_{3-i}^* \right). \tag{10}$$

Note that the bound in (9) is sharp if neither $\widetilde{V}_1$ nor $\widetilde{V}_2$ contains a triangle. On the other hand, the following lemma gives us some converse to the above inequality, by replacing $\widetilde{B}_i$ by a graph B_i^* that has the same number of edges and, additionally, is triangle-free and almost regular.

Lemma 16. *The following holds for all sufficiently large n. Let $V_1^* \cup V_2^* = V$ be a partition of V and for $i = 1, 2$ set $v_i^* = |V_i^*|$. Let $\phi : V \to \mathbb{N}$ be a function such that, for $i = 1, 2$, $b_i^* = \sum_{v \in V_i^*} \phi(v)/2$ is*

an integer and $b_i^* \leq (v_i^*)^2/16$. Then there is a graph H^* with n vertices and $v_1^* v_2^* + b_1^* + b_2^*$ edges such that

$$\#F(H^*) \leq f\left((\phi(v))_{v \in V_1^*}, (\phi(v))_{v \in V_2^*}\right). \tag{11}$$

Furthermore, if for some $i = 1, 2$, two values of ϕ on V_i^* differ by more than 1, then the inequality in (11) is strict.

Proof. For $i = 1, 2$, pick integers $d_{i,1}, \ldots, d_{i,v_i^*} \in \{\lfloor 2b_i^*/v_i^* \rfloor, \lceil 2b_i^*/v_i^* \rceil\}$ with sum $2b_i^*$. Lemma 24 shows that there is a triangle-free graph B_i^* on V_i^* whose degree sequence is $(d_{i,1}, \ldots, d_{i,v_i}^*)$. Let H^* be obtained by adding every edge between the graphs B_1^* and B_2^*. Clearly, H^* has the stated order and size while $\#F(H^*) = f((d_{1,j})_{j=1}^{v_1^*}, (d_{2,j})_{j=1}^{v_2^*})$. Using the convexity of the function $x \mapsto \binom{x}{2}$ on $\mathbb{N}$, we see that this is at most the right-hand side of (11), as required.

The second part of the lemma follows since the function $x \to \binom{x}{2}$ is *strictly* convex on $\mathbb{N}$. □

If we let $\widetilde{H}$ be our extremal graph H and let $\phi(v)$ be $d_B(v)$ for $v \in V$, then, in view of $v_i = (1/2 + o(1))n = \Omega(n)$ and $b_i = o(n^2)$, Lemma 16 applies, providing another extremal graph H^*. Thus, both (9) and (11) are equalities. This means that, for $i = 1, 2$, the part V_i does not induce a triangle and

$$|d_B(u) - d_B(v)| \leq 1, \quad \text{for every } u, v \in V_i. \tag{12}$$

For future reference, let us repeat the formula for the number of bowties in H:

$$\#F(H) = 2(n-4)b_1 b_2 + \sum_{i=1}^{2} \left(\sum_{v \in V_i} \binom{d_B(v)}{2} v_{3-i}(v_{3-i} - 2) + \binom{b_i}{2} v_{3-i} \right). \tag{13}$$

Note that the number of vertices which have degree $\lfloor 2b_i/v_i \rfloor$ or $\lceil 2b_i/v_i \rceil$ in the graph (V, B) is uniquely determined by b_i and v_i. Therefore (13) depends only on b_1, b_2, v_1, v_2. In fact, we only need to establish the values of these parameters which lead to the minimal number of bowties. However, there is some dependence between the parameters. We only require one parameter to track both part sizes. Let

$$v_1 = \lceil n/2 \rceil + a \quad \text{and} \quad v_2 = \lfloor n/2 \rfloor - a \tag{14}$$

for some $a \in \mathbb{Z}$. Proposition 3 implies that $a = o(n)$. Note that $b = q + 1 + (\lceil n/2 \rceil - \lfloor n/2 \rfloor)a + a^2$, thus it suffices to determine one of b_1, b_2, once the value of a has been established.

Theorem 1 follows if we can show that $a = 0$ when n is even and that $a = 0$ or -1 when n is odd. We will show that if neither of these holds, then moving a vertex from the larger part to the smaller decreases the number of bowties. Ideally, we would move the vertex in such a way that the number of neighbours of the vertex within its part remains unchanged and every edge between the new parts is still present. This leaves the total number of edges spanned by the parts unchanged, but increases the number of edges between V_1 and V_2 as $|a|$ is reduced. Thus, in order to leave the total number of edges unchanged, we need to remove some additional edges. Although the previous argument is only applicable if the degree of the vertex moved between the parts is even, it can be adapted to work for odd degrees as well. This is achieved by removing an edge before the vertex is moved, in particular an edge adjacent to the vertex about to be moved, resulting in an even degree for the vertex. Further superfluous edges are removed after the vertex has been moved.

We need to estimate the change in the number of bowties after moving a vertex between the parts and removing superfluous edges. Later we will see that the number of bowties destroyed by removing edges outnumbers the number of bowties created when moving the vertex.

In order to prove Theorem 1 we need three auxiliary lemmas, the proofs of which can be found in Section 5. We first estimate the number of bowties destroyed when a well-chosen set of k edges is removed from the graph H, after a vertex has been moved between its parts.

Lemma 17. *Let $U \subseteq V$ such that $|U| \geq n - 2$ and $\widehat{H}$ a graph created from H by adding and removing edges, such that no edge is removed from $E(H[U])$. If $b \geq 10$ for every $k \leq b/3$ there exists $D \subseteq B \cap E(H[U])$ with $|D| = k$ such that the removal of all edges of D from the graph $\widehat{H}$ destroys at least*

$$k \frac{bn}{8} - \frac{nk^2}{2}$$

bowties.

The number of bowties created when moving a vertex depends on the difference of b_1 and b_2. In particular, the closer the two values are, the smaller the change in the number of bowties is. In the following lemma we provide an upper bound on the difference of b_1 and b_2.

Lemma 18. *We have that $|b_1 - b_2| \leq n/4 + 33(|a| + 1)q/n$.*

As a consequence, we can derive that the degrees in the graph (V, B) are also close. By (12), this holds in a very strong form for vertices that are in the same part V_i. So the following lemma says something new, only if w is not in the same part as u or v.

Lemma 19. *If there exists a pair of vertices $u, v \in V$ such that $d_B(u) = d_B(v)$, then every vertex $w \in V$, but at most one, satisfies $d_B(w) \geq d_B(u) - 1 - 900(|a| + 1)(d_B(u) - 1)/n$.*

Using Proposition 3 and Lemmas 17–19 we prove Theorem 1.

Proof of Theorem 1. Assume for contradiction that the theorem is false. Then for each $\delta > 0$ there exists a graph H that violates Theorem 1. Let δ be sufficiently small so that the order $n \geq 1/\delta$ of H and the surplus $q = |E(H)| - ex(n, F) \leq \delta n^2$ satisfy all forthcoming inequalities. Since the constant δ given by our proof is very small, we do not write it explicitly nor try to optimise the dependencies between other constants. For notational convenience, we use asymptotic notation, e.g. writing $q = o(n^2)$.

Proposition 3 applies and gives that H contains a spanning complete bipartite graph $K(V_1, V_2)$. Define $v_i = |V_i|$ for $i = 1, 2$. Without loss of generality we may assume that $v_1 \geq v_2$. Let $a = v_1 - \lceil n/2 \rceil$. By Proposition 3, we have that $a = o(n)$. Since H is a counterexample, we have $a \geq 1$. The larger part V_1 must contain at least one edge, otherwise selecting $U \subseteq V_1$ with $|U| = v_2$ and moving the graph spanned by V_2 to U strictly decreases the number of bowties by (13).

