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Supersaturation Problem for the Bowtie

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Abstract

The Turán function $ex(n, F)$ denotes the maximal number of edges in an F-free graph on n vertices. However if $e > ex(n, F)$, many copies of F appear. We study the function $h_F(n, q)$, the minimal number of copies of F in a graph on n vertices with $ex(n, F) + q$ edges. The value of $h_F(n, q)$ has been extensively studied when F is colour critical. In this paper we consider a simple non-colour-critical graph, namely the bowtie and establish bounds on $h_F(n, q)$ for different ranges of q.

Keywords: Extremal graph theory, Removal lemma, Supersaturation, Turán function.

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1 Introduction

The *Turán function* $ex(n, F)$ of a graph F is the maximum number of edges in an F-free graph on n vertices. In 1907, Mantel [9] proved that $ex(n, K_3) = \lfloor n^2/4 \rfloor$, where K_r denotes the complete graph on r vertices. The fundamental paper of Turán [16] solved this extremal problem for cliques: the *Turán graph* $T_r(n)$, the complete r-partite graph on n vertices with parts of size $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$, is the unique maximum K_{r+1}-free graph on n vertices. Thus the Turán function satisfies $ex(n, K_{r+1}) = |E(T_r(n))|$.

Stated in the contrapositive, this implies that a graph with $ex(n, K_{r+1})+1$ edges (where, by default, n denotes the number of vertices) contains at least one copy of K_{r+1}. Rademacher (1941, unpublished) showed that a graph with $\lfloor n^2/4 \rfloor + 1$ edges contains not just one but at least $\lfloor n/2 \rfloor$ copies of a triangle. This is perhaps the first result in the so-called "theory of supersaturated graphs" that focuses on the function

$$h_F(n, q) = \min\{\#F(H) : |V(H)| = n, |E(H)| = ex(n, F) + q\},$$

the minimum number of F-subgraphs in a graph on n vertices and $ex(n, F)+q$ edges. (We say that G is a *subgraph* of H if $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$; we call G an *F-subgraph* if it is isomorphic to F.)

One possible construction for graphs with minimal number of copies of F is to add some q edges to a maximum F-free graph. Denote by $t_F(n, q)$ the smallest number of F-subgraphs that can be achieved this way. Clearly, $h_F(n, q) \le t_F(n, q)$. In fact, this bound is sharp, when q is small. Erdős [1] extended Rademacher's result by showing that $h_{K_3}(n, q) = t_{K_3}(n, q) = q\lfloor n/2 \rfloor$ for $q \le 3$. Later, he [2,3] showed that there exists some small constant $\varepsilon_{K_r} > 0$ such that $h_{K_r}(n, q) = t_{K_r}(n, q)$ for all $q \le \varepsilon_{K_r} n$. Lovász and Simonovits [7,8] found the best possible value of ε_{K_r} as $n \to \infty$, settling a long-standing conjecture of Erdős [1]. In fact, the second paper [8] completely solved the $h_{K_r}(n, q)$-problem when $q = o(n^2)$. The case $q = \Omega(n^2)$ of the supersaturation problem for cliques has been actively studied and proved notoriously difficult. Only recently was an asymptotic solution found: by Razborov [13] for K_3 (see also Fisher [5]), by Nikiforov [11] for K_4, and by Reiher [14] for general K_r.

The value of $h_F(n, q)$ has also been considered for general colour-critical graphs. A graph is r-critical if its chromatic number is $r + 1$ and in addition removing a certain edge from the graph reduces its chromatic number. Simonovits [15] established that if F is r-critical, then the unique maximal F-free graph is $T_r(n)$. Pikhurko and Yilma [12] extending the results of Mubayi [10] established that, similarly to cliques, for every colour-critical graph F there exists $\varepsilon_F > 0$ such that when $q \leq \varepsilon_F n$, we have $h_F(n,q) = t_F(n,q)$. In addition, they established the asymptotic size of $h_F(n,q)$ when $q = o(n^2)$.

