Anti-Ramsey Numbers of Doubly Edge-Critical Graphs

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Abstract: Given a graph H and a positive integer n, *Anti-Ramsey number* $\mathrm{AR}(n,H)$ is the maximum number of colors in an edge-coloring of K_n that contains no polychromatic copy of H. The anti-Ramsey numbers were introduced in the 1970s by Erdős, Simonovits, and Sós, who among other things, determined this function for cliques. In general, few exact values of $\mathrm{AR}(n,H)$ are known. Let us call a graph H *doubly edge-critical* if $\chi(H-e)\geq p+1$ for each edge $e\in E(H)$ and there exist two edges e_1, e_2 of H for which $\chi(H-e_1-e_2)=p$. Here, we obtain the exact value of $\mathrm{AR}(n,H)$ for any doubly

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edge-critical H when $n \geq n_0(H)$ is sufficiently large. A main ingredient of our proof is the stability theorem of Erdős and Simonovits for the Turán problem. J Graph Theory 61: 210–218, 2009

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1. INTRODUCTION

We will use the standard notation of graph theory, which can be found e.g. in Bollobás' book [3].

A subgraph of an edge-colored graph is *rainbow* (or *polychromatic*) if all of its edges have different colors. Let $\mathcal{F}$ be a family of graphs. For the purpose of this paper, we call an edge-coloring that contains no rainbow copy of any graph in $\mathcal{F}$ an *$\mathcal{F}$-free coloring*. The *anti-Ramsey number* $\mathrm{AR}(n,\mathcal{F})$ is the maximum number of colors in an $\mathcal{F}$-free edge-coloring of K_n, the complete graph on n vertices. Anti-Ramsey numbers were introduced by Erdős et al. [8]. Various results about this extremal function have been obtained since then: [1, 10, 14, 11, 2, 12, 17, 15, 13, 16, 4] to name a few.

Erdős et al. [8] showed that these numbers are very closely related to Turán numbers as follows.

Given a family $\mathcal{F}$ of graphs, let us call a graph G containing no graph in $\mathcal{F}$ as a subgraph an *$\mathcal{F}$-free* graph. Let $\mathrm{EX}(n,\mathcal{F})$ denote the set of $\mathcal{F}$-free graphs on n vertices. (When $\mathcal{F}$ consists of a single graph F, we will write $\mathrm{EX}(n,F)$ instead of $\mathrm{EX}(n,\{F\})$, etc.) The *Turán number* $\mathrm{ex}(n,\mathcal{F})$ is the maximum number of edges of a graph in $\mathrm{EX}(n,\mathcal{F})$. Recall that the *Turán graph* $T_{n,p}$ is a complete p-partite graph on n vertices in which each part has size $\lfloor n/p \rfloor$ or $\lceil n/p \rceil$. Let $t(n,p)=e(T_{n,p})$. Turán [21] showed that $\mathrm{ex}(n,K_{p+1})=t(n,p)$ and that $T_{n,p}$ is the unique maximum graph in $\mathrm{EX}(n,K_{p+1})$. More generally, the celebrated result of Erdős et al. [9, 7] states that for any $\mathcal{F}$ we have

$$\mathrm{ex}(n,\mathcal{F})=t(n,p)+o(n^2)=\left(1-\frac{1}{p}\right)\binom{n}{2}+o(n^2), \tag{1}$$

where $p=\Psi(\mathcal{F})$ is the *subchromatic number*:

$$\Psi(\mathcal{F})=\min\{\chi(F):F\in\mathcal{F}\}-1. \tag{2}$$

Let us give a few other definitions needed to state the results. The *derived family* of $\mathcal{F}$ is

$$\mathcal{F}^-=\{H-e:H\in\mathcal{F},\ e\in E(H)\}.$$

Let xy be an edge in a graph H. Let H_1,H_2 be two copies of $H-xy$. For $i=1,2$, let x_i denote the image of x in H_i and y_i the image of y in H_i. Let H^*_{xy} denote the graph obtained by taking H_1 and H_2 and identifying x_1 with x_2 and y_1 with y_2.