Recall that, by (12), all bad degrees inside a part differ by at most 1. Let d be the maximal integer such that there exist two vertices $v, w \in V_1$ such that $d_B(v) = d_B(w) = d$. Since V_1 contains at least one edge, we have $d > 0$.

Note that each bad degree is at most $d + 2 + o(d)$. Indeed, this is true if V_2 spans no edges. Otherwise, V_2 has two vertices u', v' of the same positive degree d' and in view of $a = o(n)$ from Lemma 19 applied to these two vertices we infer $d' \leq d + 1 + o(d)$. The claim follows from (12). Thus, for every $z \in V$, we have, for example, $d_B(z) \leq 2d + 2$. Since $v_1, v_2 \leq n$, we have

$$b_1, b_2 = O(dn). \tag{15}$$

We consider two cases depending on the parity of d.

Case 1: *d is even.*

Roughly speaking, we move v from V_1 to V_2 in such a way that for every vertex in $V \setminus \{v\}$ the number of neighbours within its part remains unchanged, but for v the number of neighbours within its part changes from d to $d - 2$, i.e. an edge is removed from B. In addition, every edge between the new parts is present. More formally, set $V_1^* = V_1 \setminus \{v\}$, $V_2^* = V_2 \cup \{v\}$ and define $\phi : V \to \mathbb{N}$ such that for $u \in V \setminus \{v\}$ we have $\phi(u) = d_B(u)$ and $\phi(v) = d_B(v) - 2$. Let H^* be the graph provided

by [Lemma 16](#) for this function ϕ. We have that

$$
\begin{aligned}
\#F(H^*) \leq\ & 2(n-4)\left(b_1 - \frac{d_B(v)}{2}\right)\left(b_2 + \frac{d_B(v)-2}{2}\right) \\
& + \binom{d_B(v)-2}{2}(v_1-1)(v_1-3) \\
& + \binom{b_1 - d_B(v)/2}{2}(v_2+1) + \binom{b_2 + (d_B(v)-2)/2}{2}(v_1-1) \\
& + \sum_{u \in V_1 \setminus \{v\}} \binom{d_B(u)}{2}(v_2+1)(v_2-1) + \sum_{u \in V_2} \binom{d_B(u)}{2}(v_1-1)(v_1-3).
\end{aligned}
$$

By [(13)](#) the number of bowties in H is

$$
\begin{aligned}
\#F(H) =\ & 2(n-4)b_1 b_2 + \binom{d_B(v)}{2}v_2(v_2-2) + \binom{b_1}{2}v_2 + \binom{b_2}{2}v_1 \\
& + \sum_{u \in V_1 \setminus \{v\}} \binom{d_B(u)}{2}v_2(v_2-2) + \sum_{u \in V_2} \binom{d_B(u)}{2}v_1(v_1-2).
\end{aligned}
$$

We are interested in a lower bound on $\#F(H) - \#F(H^*)$, more precisely the difference of the above two bounds. We will examine the difference of each of the six terms in order to determine the overall change. Recall that $d_B(v) = d$.

We start with

$$
\begin{aligned}
2(n-4)b_1 b_2 - 2(n-4)&\left(b_1 - \frac{d}{2}\right)\left(b_2 + \frac{d-2}{2}\right) \\
&= 2(n-4)\left(b_2 \frac{d}{2} - b_1 \frac{d-2}{2} + \frac{d(d-2)}{4}\right) \\
&\overset{(15)}{\geq} ndb_2 - n(d-2)b_1 + O(d^2 n). \tag{16}
\end{aligned}
$$

Recall that $v_1, v_2 = n/2 + O(a)$. Thus for the following term we have

$$
\begin{aligned}
\binom{d}{2}v_2(v_2-2) - \binom{d-2}{2}&(v_1-1)(v_1-3) \\
&= \left(\binom{d}{2} - \binom{d-2}{2}\right)\left(\frac{n}{2}\right)^2 + O(ad^2 n) \\
&= (2d-3)\left(\frac{n}{2}\right)^2 + O(ad^2 n). \tag{17}
\end{aligned}
$$

By the definition of d and [(12)](#), every vertex in V_1, except at most one, has degree at most d. Therefore $b_1 - 1 \leq dn/4 + O(ad)$ and thus we have

$$
\begin{aligned}
\binom{b_1}{2}v_2 - \binom{b_1 - d/2}{2}(v_2+1) &\geq b_1 \frac{d}{2}v_2 - \binom{b_1}{2} + O(d^2 n) \\
&= b_1 \frac{d}{2}v_2 - b_1 \frac{b_1-1}{2} + O(d^2 n) \\
&\overset{(15)}{\geq} b_1\left(\frac{d}{2}\frac{n}{2} - \frac{dn}{8}\right) + O(ad^2 n) \\
&= b_1 \frac{dn}{8} + O(ad^2 n). \tag{18}
\end{aligned}
$$

Lemma 19 implies that for every $u \in V_2$, except at most one, we have $d_B(u) \geq d - 1 + O(ad/n)$. Thus $b_2 - 1 \geq (d-1)n/4 + O(ad)$ and similarly as before

$$\binom{b_2}{2} v_1 - \binom{b_2 + (d-2)/2}{2}(v_1 - 1) = -b_2 \frac{d-2}{2} \frac{n}{2} + \binom{b_2}{2} + O(ad^2 n)$$

$$= -b_2 \frac{d-2}{2} \frac{n}{2} + b_2 \frac{b_2 - 1}{2} + O(ad^2 n)$$

$$\geq -b_2 \frac{(d-3)n}{8} + O(ad^2 n). \tag{19}$$

All that is left to estimate are the two sums. The first of these is

$$\sum_{u \in V_1 \setminus \{v\}} \binom{d_B(u)}{2} v_2(v_2 - 2) - \sum_{u \in V_1 \setminus \{v\}} \binom{d_B(u)}{2}(v_2 + 1)(v_2 - 1)$$

$$= (-2v_2 + 1) \sum_{u \in V_1 \setminus \{v\}} \binom{d_B(u)}{2}.$$

Recall that for every $u \in V_1 \setminus \{v\}$, except at most one, we have that $d_B(u) \leq d$ and by (12) for the one exception we have $d_B(u) \leq d + 1$. When $d_B(u) \leq d$, we have

$$\binom{d_B(u)}{2} \leq \frac{d_B(u)(d-1)}{2},$$

on the other hand, if $d_B(u) = d + 1$, then

$$\binom{d_B(u)}{2} = \frac{d_B(u)(d-1)}{2} + \frac{d+1}{2}.$$

Thus

$$(-2v_2 + 1) \sum_{u \in V_1 \setminus \{v\}} \binom{d_B(u)}{2} \geq (-2v_2 + 1) \left(\frac{d+1}{2} + \sum_{u \in V_1 \setminus \{v\}} \frac{d_B(u)(d-1)}{2} \right)$$

$$= (-2v_2 + 1)(d-1) \sum_{u \in V_1 \setminus \{v\}} \frac{d_B(u)}{2} + O(ad^2 n)$$

$$\geq (-2v_2 + 1)(d-1)b_1 + O(ad^2 n)$$

$$\overset{(15)}{=} -b_1(d-1)n + O(ad^2 n). \tag{20}$$

Finally, we have

$$\sum_{u \in V_2} \binom{d_B(u)}{2} v_1(v_1 - 2) - \sum_{u \in V_2} \binom{d_B(u)}{2}(v_1 - 1)(v_1 - 3) = (2v_1 - 3) \sum_{u \in V_2} \binom{d_B(u)}{2}.$$