Obviously, if we do not know the exact value of $ex(n,F)$, then it is difficult to say much about the supersaturation problem. In this paper we investigate one of the remaining graphs for which this is known, namely the bowtie (two copies of K_3 joined at a vertex). From this point on F denotes the bowtie.

The main contribution of this paper is threefold. First we establish that when $q = o(n^2)$ any graph with minimal number of copies of F contains a spanning complete bipartite graph. Second we establish the exact number of bowties contained in the graph when $q \leq n/4 - 1$. Finally we establish the asymptotic size of $h_F(n,q)$ when $q = \omega(n)$ and $q = o(n^2)$.

2 Main results

The Turán function of the bowtie is $ex(n,F) = \lfloor n^2/4 \rfloor + 1$. In addition, the maximal F-free graph is known, it is a copy of $T_2(n)$ with an arbitrarily added edge. Recall that $T_2(n)$ is a bipartite graph, where the partitions have size $\lceil n/2 \rceil$ and $\lfloor n/2 \rfloor$ vertices respectively. Thus if n is odd, the two partitions have different sizes and thus, depending on which partition the additional edge is added to, distinct graphs are created.

Due to the existence of multiple maximal F-free graphs we define $t_F(n,q)$ as the minimal number of copies of F contained in the graph created by adding q edges to any maximal F-free graph.

We first show that as long as $q = o(n^2)$ any graph with minimal number of copies of F contains a spanning complete bipartite graph.

Theorem 2.1 *Let H be a graph on n vertices with $ex(n,F) + q$ edges containing $h_F(n,q)$ copies of the bowtie. If $q = o(n^2)$, then the vertex set of H can be partitioned into two parts V_1 and V_2 such that $|V_1|, |V_2| = (1+o(1))n/2$ and every edge between V_1 and V_2 is present.*

In fact, when $q < n/4 - 1$, a stronger result is true, namely $|V_1| = \lceil n/2 \rceil$ and $|V_2| = \lfloor n/2 \rfloor$. Based on this we can establish the exact value of $h_F(n,q)$.

Theorem 2.2 *Let $q \leq n/4 - 1$. Then we have $h_F(n,q) = t_F(n,q)$ and*

$$h_F(n,q) = \binom{q+1}{2} \lfloor n/2 \rfloor.$$

For large q we establish the asymptomatics of $h_F(n,q)$.

Theorem 2.3 *Let $q = \omega(n)$ and $q = o(n^2)$. Then we have*

$$h_F(n, q) = (1 + o(1))\frac{9}{8}q^2 n.$$

3 Proof outline

We start by showing an upper bound on $h_F(n, q)$. This can be achieved by considering an arbitrary graph on n vertices with $ex(n, F) + q$ edges. This not only gives us the upper bounds in Theorem 2.2 and Theorem 2.3, but also allows us to use the graph removal lemma (see e.g. [6, Theorem 2.9]).

The graph removal lemma implies that since the number of bowties in the graph is small, the graph can be made bowtie-free by removing a small number of edges. In addition, since the chromatic number of the bowties is 3, the stability result of Erdős [4] and Simonovits [15] implies that the vertex set of a bowtie-free graph can be partitioned into two sets such that almost $|E(T_2(n))|$ edges are present between the two partitions. Denote these partitions with V_1 and V_2 and note that both V_1 and V_2 contain roughly $n/2$ vertices.

The key step is to establish that every edge between V_1 and V_2 is present. We show that for every graph, where at least one edge e between V_1 and V_2 is missing, we can find another graph, where in addition to the previous edges between V_1 and V_2, e is also present and the second graph has fewer bowties.