Theorem 1 (Erdős et al. [8]). *Let $\mathcal{F}$ be an arbitrary graph family. Let $H\in\mathcal{F}$ and $xy\in E(H)$ be such that $\chi(H-xy)=\min\{\chi(F):F\in\mathcal{F}^-\}$. Then*

$$\mathrm{ex}(n,\mathcal{F}^-)+1\leq \mathrm{AR}(n,\mathcal{F})\leq \mathrm{ex}(n,H^*_{xy}).$$

It follows from (1) that $\mathrm{AR}(n,\mathcal{F})=t(n,p)+o(n^2)$, where $p=\Psi(\mathcal{F}^-)$.

Proof. We recall the proof because we will need it later.

Let G be a graph in $\mathrm{EX}(n,\mathcal{F}^-)$ of maximum size. Assign distinct colors to all of $E(G)$ and a new color to all of $E(\overline{G})$. This yields an $\mathcal{F}$-free coloring of $E(K_n)$. So $\mathrm{AR}(n,\mathcal{F})\geq \mathrm{ex}(n,\mathcal{F}^-)+1$.

Next, suppose that c is an edge-coloring of K_n that uses more than $\mathrm{ex}(n,H^*_{xy})$ colors. Let L be a *representing graph* of c, that is, it is a spanning subgraph of K_n obtained by taking one edge of each color in c (where L may contain isolated vertices). Then L must contain a copy F of H^*_{xy}. This is a rainbow copy of H^*_{xy} in c. Let H_1 and H_2 denote the two copies of $H-xy$ in F as in the definition of H^*_{xy}. Note that H_1 and H_2 are edge-disjoint and since F is rainbow, no color appears in both H_1 and H_2. Suppose without loss of generality that color $c(xy)$ does not appear in H_1. Now $H_1\cup xy$ is a rainbow copy of H. This shows that $\mathrm{AR}(n,\mathcal{F})\leq \mathrm{ex}(n,H^*_{xy})$. ■

Theorem 2 (Erdős et al. [8]). *For any $p\geq 2$ and all sufficiently large n, $n\geq n_0(p)$, we have*

$$\mathrm{AR}(n,K_{p+2})=t(n,p)+1, \tag{3}$$

and any coloring achieving this bound is obtained by taking a rainbow $T_{n,p}$ and coloring all edges in its complement with the same (*extra*) *color.*

Montellano-Ballesteros and Neumann-Lara [14] and independently Schiermeyer [17] showed that, in fact, (3) holds for every $n\geq p+1$ but did not characterize extremal colorings.

Let p be a positive integer. For $0\leq i\leq p$ and $n\geq p+i$, let $T^i_{n,p}$ be obtained from the Turán graph by adding a single edge into each of the i largest vertex parts. (Thus, $e(T^i_{n,p})=t(n,p)+i$.)

We say that a graph family $\mathcal{F}$ is *doubly edge-p-critical* if $\Psi(\mathcal{F}^-)\geq p$ (i.e. $\chi(H)\geq p+1$ for every $H\in\mathcal{F}^-$) but there exists a graph $H\in\mathcal{F}$ and two edges $e_1,e_2\in E(H)$ such that $\chi(H-e_1-e_2)=p$. In other words, we require that $T^1_{n,p}$ is $\mathcal{F}$-free for every n but we can add two edges to some $T_{n,p}$ (into two different parts or into the same part) and obtain an element of $\mathcal{F}$ as a subgraph. Furthermore, we subdivide all such families into the following *types*: a doubly edge-p-critical $\mathcal{F}$ has *Type* j, where $j\geq 1$ is the maximum element of $[p]$ such that $T^j_{n,p}$ is $\mathcal{F}$-free for every n. This definition implies that if $\mathcal{F}$ is of Type j, then, for every $n\geq p+j$, we have

$$\mathrm{AR}(n,\mathcal{F})\geq t(n,p)+j. \tag{4}$$

Indeed, the following coloring of K_n is $\mathcal{F}$-free and uses $t(n,p)+j$ colors.