Recall that for every $u \in V_2$, except at most one, we have $d_B(u) \geq d - 1 - O(ad/n)$ and by (12) for the one exception we have $d_B(u) \geq d - 2 - O(ad/n)$, thus

$$(2v_1 - 3) \sum_{u \in V_2} \binom{d_B(u)}{2} \geq (2v_1 - 3)(d-2) \sum_{u \in V_2} \frac{d_B(u)}{2} + O(ad^2 n)$$

$$= b_2(d-2)n + O(ad^2 n). \tag{21}$$

Combining (16)–(21) we have

$$\#F(H) - \#F(H^*) \geq (2d - 3)\left(\frac{n}{2}\right)^2 + \left(\frac{15d - 13}{4} b_2 - \frac{15d - 24}{4} b_1 \right) \frac{n}{2} + O(ad^2 n)$$

$$\overset{\text{Lemma 18}}{\geq} (2d - 3)\left(\frac{n}{2}\right)^2 - \frac{15d - 24}{8}\left(\frac{n}{2}\right)^2 + O(ad^2 n)$$

$$\overset{d=o(n)}{=} \frac{d}{8}\left(\frac{n}{2}\right)^2 + o(adn^2) \tag{22}$$

when n is large enough.

Note that the number of edges in H^* exceeds the number of edges in H by

$$\begin{aligned} E(H^*) - E(H) &\geq (\lceil n/2 \rceil + (a-1))\,(\lfloor n/2 \rfloor - (a-1)) \\ &\quad - (\lceil n/2 \rceil + a)\,(\lfloor n/2 \rfloor - a) - 1 \\ &\geq a^2 - (a-1)^2 - 1 = 2(a-1). \end{aligned}$$

When $a < 10$,

$$\#F(H) - \#F(H^*) \overset{(22)}{\geq} \frac{d}{8}\left(\frac{n}{2}\right)^2 + o(adn^2) > 0$$

for large enough n and thus already H^* has fewer bowties than H and removing the additional edges only decreases this number. On the other hand, if $a \geq 10$, then removing these $2(a-1)$ additional edges plays a significant role, because by Lemma 17 this destroys at least $(a-1)bn/4 - 2(a-1)^2 n$ bowties. Recall that $b \geq q + 1 + a^2$ and when $a \geq 10$ then $b \geq q + 1 + a^2 \geq 10(a-1)$ leading to the destruction of at least

$$(a-1)n\left(\frac{b}{4} - 2(a-1)\right) \geq (a-1)n\left(\frac{b}{4} - \frac{2b}{10}\right) = (a-1)\frac{b}{20}n \overset{b \geq (d-1)n/2, d \geq 2}{\geq} \frac{adn^2}{100} \tag{23}$$

bowties. This together with (22) leads to a contradiction.

Case 2: d is odd.

Recall that $v, w \in V_1$ are such that $d_B(v) = d_B(w) = d$. Roughly speaking, we will move v from V_1 to V_2 in such a way that for every vertex in $V \setminus \{v, w\}$ the number of neighbours within its part remains unchanged, but for v and w the number of neighbours within its part changes from d to $d-1$. Formally, set $V_1^* = V_1 \setminus \{v\}$, $V_2^* = V_2 \cup \{v\}$ and define $\phi : V \to \mathbb{N}$ such that $\phi(v) = d_B(v) - 1$, $\phi(w) = d_B(w) - 1$ and for $u \in V \setminus \{v, w\}$ we have $\phi(u) = d_B(u)$. Let H^* be the graph returned by Lemma 16. The number of bowties in H^* satisfies

$$\begin{aligned} \#F(H^*) \leq\; &2(n-4)\left(b_1 - \frac{d_B(v)+1}{2}\right)\left(b_2 + \frac{d_B(v)-1}{2}\right) \\ &+ \binom{d_B(v)-1}{2}(v_1-1)(v_1-3) + \binom{d_B(w)-1}{2}(v_2+1)(v_2-1) \\ &+ \binom{b_1 - (d_B(v)+1)/2}{2}(v_2+1) + \binom{b_2 + (d_B(v)-1)/2}{2}(v_1-1) \\ &+ \sum_{u \in V_1 \setminus \{v,w\}} \binom{d_B(u)}{2}(v_2+1)(v_2-1) + \sum_{u \in V_2} \binom{d_B(u)}{2}(v_1-1)(v_1-3). \end{aligned}$$

Clearly, the number of bowties in H is the same as earlier

$$\begin{aligned} \#F(H) =\; &2(n-4)b_1 b_2 \\ &+ \binom{d_B(v)}{2}v_2(v_2-2) + \binom{d_B(w)}{2}v_2(v_2-2) \\ &+ \binom{b_1}{2}v_2 + \binom{b_2}{2}v_1 \\ &+ \sum_{u \in V_1 \setminus \{v,w\}} \binom{d_B(u)}{2}v_2(v_2-2) + \sum_{u \in V_2} \binom{d_B(u)}{2}v_1(v_1-2). \end{aligned}$$

The calculations, for most part, are analogous to Case 1. In (16) we just need to replace d with $d+1$

$$2(n-4)b_1 b_2 - 2(n-4)\left(b_1 - \frac{(d+1)}{2}\right)\left(b_2 + \frac{d-1}{2}\right)$$

$$\geq n(d+1)b_2 - n(d-1)b_1 + O(d^2n). \tag{24}$$

On the other hand, in (17) the vertex degree decreases only by one, leading to

$$\binom{d}{2}v_2(v_2-2) - \binom{d-1}{2}(v_1-1)(v_1-3)$$

$$= \binom{d}{2}\left(\frac{n}{2}\right)^2 - \binom{d-1}{2}\left(\frac{n}{2}\right)^2 + O(ad^2n)$$

$$= (d-1)\left(\frac{n}{2}\right)^2 + O(ad^2n).$$

However, this time the degree of another vertex in V_1 also decreases

$$\binom{d_B(w)}{2}v_2(v_2-2) - \binom{d_B(w)-1}{2}(v_2+1)(v_2-1) = (d-1)\left(\frac{n}{2}\right)^2 + O(ad^2n).$$

For (18) and (19) our bounds on b_1 and b_2 still hold, leading to

$$\binom{b_1}{2}v_2 - \binom{b_1-(d+1)/2}{2}(v_2+1) \geq b_1\left(\frac{d+1}{2}\frac{n}{2} - \frac{dn}{8}\right) + O(ad^2n)$$

$$= b_1\frac{(d+2)n}{8} + O(ad^2n)$$

and

$$\binom{b_2}{2}v_1 - \binom{b_2+(d-1)/2}{2}(v_1-1) \geq -b_2\frac{d-1}{2}\frac{n}{2} + b_2\frac{(d-1)n}{8} + O(ad^2n)$$

$$\geq -b_2\frac{(d-1)n}{8} + O(ad^2n).$$

Finally, note that removing an additional vertex from the sum in (20) has no affect on the lower bound and (21) remains unchanged. Therefore, we have

$$\#F(H) - \#F(H^*) \geq 2(d-1)\left(\frac{n}{2}\right)^2 + \left(\frac{15d-7}{4}b_2 - \frac{15d-18}{4}b_1\right)\frac{n}{2} + O(ad^2n)$$

$$\overset{\text{Lemma 18}}{\geq} 2(d-1)\left(\frac{n}{2}\right)^2 - \frac{15d-18}{8}\left(\frac{n}{2}\right)^2 + O(ad^2n)$$

$$\geq \frac{d+2}{8}\left(\frac{n}{2}\right)^2 + O(ad^2n)$$

when n is large enough. Similarly as before we have at least $2(a-1)$ additional edges in the graph. If $a < 10$ or $d = 1$, then the number of bowties has already decreased even before removing these edges. In the remaining cases, a calculation analogue to (23) implies that the removal of the additional edges decreases the number of bowties. □