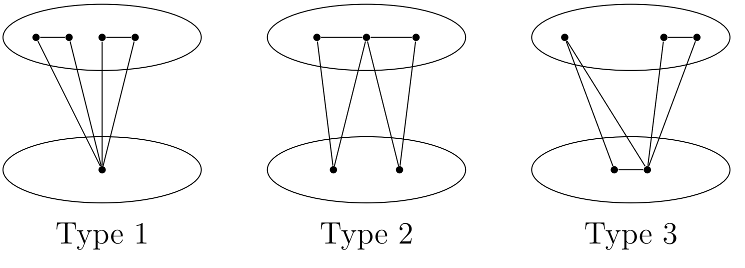

Fig. 1. Types of bowties

Once we know the existence of such a partition we can express the number of bowties. Ignore the triangles spanned by V_1 and V_2 for the moment, later we will see that no such triangles exist in the extremal graph. A bowtie can be formed in three different ways (see Figure 1). Since we are ignoring triangles spanned by V_1 and V_2, any triangle in the graph contains exactly one edge spanned by V_1 or V_2. As a bowtie consists of two triangles it has to have exactly two edges spanned by V_1 or V_2 and we distinguish the following cases, depending on whether both edges are spanned by the same set or not, in addition if they are spanned by the same set we consider the cases when the two edges are adjacent or non-adjacent.

Recall that every edge between V_1 and V_2 is present. Therefore any pair of disjoint edges in V_1 is contained in $|V_2|$ bowties. If the two edges are adjacent, then they are contained in $|V_2|(|V_2|-1)$ bowties. Finally any two edges, where one edge is spanned by V_1 and the other edge is spanned by V_2, is contained in $2(n-4)$ bowties.

The previous argument implies that the number of bowties depends only on the degree sequence of the graphs spanned by V_1 and V_2. In addition, due to the small number of edges spanned by V_1 and V_2 one can rearrange the edges in such a way that the degree sequence remains unchanged, but V_1 and V_2 are triangle-free. Destroying these triangles can only decrease the number of bowties, justifying our earlier decision to ignore them.

All that is left to show for Theorem 2.1 is that $|V_1|, |V_2| = (1+o(1))n/2$. Note that if $|V_1| = \lceil n/2 \rceil$ and $|V_2| = \lfloor n/2 \rfloor$, the number of edges spanned by V_1 and V_2 is exactly $q+1$. If we increase the number of vertices in V_1 by a and, in order to leave the total number of vertices unchanged, at the same time we decrease the number of vertices in V_2 by the same amount, then the number of edges spanned by V_1 and V_2 increases by at least a^2. However, for large a, due to these additional a^2 edges, the number of bowties in the graph increases significantly implying $a^2 = O(q) = o(n^2)$ and thus our result.

Now we consider the lower bound in Theorem 2.2. Since q is small, a more precise analysis in determining the value of a is possible, and in this case the optimal solution is $a = 0$. From the different types of edge pairs (see Figure 1) it turns out that type 1 creates the least number of bowties. When $q < n/4-1$ we can avoid type 2 and 3 bowties, as every edge can be placed into V_1 in such a way that it forms a matching. Theorem 2.2 follows once one counts the number of bowties.

Finally we establish the lower bound in Theorem 2.3. Due to the large number of bowties created by any pair of adjacent edges, we want to minimise the number of adjacent pairs. Let e_1 denote the number of edges spanned by V_1 and e_2 denote the number of edges spanned by V_2. Note that by removing some edges we can achieve that $e_1+e_2 = q+1$ and that this only decreases the number of bowties. For fixed e_1 and e_2 the number of adjacent edge pairs is minimised, if every vertex in V_1 is adjacent to roughly $2e_1/n$ edges and every vertex in V_2 is adjacent to roughly $2e_2/n$ edges, i.e. the vertex degrees can be defined as a function of e_1 and e_2. Since we fixed the sum of e_1 and e_2 as $q+1$, the number of bowties present in the graph can be approximated by a function of e_1, which after optimisation leads to the asymptotic result.

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