Construction 3. *Take a copy H of $T_{n,p}$, color its edges with distinct colors and then color its complement $\overline{H}$ with j new colors so that each of the p parts of H is monochromatic and such that each of these j new colors is used on at least one part.*

The main result of our paper shows that this is best possible:

Theorem 4. *Let $p\geq 2$. For an arbitrary* (*possibly infinite*) *doubly edge-p-critical family $\mathcal{F}$ of Type j there is an n_0 such that for all $n\geq n_0$ we have*

$$\mathrm{AR}(n,\mathcal{F})=t(n,p)+j, \tag{5}$$

and all $\mathcal{F}$-free edge-colorings of K_n that achieve this bound are described in Construction 3.

This theorem generalizes Theorem 2 as K_{p+2} (more precisely, $\{K_{p+2}\}$) is doubly edge-p-critical of Type 1 for $p \geq 2$. One can easily generate many examples of doubly edge-critical graphs. One is $T^2_{n,p}$. Further examples can be obtained by taking the disjoint union of two edge-p-critical graphs. (A graph H is *edge-p-critical* if for some edge $e \in E(H)$ we have $\chi(H-e)=p<\chi(H)$.) Also, one can start with an arbitrary doubly edge-p-critical graph H and add a new set A of at least 2 vertices so that A spans no edge while we have a complete bipartite graph between $V(H)$ and A. Some variations of the last construction are possible: for example, one can let A span certain graphs F although not every F would do here (e.g. $F=K_1$, that is, $|A|=1$, does not work for some H).

If $\mathcal{F}$ is doubly edge-p-critical, then we can choose a graph $H \in \mathcal{F}^-$ in (3) which is edge-p-critical. An important result of Simonovits [18] states that $\mathrm{ex}(n,H)=t(n,p)$ for all large n. This was one of our motivations: to prove the analog of Simonovits' result to anti-Ramsey numbers. Our result is a bit more complicated to prove (and to state) but the main idea is the same: we obtain the exact result using the stability approach. The proof can be found in Section 3.

An unpublished conjecture of the first author states that $\mathrm{AR}(n,\mathcal{F}) \leq \mathrm{ex}(n,\mathcal{F}^-)+O(n)$ for any family $\mathcal{F}$ (see [10] for some discussions). Our Theorem 4 confirms the conjecture in the case when, for some $p \geq 2$,

$$\mathrm{ex}(n,\mathcal{F}^-)=t(n,p) \quad \text{for all } n \geq n_0(p,\mathcal{F}). \tag{6}$$

Indeed, any $\mathcal{F}^-$ satisfying (6) has subchromatic number p while some $T^1_{N,p}$ is not $\mathcal{F}^-$-free (for otherwise $\mathrm{ex}(n,\mathcal{F}^-) \geq t(n,p)+1$ for all n). This shows that $\mathcal{F}$ is doubly edge-p-critical. It is possible that our method might apply in some other cases when $\mathrm{ex}(n,\mathcal{F}^-)$ can be determined via the stability approach. However, it is not clear how to proceed in general.

2. PRELIMINARIES

Here, we present some results and definitions that are needed for our proof of Theorem 4 in Section 3.

For a set X and an integer k, let $\binom{X}{k}=\{A \subseteq X : |A|=k\}$. Given a graph G and disjoint subsets $A_1,\ldots,A_p$ of vertices in G, we use $G(A_1,\ldots,A_p)$ to denote the p-partite subgraph of G with parts $A_1,\ldots,A_p$ containing all the edges of G connecting different A_i's. Let $K(A_1,\ldots,A_p)$ denote the complete p-partite graph with parts $A_1,\ldots,A_p$. If $S \subseteq V(G)$, then $G[S]$ denotes the subgraph of G induced by S.