4.2. Proof of Theorem 2

In order to prove Theorem 2 we need to show

$$h_F(n,q) = (1 \pm c)\frac{n}{2}\left[\binom{e_1}{2} + \binom{e_2}{2} + m\binom{d+1}{2}\frac{n}{2} + (n-m)\binom{d}{2}\frac{n}{2} + 4e_1e_2\right],$$

where

$$e_1 = \left\lfloor \frac{dn}{4} + \frac{\min\{m, n/2\}}{2} \right\rfloor \quad \text{and} \quad e_2 = q + 1 - e_1.$$

In the previous subsection we established the values of v_1 and v_2, thus we only need to determine the values of b_1 and b_2. More precisely, the asymptotics of these two values suffice, which we achieve by analysing the degree sequence in the graphs spanned by V_1 and V_2. We first show that

the bad degree of almost every vertex must take one of two values (Lemma 20). In addition in one of the partitions almost every vertex must have the same bad degree (Lemma 21). The proofs of these two lemmas can be found in Section 5. Lemmas 20 and 21 are sufficient to establish the asymptotics of b_1 and b_2 and complete the proof of Theorem 2.

Lemma 20. *Let $d = \lfloor 2(q+1)/n \rfloor$. For every vertex $v \in V$, we have $d - 1 \le d_B(v) \le d + 2$. In addition, both the number of vertices with bad degree $d - 1$ and $d + 2$ are at most one.*

This lemma shows that almost every vertex has degree $\lfloor 2(q + 1)/n \rfloor$ or $\lfloor 2(q + 1)/n \rfloor + 1$ in the graph (V, B). We have yet to establish how many of these vertices are contained in V_1 and V_2. In the following lemma we show that in one of the two parts almost every vertex has the same degree. In fact, we show a more general statement. By (12), pick integers k and ℓ such that every vertex in V_1 has degree k or $k + 1$ and every vertex in V_2 has degree ℓ or $\ell + 1$. Let C_i denote the set of vertices of degree i in V_1 and D_i denote the set of vertices of degree i in V_2.

Lemma 21. *In H, at least one of $|C_k|, |C_{k+1}|, |D_\ell|, |D_{\ell+1}|$ is at most 1.*

The previous two lemmas give us a satisfactory estimate for the degree sequence of (V, B), allowing us to determine the asymptotics of $h_F(n, q)$.

Proof of Theorem 2. By (13) and since $v_i = (1 + o(1))n/2$, the number of bowties is

$$(1 + o(1))\frac{n}{2}\left[\binom{b_1}{2} + \binom{b_2}{2} + \frac{n}{2}\sum_{v \in V}\binom{d_B(v)}{2} + 4b_1 b_2\right].$$

When $q < \lfloor n/4 \rfloor - 1$, Lemma 21 implies that the optimal solution is when V_1 or V_2 contains a matching of size $q + 1$ and the other part does not span any edge. Therefore the statement holds for this range of q.

Now assume that $q \ge \lfloor n/4 \rfloor - 1$. Recall from the statement of the theorem that $2(q + 1) = dn + m$ for $d, m \in \mathbb{N}$ with $m < n$ and $d = \lfloor 2(q + 1)/n \rfloor$. By Lemma 20 we have that almost every vertex has degree d or $d + 1$ in (V, B) and any other vertex has degree $d - 1$ or $d + 2$. Thus we obtain

$$\sum_{v \in V}\binom{d_B(v)}{2} = (n - m)\binom{d}{2} + m\binom{d + 1}{2} + O(d + 1).$$

Since $d = o(n)$, we have $(d + 1)n = o(n^2)$. In addition, $q = \Omega(n)$ and thus $b_1^2 + b_2^2 = \Omega(n^2)$. Therefore, the total number of bowties is

$$(1 + o(1))\frac{n}{2}\left[\binom{b_1}{2} + \binom{b_2}{2} + \frac{n}{2}\left((n - m)\binom{d}{2} + m\binom{d + 1}{2}\right) + 4b_1 b_2\right].$$

All that is left to show is that one of the parts contains $(1 + o(1))e_1$ edges. Recall that the sum of the bad degrees is $dn + m$. Let $m > n/2$. By Lemma 20 there exist at least $n/2 - 3$ vertices of bad degree $d + 1$. If each part has at least 2 vertices with bad degree $d + 1$, then by Lemma 21 all but at most one vertex must have bad degree $d + 1$ in one of the parts. Together with Theorem 1 this implies that there exists a part with at least $n/2 - 2$ vertices of bad degree $d + 1$. Otherwise, one part has at most one vertex of bad degree $d + 1$ and thus the other must have at least $n/2 - 4$ vertices of bad degree $d + 1$. So in either case we have a part containing at least $n/2 - 4$ vertices of bad degree $d + 1$ and without loss of generality we may assume that this is V_1. Then $b_1 = (1 + o(1))(d + 1)n/2 = (1 + o(1))e_1$.

Now consider the case when $m \le n/2$. A similar argument as before, implies the existence of a part containing $n/2 - 4$ vertices with bad degree d and without loss of generality assume that this is V_2. Therefore,

$$b_1 = \frac{dn + m}{2} - (1 + o(1))\frac{dn}{4} - o\left(\frac{(d + 1)n}{4}\right) = (1 + o(1))e_1$$

as required. $\square$

Remark. Lemmas 20 and 21 leave only a limited number of options for the degree sequence of the extremal graph H. The exact value of $h_F(n, q)$ can be deduced after a precise analysis. In particular when n is divisible by 4, since the number of vertices in both parts is even the aforementioned lemmas imply that every vertex has bad degree d or $d + 1$ and every vertex in one of the parts has the same bad degree. Therefore the formula in Theorem 2 holds exactly, i.e. without the $1 \pm c$ multiplier.

5. Proof of technical lemmas

We start with results on the existence of triangle-free graphs with a given degree sequence, culminating in Lemma 24 which was used in the proof of Lemma 16. While some of the intermediate steps can be derived from the Gale–Ryser theorem [10,30] that characterises possible degree sequences of bipartite graphs, we present simple direct constructions instead. Let an (α, a, β, b)-*graph* mean a triangle-free graph with $\alpha + \beta$ vertices which has α vertices of degree a and β vertices of degree b if $a \neq b$ and is a-regular if $a = b$. Trivially an (α, a, β, b) graph is also a (β, b, α, a) graph.