Lemma 5. *Let $N,m,p \geq 2$ and $s \geq 0$ be integers. Let G be a p-partite graph with parts $A_1 \cup S, A_2,\ldots,A_p$ where $|S|=s$, $|A_i|=N$ for $i=1,\ldots,p$, and $S,A_1,\ldots,A_p$ are pairwise disjoint. Suppose that there is no selection of subsets $D_1 \subseteq A_1, D_2 \subseteq A_2,\ldots,D_p \subseteq A_p$ with $|D_i|=m$ for each i such that $G[D_1 \cup S, D_2,\ldots,D_p]$ is a complete*

p-partite graph. Let $K = K(A_1 \cup S, A_2, \ldots, A_p)$. Let $E^*(S)$ denote the set of edges of K incident to S. Then

1. $|E(K) \setminus E(G)| \geq \frac{1}{2}(\frac{N}{m})^2$, *or*
2. $|(E(K) \setminus E(G)) \cap E^*(S)| \geq \frac{1}{2}(\frac{N}{m})$.

Proof. There are $M_1 = \binom{N}{m}^p$ ways to select the D_i's. By our assumption, for each selection, the p-partite graph $G' = G[D_1 \cup S \cup D_2 \cup \cdots \cup D_p]$ misses some edge of K whose two endpoints are in $D_1 \cup S \cup D_2 \cup \cdots \cup D_p$. Suppose first that for at least $\frac{1}{2}M_1$ of these selections, G' misses an edge not incident to S. For each edge $xy \in (E(K) \setminus E(G)) \setminus E^*(S)$, the number ways to select the D_i's such that $x, y \in D_1 \cup \cdots \cup D_p$ is $M_2 = \binom{N}{m}^{p-2} \binom{N-1}{m-1}^2$. So, we have

$$|E(K) \setminus E(G)| \geq |(E(K) \setminus E(G)) \setminus E^*(S)| \geq \frac{(1/2)M_1}{M_2} = \frac{1}{2}\left(\frac{N}{m}\right)^2.$$

For the rest of the proof, suppose that for at least $\frac{1}{2}M$ of the ways to select the D_i's, G' misses an edge of $E(K) \setminus E(G)$ that is incident to S. Fixing an edge $xy \in (E(K) \setminus E(G)) \cap E^*(S)$, where $x \in S$, the number of ways to select the D_i's such that $y \in D_2 \cup \cdots \cup D_p$ is $M_3 = \binom{N}{m}^{p-1} \binom{N-1}{m-1}$. This implies that

$$|(E(K) \setminus E(G)) \cap E^*(S)| \geq \frac{(1/2)M_1}{M_3} = \frac{1}{2}\left(\frac{N}{m}\right). \quad \blacksquare$$

Also, we will need the following very useful result.

Theorem 6 (The Stability Theorem, Erdős [5, 6], Simonovits [18]). *Let $\mathcal{F}$ be a family of graphs with subchromatic number p. For every $\varepsilon > 0$, there exist $\delta > 0$ and n_ε such that for $n > n_\varepsilon$ if G is a graph in* $\mathrm{EX}(n, \mathcal{F})$ *with*

$$e(G) > \mathrm{ex}(n, \mathcal{F}) - \delta n^2,$$

then G can be obtained from $T_{n,p}$ by changing at most εn^2 edges.

3. PROOF OF THEOREM 4

Let $\mathcal{F}$ be a doubly edge-p-critical family of graphs. We have already demonstrated the lower bound in (5). Let us prove the upper bound and characterize the extremal colorings.