Lemma 22. *For any non-negative integers d, i, m satisfying $d < i + m$ there exists a $(2i, d + 1, 2m, d)$-graph. In addition this is a balanced bipartite graph.*

Proof. Partition the vertices into two sets $\{v_{1,j} : j = 1, \ldots, i + m\}$ and $\{v_{2,j} : j = 1, \ldots, i + m\}$. Join $v_{1,j}$ with $v_{2,j}$ by an edge for every $j \leq i$. Let M_ℓ be a perfect matching where for every $j = 1, \ldots, i + m$ the vertex $v_{1,j}$ is connected to $v_{2,k}$ where $k \equiv j + \ell \pmod{i + m}$. In order to complete the graph, insert M_ℓ for $\ell = 1, \ldots, d$. The obtained graph has the desired degree sequence and is also bipartite (and thus triangle-free). $\square$

Lemma 23. *For any integers k, m satisfying $k \geq m \geq 0$ there exists a $(4k, k, 1, 2m)$-graph.*

Proof. Denote by u the vertex of degree $2m$ and partition the remaining vertices into 4 sets of size k: $U_1, \ldots, U_4$. Denote the vertices in U_i by $u_{i,j}$ for $j = 1, \ldots, k$. Join u to vertices $u_{i,j}$ by an edge, where $i = 1, 3$ and $j = 1, \ldots, m$, i.e. we join u to m vertices in U_1 and to m vertices in U_3. In addition, for $j = m + 1, \ldots, k$ insert an edge between $u_{1,j}$ and $u_{3,j}$. Also for $j = 1, \ldots, k$ insert an edge between $u_{2,j}$ and $u_{4,j}$. Finally insert every edge between U_1 and U_2 and every edge between U_3 and U_4 except the ones in the set $\{\{u_{i,j}, u_{i+1,j}\} : i = 1, 3, j = 1, \ldots, k\}$.

Note that the obtained graph G has the required degree sequence. In addition u is not contained in a triangle, while, $G - u$ is bipartite with parts $U_1 \cup U_4$ and $U_2 \cup U_3$. Thus, G is triangle-free, as required. $\square$

Lemma 24. *Let a, b, α, β be non-negative integers such that $|a - b| = 1$, $\alpha a + \beta b$ is even and $3a + 3b < \alpha + \beta - 1$. Then there is an (α, a, β, b)-graph.*

Proof. If both α and β are even, then Lemma 22 directly gives the desired graph. Assume that at least one of α and β is odd. In fact, exactly one of them is odd, because a and b have different parities while $\alpha a + \beta b$ is even by our assumption. By symmetry, assume that α is odd and β is even. Then, necessarily, a is even.

By the remaining assumption of the lemma, we have that $6a < \alpha + \beta - 1$ or $6b < \alpha + \beta - 1$. In the former case we take the disjoint union of the $(4a, a, 1, a)$-graph given by Lemma 23 (with $k = a$ and $m = a/2$) and the $(\alpha - 4a - 1, a, \beta, b)$-graph given by Lemma 22. In the latter case we take the $(4b, b, 1, a)$-graph of Lemma 23 (with $k = b$ and $m = a/2$) and the $(\alpha - 1, a, \beta - 4b, b)$-graph of Lemma 22. $\square$

Now we provide the proofs of the auxiliary lemmas of Section 4.

Proof of Lemma 17. The number of bowties destroyed by removing the edges in D from the graph $\widehat{H}$, is at least the number of bowties destroyed by removing D from the graph $H[U]$. We will analyse the latter.

Note that

$$|B \cap E(H[U])| \overset{\text{Lemma 10}}{\geq} b - 2\left(\frac{80b}{n} + 1\right) \geq \frac{2b}{3},$$

where the last inequality follows from our assumption that $b \geq 10$ and because n is large enough.

For simplicity of notation let $U_1 = U \cap V_1$ and $U_2 = U \cap V_2$. Without loss of generality we assume that U_1 spans at least as many edges in $B \cap E(H[U])$ as U_2. Select an arbitrary pair of edges $\{w_1, w_2\}, \{w_3, w_4\} \in B \cap E(H[U_1])$. If $\{w_3, w_4\}$ is disjoint from $\{w_1, w_2\}$, then for every shared neighbour of w_1, w_2, w_3, w_4 we have a bowtie. On the other hand, if $\{w_3, w_4\}$ is adjacent to $\{w_1, w_2\}$, then every pair of shared neighbours results in a bowtie. By Proposition 3 every edge between U_1 and U_2 is present. Since only two vertices were removed from the graph we have $|U_1|, |U_2| = (1 + o(1))n/2$. Thus the number of bowties containing $\{w_1, w_2\}$ and $\{w_3, w_4\}$ is at least $(1 + o(1))n/2$.

Therefore, removing arbitrary k edges from $B \cap E(H[U_1])$ destroys at least

$$(1 + o(1))\frac{n}{2}\left(k\left(\frac{|B \cap E(H[U])|}{2} - k\right)\right) \geq (1 + o(1))\frac{n}{2}\left(k\left(\frac{b}{3} - k\right)\right)$$

$$\geq k\frac{bn}{8} - \frac{nk^2}{2}$$

bowties.   □

Proof of Lemma 18. Assume for contradiction that $|b_1 - b_2| > n/4 + 33(|a| + 1)q/n$. Without loss of generality assume that

$$b_1 > b_2 + n/4 + 33(|a| + 1)q/n. \tag{25}$$

Roughly speaking, we move $e_M := \lfloor n/4 \rfloor - |a|$ edges from V_1 to V_2 and show that the resulting graph has fewer copies of bowties. Let R_1 be a set of vertices in V_1 of size $2\lfloor n/4 \rfloor - 2|a|$ maximising the function $\sum_{v \in R_1} d_B(v)$, i.e. it contains the $2\lfloor n/4 \rfloor - 2|a|$ vertices $v \in V_1$ with the largest value of $d_B(v)$. Note that

$$|V_1| - |R_1| \geq \left\lfloor\frac{n}{2}\right\rfloor - |a| - 2\left\lfloor\frac{n}{4}\right\rfloor + 2|a| \geq |a| \geq 0.$$

On the other hand, let R_2 be the set of vertices in V_2 of size $2\lfloor n/4 \rfloor - 2|a|$ minimising the function $\sum_{v \in R_2} d_B(v)$. In order to move the e_M edges we reduce the value of $d_B(v)$ for $v \in R_1$ by one and increase the value of $d_B(v)$ for $v \in R_2$ by one. Define $\phi : V \to \mathbb{N}$ by

$$\phi(v) = \begin{cases} d_B(v) - 1, & \text{if } v \in R_1, \\ d_B(v), & \text{if } v \in V \setminus (R_1 \cup R_2), \\ d_B(v) + 1, & \text{if } v \in R_2. \end{cases}$$

Since by (12) the value of $d_B(v)$ differs by at most one for vertices in V_1 and R_1 contains the vertices v with the largest value of $d_B(v)$, we also have that the value of $\phi(v)$ differs by at most one for vertices in V_1. The same argument also gives us that the value of $\phi(v)$ differs by at most one for vertices in V_2.

Denote by H^* the graph returned by Lemma 16 for $V_1^* = V_1$ and $V_2^* = V_2$. By (13) and Lemma 16 we have

$$\#F(H) - \#F(H^*) \geq \sum_{u \in R_1} \phi(u)v_2(v_2 - 2) + e_M b_2(2n - 8) + v_2\left(e_M b_1 - \frac{e_M^2}{2} - \frac{e_M}{2}\right)$$

$$- \sum_{u \in R_2} d_B(u)v_1(v_1 - 2) - e_M(b_1 - e_M)(2n - 8) - v_1\left(b_2 e_M + \frac{e_M^2}{2} - \frac{e_M}{2}\right).$$

By Proposition 3 and (25) we have $4q + 4 \geq b_1 \geq n/4$, implying $b_1 \leq (1 + o(1))4q$ and $16q/n \geq 1 + o(1)$. Therefore $\sum_{u \in R_1} \phi(u) \leq \sum_{u \in V_1} \phi(u) \leq 2b_1 \leq (1 + o(1))8q \leq 9q$. Thus

$$
\sum_{u \in R_1} \phi(u) v_2 (v_2 - 2) \geq \sum_{u \in R_1} \phi(u) \left(\frac{n}{2} - |a| - \frac{1}{2} \right) \left(\frac{n}{2} - |a| - \frac{5}{2} \right)
$$
$$
\geq \sum_{u \in R_1} \phi(u) \left(\frac{n}{2} \right)^2 - 14(|a| + 1)qn.
$$

Proposition 3 also implies that $v_1 = (1 + o(1))n/2$, therefore the average vertex degree in $H[V_1]$ is $2b_1/v_1 \leq (1 + o(1))16q/n$. Together with (12), for every vertex $v \in V_1$ we have $d_B(v) \leq (1 + o(1))16q/n + 1$ for large enough n. Recall that $16q/n \geq 1 + o(1)$. Thus for $v \in V_1$ we have $d_B(v) \leq (1 + o(1))32q/n \leq 33q/n$. Therefore, we have

$$
\sum_{u \in R_1} \phi(u) \left(\frac{n}{2} \right)^2 - 14(|a| + 1)qn
$$
$$
\geq \left[\sum_{u \in V_1} \phi(u) - |V_1 \setminus R_1| \frac{33q}{n} \right] \left(\frac{n}{2} \right)^2 - 14(|a| + 1)qn
$$
$$
\geq \left(\frac{n}{2} \right)^2 \sum_{u \in V_1} \phi(u) - 39(|a| + 1)qn,
$$

as $|V_1 \setminus R_1| \leq \lceil n/2 \rceil + |a| - 2\lfloor n/4 \rfloor + 2|a| \leq 3(|a| + 1)$.

Since $q \geq (1 + o(1))n/16$ we have $b_2 \leq 4q + 4 \leq 5q$ for large enough n. Together with $e_M \leq n/4$ this implies

$$
e_M b_2 (2n - 8) \geq e_M b_2 2n - 10(|a| + 1)qn
$$
$$
\overset{e_M \geq n/4 - |a| - 1}{\geq} \left(\frac{n}{2} \right)^2 \sum_{u \in V_2} d_B(u) - 20(|a| + 1)qn.
$$

Finally using $e_M = \lfloor n/4 \rfloor - |a|$ and $e_M + 1 \leq n/4 + 1 \leq b_1 \leq (1 + o(1))4q$ we have

$$
v_2 \left(e_M b_1 - \frac{e_M^2}{2} - \frac{e_M}{2} \right) \geq \frac{n}{2} \left(e_M b_1 - \frac{e_M^2}{2} - \frac{e_M}{2} \right) - 2(|a| + 1)qn
$$
$$
\geq \frac{n}{2} b_1 \left(\frac{n}{4} - |a| - 1 \right) - \frac{n}{4} \left(\left(\frac{n}{4} \right)^2 + \frac{n}{4} \right) - 2(|a| + 1)qn
$$
$$
\geq \frac{n^2}{8} b_1 - \frac{n^3}{64} - \frac{n^2}{16} - 5(|a| + 1)qn.
$$

Therefore, we obtain

$$\sum_{u \in R_1} \phi(u) v_2 (v_2 - 2) + e_M b_2 (2n - 8) + v_2 \left(e_M b_1 - \frac{e_M^2}{2} - \frac{e_M}{2} \right)$$

$$\geq \left(\frac{n}{2} \right)^2 \sum_{u \in V_1} \phi(u) + \left(\frac{n}{2} \right)^2 \sum_{u \in V_2} d_B(u) + \frac{n^2}{8} b_1 - \frac{n^3}{64} - \frac{n^2}{16} - 64(|a| + 1)qn.$$

A similar argument shows that

$$\sum_{u \in R_2} d_B(u) v_1 (v_1 - 2) + e_M (b_1 - e_M)(2n - 8) + v_1 \left(b_2 e_M + \frac{e_M^2}{2} - \frac{e_M}{2} \right)$$

$$\leq \left(\frac{n}{2} \right)^2 \sum_{u \in V_2} d_B(u) + \left(\frac{n}{2} \right)^2 \sum_{u \in V_1} \phi(u) + \frac{n^2}{8} b_2 + \frac{n^3}{64} + \frac{n^2}{16} + 64(|a| + 1)qn.$$

Summing up, we have

$$\#F(H) - \#F(H^*) \geq \frac{n^2}{8} \left[b_1 - b_2 - \frac{n}{4} - 1 - 16 \frac{(|a| + 1)q}{n} \right]$$

$$\overset{16q/n \geq 1 + o(1)}{\geq} \frac{n^2}{8} \left[b_1 - b_2 - \frac{n}{4} - 33 \frac{(|a| + 1)q}{n} \right],$$

which is positive due to (25), leading to a contradiction on minimality. □

Proof of Lemma 19. Let $d := d_B(u) - 1 = d_B(v) - 1$. We start with the case when $u, v \in V_1$, $d \geq 1$ and at most one vertex in V_1 has degree $d + 2$ in (V, B).

Assume for contradiction that there exist 2 vertices $w_1, w_2 \in V$ with $d_B(w_1), d_B(w_2) < d - 900(|a| + 1)d/n$. By (12) for any vertex $w \in V_1$ we have that $d_B(w) \geq d$ so in fact $w_1, w_2 \in V_2$. In addition, we may choose w_1, w_2 such that for every $w \in V_2 \setminus \{w_1, w_2\}$ we have $d_B(w_1), d_B(w_2) \leq d_B(w)$. Let $\phi : V \to \mathbb{N}$ be the function defined by

$$\phi(z) = \begin{cases} d_B(z) - 1, & \text{if } z = u, v, \\ d_B(z), & \text{if } v \in V \setminus \{u, v, w_1, w_2\}, \\ d_B(z) + 1, & \text{if } z = w_1, w_2. \end{cases}$$

Let H^* be the graph returned by Lemma 16 for $V_1^* = V_1$ and $V_2^* = V_2$. Thus, by (13), we have

$$\#F(H) - \#F(H^*) \geq (d_B(u) + d_B(v) - 2) v_2 (v_2 - 2) + b_2 (2n - 8) + (b_1 - 1) v_2$$
$$- (d_B(w_1) + d_B(w_2)) v_1 (v_1 - 2) - (b_1 - 1)(2n - 8) - b_2 v_1.$$

Note that $d_B(v), d_B(w_1) \leq d + 1$ and together with (12) we have that $d_B(w) \leq d + 2$ for any $w \in V$. Therefore, $b_1, b_2 \leq (1 + o(1))(d + 2)n/4 \leq dn$ as $d \geq 1$. Thus we have

$$\#F(H) - \#F(H^*) \geq (2d - d_B(w_1) - d_B(w_2)) \frac{n^2}{4} + \frac{3}{2} n(b_2 - b_1) - 26(|a| + 1)dn.$$