Fix a large integer m, a graph $H \in \mathcal{F}$, and edges $e_1, e_2 \in E(H)$ with $H - e_1 - e_2 \subseteq T_{pm,p}$. Let $\mathcal{F}$ be of Type j. If $j < p$, then (enlarging m if necessary) fix $F \in \mathcal{F}$ such that $F \subseteq T_{pm,p}^{j+1}$. If $j = p$, then we can let e.g. $F = H$.

Given F, H, and m, choose, in this order, small positive constants

$$\frac{1}{m} \gg \varepsilon \gg \delta \gg \frac{1}{n_0},$$

where $\alpha \gg \beta$ means that β is sufficiently small depending on α. Let $n \geq n_0$ be arbitrary. We do not specify the dependencies between constants explicitly, since we use Theorem 6 as a "black box." In particular, we make no attempt to optimize the constants.

Let c be an extremal $\mathcal{F}$-free edge-coloring of the host graph K_n on the vertex-set V. Let L be a representing graph of c, that is, it is obtained by picking one edge from each color class. By the extremality of c we have

$$e(L) \geq t(n,p) + j. \tag{7}$$

By the proof of Theorem 1, L cannot contain H^*_{xy}. Since $e(L) > t(n,p) > \mathrm{ex}(n, H^*_{xy}) - \delta n^2$, by Theorem 6 L can be obtained from $T_{n,p}$ by changing at most εn^2 edges.

Among all complete p-partite graphs with vertex-set V, choose T so that $|E(T) \cap E(L)|$ is maximum. Let the graph T' on V have $E(T) \cap E(L)$ as the edge-set. Our choice of T implies that

$$e(T') \geq t(n,p) - \varepsilon n^2. \tag{8}$$

Since $e(T) \leq t(n,p)$, we have

$$|E(T) \setminus E(L)| \leq \varepsilon n^2. \tag{9}$$

Let $A_1, A_2, \ldots, A_p$ denote the parts of T. We have, for example, $|A_i| \geq n/2p$ for all $i \in [p]$, for otherwise it is easy to see that the imbalance of the $|A_i|$'s forces $e(T) < t(n,p) - \varepsilon n^2$, contradicting (8).

Suppose first that for each $i = 1, \ldots, p$, only one color is used inside A_i in the coloring c.

We apply Lemma 5 with $S = \emptyset$ to T' (or rather to an arbitrary equipartite subgraph of T' with each part having size $N = \lfloor n/2p \rfloor$). By (9), this gives a complete p-partite subgraph $K(D_1, \ldots, D_p) \subseteq T'$ with $|D_i| \geq m + p$ for each $i \in [p]$. (Note that we apply Lemma 5 with respect to $m + p$ rather than m.) Let c_i be the common color of $\binom{D_i}{2}$. Suppose there are exactly $r \leq p$ different elements in $\{c_1, \ldots, c_p\}$. By removing p vertices from each D_i, we can find a new complete p-partite graph $K' = T'[D_1 \cup \cdots \cup D_p]$ such that $c(E(K')) \cap \{c_1, \ldots, c_p\} = \emptyset$. It follows that $r \leq j$ for otherwise $j < p$ and G contains a rainbow $T^{j+1}_{mp,p}$, and thus a rainbow F, a contradiction.

Now, (7) implies that $r = j$, $e(T) = t(n,p)$ (and so $T \cong T_{n,p}$), and the coloring c is indeed described by Construction 3.

So, let us assume that some part A_i has at least 2 different colors. We will derive a contradiction. The following observation will be useful in our quest.

Claim 1. *There is no complete p-partite graph $K(D_1, \ldots, D_p) \subseteq T'$ such that $|D_i| \geq m + 7$ for each $i \in [p]$ and $c(uv) \neq c(xy)$ for some uv, xy in one part D_i.*

Proof of Claim. Suppose that the claim is false. Without loss of generality, $u, v, x, y \in D_1$. We can assume that the pairs uv and xy are disjoint. (Otherwise pick a third pair disjoint from both uv and xy; its color is different from at least one of $c(uv)$ or $c(xy)$.)