Recall that $d_B(w) \leq d + 2$ for any $w \in V$. Therefore $q \leq (d + 2)n/2 \overset{d \geq 1}{\leq} 3dn/2$. This together with Lemma 18 implies $b_2 \leq b_1 + n/4 + 33(|a| + 1)q/n \leq b_1 + n/4 + 50(|a| + 1)d$ and thus

$$\#F(H) - \#F(H^*) \geq (2d - d_B(w_1) - d_B(w_2)) \frac{n^2}{4} - \frac{3}{2} n \frac{n}{4} - 101(|a| + 1)dn. \tag{26}$$

We first consider the case when $900(|a| + 1)d/n < 1$. In this case any vertex with degree less than $d - 900(|a| + 1)d/n$ has degree at most $d - 1$. Therefore, $d_B(w_1), d_B(w_2) \leq d - 1$ and thus

$$\#F(H) - \#F(H^*) \overset{(26)}{\geq} 2 \frac{n^2}{4} - \frac{3n^2}{8} - 101(|a| + 1)dn > 0$$

where the last inequality follows because our conditions imply $101(|a| + 1)q < n^2/8$, resulting in a contradiction.

Next we consider the case when $900(|a| + 1)d/n \geq 1$. Recall that by our assumption $d_B(w_1)$, $d_B(w_2) < d - 900(|a| + 1)d/n$ and thus

$$\#F(H) - \#F(H^*) \overset{(26)}{\geq} 450(|a| + 1)dn - \frac{3n^2}{8} - 101(|a| + 1)dn > 0,$$

where the last inequality follows because our conditions imply $900(|a| + 1)dn \geq n^2$, leading to a contradiction.

Now we consider the remaining cases. An analogous proof works if $u, v \in V_2$. Should u and v be in different parts, by (12) we have for all $w \in V$ that $d_B(w) \geq d$ giving the required bound. Also the statement is trivial when $d + 1 = 1$, as the degree of any vertex is non-negative. Finally, if more than two vertices of degree $d + 2$ exist, then u and v can be replaced by two vertices of degree $d + 2$ and our earlier argument implies the result. $\square$

Proof of Lemma 20. Note that $dn \leq 2(q + 1) \leq (d + 1)n$. We start by showing that in the graph (V, B) at most one vertex of degree at least $d + 2$ exists. Suppose that this is false. Let W_{d+1} be the set of vertices $w \in V$ with $d_B(w) < d + 1$. By Theorem 1 we have $|a| \leq 1$, which together with Lemma 19, $d = o(n)$ and the fact that $d_B(w)$ is an integer imply that $|W_{d+1}| \leq 1$. In fact, by (12) we have that if there exists a vertex $w \in W_{d+1}$, then $d_B(w) \geq d$. Therefore,

$$2(q + 1) = \sum_{w \in V} d_B(w) \geq 2(d + 2) + d + (n - 3)(d + 1) > (d + 1)n,$$

contradicting our earlier observation.

Note that if there exists in (V, B) a vertex of degree at least $d + 3$, then by (12) there exists a pair of vertices $u, v \in V$ such that $d_B(u) = d_B(v) \geq d + 2$, a contradiction. Therefore, there is at most one vertex with degree larger than $d + 1$ in (V, B) and by (12) this vertex, if it exists, has degree exactly $d + 2$ in (V, B). It only remains to show that there is at most one vertex with degree less than d in this graph and should such a vertex exist it has degree $d - 1$.

Lemma 19 implies that this is in fact true if (V, B) contains two vertices of degree $d + 1$. Now assume that (V, B) has at most one vertex of degree $d + 1$. By (12) the presence of a vertex of degree $d + 2$ in (V, B) would imply that many vertices of degree $d + 1, d + 2$ or $d + 3$ should be present. However, by our assumption there is only one vertex of degree $d + 1$ in (V, B) and previously we have shown that there is at most one vertex of degree at least $d + 2$ in (V, B). Thus (V, B) contains no vertex with degree larger than $d + 1$. Let W_d be the set of vertices $w \in V$ with $d_B(w) < d$. Then

$$dn \leq 2(q + 1) = \sum_{w \in V} d_B(w) = d + 1 + (n - |W_d| - 1)d + \sum_{w \in W_d} d_B(w).$$

Rearranging the terms gives us

$$|W_d|d - 1 \leq \sum_{w \in W_d} d_B(w) \leq |W_d|(d - 1),$$

where the right hand inequality follows from the definition of W_d. The inequality holds only if $|W_d| \leq 1$ and for $w \in W_d$ we have $d_B(w) = d - 1$. $\square$

Proof of Lemma 21. Assume for contradiction that the size of each of these sets is at least two. Therefore there exist vertices $u_1, u_2 \in C_{k+1}, v_1, v_2 \in C_k, w_1, w_2 \in D_{\ell+1}, z_1, z_2 \in D_\ell$. Roughly speaking, we want to show that either moving an edge from C_{k+1} to D_ℓ decreases the number of bowties, or moving an edge from $D_{\ell+1}$ to C_k does.

Define $\phi_1 : V \to \mathbb{N}$ by

$$\phi_1(x) = \begin{cases} d_B(x) - 1, & \text{if } x = u_1, u_2, \\ d_B(x), & \text{if } v \in V \setminus \{u_1, u_2, z_1, z_2\}, \\ d_B(x) + 1, & \text{if } x = z_1, z_2, \end{cases}$$

and $\phi_2 : V \to \mathbb{N}$ by

$$\phi_2(x) = \begin{cases} d_B(x) - 1, & \text{if } x = w_1, w_2, \\ d_B(x), & \text{if } v \in V \setminus \{w_1, w_2, v_1, v_2\}, \\ d_B(x) + 1, & \text{if } x = v_1, v_2. \end{cases}$$

For $i = 1, 2$ let H_i be the graph returned by Lemma 16 with $\phi = \phi_i$, $V_1^* = V_1$ and $V_2^* = V_2$. Thus, by (13), we have

$$\begin{aligned} \#F(H_1) - \#F(H) \leq\ & 2\ell v_1(v_1 - 2) + (b_1 - 1)(2n - 8) + b_2 v_1 \\ & - 2k v_2(v_2 - 2) - b_2(2n - 8) - (b_1 - 1)v_2 \end{aligned}$$

and

$$\begin{aligned} \#F(H_2) - \#F(H) \leq\ & 2k v_2(v_2 - 2) + (b_2 - 1)(2n - 8) + b_1 v_2 \\ & - 2\ell v_1(v_1 - 2) - b_1(2n - 8) - (b_2 - 1)v_1. \end{aligned}$$

Should either one of these be negative, we are done, as we have a contradiction on the minimality of H. Clearly, this holds if the sum of the two terms is negative. Note that the $2\ell v_1(v_1 - 2)$ and the $2k v_2(v_2 - 2)$ terms cancel and thus

$$\begin{aligned} \#F(H_1) + \#F(H_2) - 2\#F(H) \leq\ & (b_1 - 1)(2n - 8) + b_2 v_1 - b_2(2n - 8) - (b_1 - 1)v_2 \\ & + (b_2 - 1)(2n - 8) + b_1 v_2 - b_1(2n - 8) - (b_2 - 1)v_1 \\ =\ & -(2n - 8) + v_1 - (2n - 8) + v_2 \\ =\ & -3n + 16 < 0 \end{aligned}$$

and the statement follows. □

6. Concluding remarks

After Proposition 3 is established (namely, that every extremal graph admits a vertex partition $\{V_1, V_2\}$ such that all cross edges are present), the rest essentially reduces to the problem of minimising the right-hand side of (13) as a function of e.g. $v_1 = |V_1|$ and $b_1 = |E(H(V_1))|$. Surprisingly, this integer optimisation problem turned out to be very delicate and we needed a lot of calculations to solve it (i.e. to derive Theorems 1–2 from Proposition 3). While our method may apply to other non-critical graphs F, we expect that similar algebraic difficulties will appear. One important difference between the cases of critical and non-critical F for $q = o(n^2)$ is that in the former case a single edge added to $K(V_1, V_2)$ already creates many copies of F, which often gives the dominant term for $h_F(n, q)$ and makes analysis easier.

One cannot expect that Theorem 1 holds for all q. For example, the Turán graph $T_r(n)$ is known to minimise the number of triangles among all graphs of the same order and size, which follows from the results by Moon and Moser [23] and, independently, Nordhaus and Stewart [26], which can be also derived from the paper of Goodman [12]. The corresponding stability result was obtained by Lovász and Simonovits [21]. Since this graph has approximately the same number of triangles per each vertex, the Cauchy–Schwarz Inequality implies that $T_r(n)$ also asymptotically minimises the number of bowties, including the stability result that all asymptotically optimal graphs are $o(n^2)$-close in the edit distance to $T_r(n)$. However, if $r \geq 3$ is a fixed odd integer, then $T_r(n)$ is $\Omega(n^2)$-away from containing $T_2(n)$. So Theorem 1 strongly fails for the corresponding value of q. It would be interesting to know how $h_F(n, q)$ behaves for larger q, in particular find the largest c such that $h_F(n, q) = (1 + o(1))t_F(n, q)$ holds for every $q \leq cn^2$.

References

[1] G. Cherlin, S. Shelah, N. Shi, Universal graphs with forbidden subgraphs and algebraic closure, Adv. Appl. Math. 22 (4) (1999) 454–491.

[2] D. Conlon, J. Fox, B. Sudakov, An approximate version of Sidorenko's conjecture, Geom. Func. Anal. 20 (2010) 1354–1366.

[3] P. Erdős, Some theorems on graphs, Riv. Lemat. 9 (1955) 13–17.

[4] P. Erdős, On the number of complete subgraphs contained in certain graphs, M. Tud. Akad. Mat. Kut. Intez. Közl. 7 (1962) 459–464.

[5] P. Erdős, Some recent results on extremal problems in graph theory. Results, in: Theory of Graphs (Internat. Sympos., Rome, 1966), Gordon and Breach, New York; Dunod, Paris, 1967, pp. 117–123, (English); pp. 124–130 (French).

[6] P. Erdős, Z. Füredi, R. Gould, D. Gunderson, Extremal graphs for intersecting triangles, Journal of Combinatorial Theory, Series B 64 (1995) 89–100.

[7] P. Erdős, M. Simonovits, Supersaturated graphs and hypergraphs, Combinatorica 3 (1983) 181–192.

[8] D. Evans, J. Hubička, J. Nešetřil, Ramsey properties and extending partial automorphisms for classes of finite structures, 2017, arXiv:1705.02379.

[9] D.C. Fisher, Lower bounds on the number of triangles in a graph, J. Graph Theory 13 (4) (1989) 505–512.

[10] D. Gale, A theorem on flows in networks, Pacific J. Math. 7 (1957) 1073–1082.

[11] R. Glebov, Extremal graphs for clique-paths, 2011, arXiv:1111.7029.

[12] A.W. Goodman, On sets of acquaintances and strangers at any party, Amer. Math. Monthly 66 (1959) 778–783.

[13] H. Hatami, Graph norms and Sidorenko's conjecture, Israel J. Math. 175 (2010) 125–150.

[14] J. Hubička, J. Nešetřil, Bowtie-free graphs have a Ramsey lift, Adv. Appl. Math. 96 (2018) 286–311.

[15] J.H. Kim, C. Lee, J. Lee, Two approaches to Sidorenko's conjecture, Trans. Amer. Math. Soc. 368 (2016) 5057–5074.

[16] P. Komjáth, Some remarks on universal graphs, Discrete Math. 199 (1–3) (1999) 259–265.

[17] J. Komlós, M. Simonovits, Szemerédi's regularity lemma and its applications in graph theory, in: Combinatorics, Paul Erdős is Eighty, Vol. 2 (Keszthely, 1993), in: Bolyai Soc. Math. Stud., vol. 2, János Bolyai Math. Soc., Budapest, 1996, pp. 295–352.

[18] J.L.X. Li, B. Szegedy, On the logarithmic calculus and Sidorenko's conjecture, 2011, arXiv:1107.1153.

[19] H. Liu, Extremal graphs for blow-ups of cycles and trees, Electron. J. Combin. 20 (2013) Paper 65, 16.

[20] L. Lovász, M. Simonovits, On the number of complete subgraphs of a graph, in: Proceedings of the Fifth British Combinatorial Conference (Univ. Aberdeen, Aberdeen, 1975), in: Congressus Numerantium, No. XV, Utilitas Math., Winnipeg, Man., 1976, pp. 431–441.

[21] L. Lovász, M. Simonovits, On the number of complete subgraphs of a graph. II, in: Studies in Pure Mathematics, Birkhäuser, Basel, 1983, pp. 459–495.

[22] W. Mantel, Problem 28, Wiskd. Opg. 10 (1907) 60–61.

[23] J.W. Moon, L. Moser, On a problem of Turán, Publ. Math. Inst. Hung. Acad. Sci. 7 (1962) 283–287.

[24] D. Mubayi, Counting substructures I: color critical graphs, Adv. Math. 225 (5) (2010) 2731–2740.

[25] V. Nikiforov, The number of cliques in graphs of given order and size, Trans. Amer. Math. Soc. 363 (3) (2011) 1599–1618.

[26] E.A. Nordhaus, B.M. Stewart, Triangles in an ordinary graph, Canad. J. Math. 15 (1963) 33–41.

[27] O. Pikhurko, Z. Yilma, Supersaturation problem for color-critical graphs, J. Combin. Theory Ser. B 123 (2017) 148–185.

[28] A. Razborov, On the minimal density of triangles in graphs, Combin. Probab. Comput. 17 (4) (2008) 603–618.

[29] C. Reiher, The clique density theorem, Ann. of Math. (2) 184 (3) (2016) 683–707.

[30] H.J. Ryser, Combinatorial properties of matrices of zeros and ones, Canad. J. Math. 9 (1957) 371–377.

[31] A.F. Sidorenko, A correlation inequality for bipartite graphs, Graphs Combin. 9 (1993) 201–204.

[32] M. Simonovits, A method for solving extremal problems in graph theory, stability problems, in: Theory of Graphs (Proc. Colloq., Tihany, 1966), Academic Press, New York, 1968, pp. 279–319.

[33] M. Simonovits, Extremal graph problems, degenerate extremal problems and super-saturated graphs, in: Progress in Graph Theory (Waterloo, Ont., 1982), Academic Press, Toronto, 1984, pp. 419–437.

[34] B. Szegedy, An information theoretic approach to Sidorenko's conjecture, 2014, arXiv:1406.6738v3.

[35] P. Turán, On an extremal problem in graph theory (in Hungarian), Mat. Fiz. Lapok 48 (1941) 436–452.