By removing 6 vertices from each D_i, we can make sure that no color present in $\binom{\{u,v,x,y\}}{2}$ appears in the new graph $K' = K(D_1, \ldots, D_p)$ (while u, v, x, y are still in D_1). The graph $H \in \mathcal{F}$ can be embedded into $T_{mp,p}+f_1+f_2$, where f_1, f_2 are two extra edges.

Suppose first that f_1, f_2 are in one part of the Turán graph. If they are disjoint, then we can map f_1, f_2 to uv, xy and extend this mapping to an arbitrary embedding $T_{mp,p} \subseteq K'$. This gives a rainbow copy of H, a contradiction. If $f_1 \cap f_2 \neq \emptyset$, then we can map f_1 to ux and f_2 to one of uv, xy, namely to a pair whose color is different from $c(ux)$. Again this gives us a rainbow H because we took care to eliminate any color in $\binom{\{u,v,x,y\}}{2}$ from $E(K')$.

So let us assume that f_1, f_2 belong to different parts of $T_{mp,p}$. Let us map f_2 to an arbitrary pair wz in D_2. If $c(wz) \in \{c(uv), c(xy)\}$, then map f_1 to one of uv, xy, namely to a pair whose color is different from $c(wz)$. Note that the c-colors of the images of f_1 and f_2 do not appear on $E(K')$, so we can extend this to a rainbow copy of H. If $c(wz) \notin \{c(uv), c(xy)\}$, then at most one edge between wz and $\{u, v, x, y\}$ can have c-color $c(wz)$, and we can safely map f_1 either to uv or xy. At most one edge f of L can have color $c(wz)$. Since we still have some freedom (each current $|D_i| \geq m+1$) we can find a copy of H avoiding f. This H-subgraph is rainbow, contradicting our assumptions. This finishes the proof of the claim. ■

Call the edges in $E(T) \setminus E(L)$ *missing* and the edges in $E(L) \setminus E(T)$ *bad*. Let M (resp. B) be the graph on V consisting of all missing (resp. bad) edges. Since $e(L) \geq e(T)$, we have

$$e(B) \geq e(M). \tag{10}$$

Let $\gamma = 1/(25pm)$ and $W = \{x : d_M(x) \geq \gamma n\}$.

Claim 2. *If uv, xy are any two edges (possibly intersecting) with different colors inside some A_i, then $W \cap \{u, v, x, y\} \neq \emptyset$.*

Proof of Claim. For notational convenience, suppose $i=1$. Let $S = \{u, v, x, y\}$. Let C_1 be a subset of $A_1 \setminus S$ of size $N = \lfloor n/3p \rfloor$; C_1 exists since $|A_1| \geq n/2p$. For $i = 2, \ldots, p$, let C_i be a subset of A_i of size N.

By Claim 1, we cannot find $(m+7)$-sets $D_i \subseteq C_i$, $i \in [p]$, such that $L(D_1 \cup S, D_2, \ldots, D_p)$ is a complete p-partite graph. By (9) and Lemma 5, S is incident to at least $N/(2(m+7))$ missing edges. Thus at least one vertex of S is incident to at least $N/(8(m+7)) \geq \gamma n$ edges, as required. ■

Claim 2 in particular implies that $e(M) \geq \gamma n$. Hence, $e(B) \geq \gamma n$. Edges in B have endpoints in the same A_i. Without loss of generality, suppose $B[A_1]$ contains the maximum number of edges of B among all A_i's. Then $e(B[A_1]) \geq e(B)/p \geq \gamma n/p \geq 2$. Let $W_1 = W \cap A_1$. Since $B[A_1]$ has at least two edges (and all edges of $B \subseteq L$ have different colors), we have $W_1 \neq \emptyset$ by Claim 2. Again, by Claim 2, at most one edge in $B[A_1]$ can be disjoint from W_1. Therefore, the edges in $B[A_1]$ can be covered by at most $|W_1|+1$ vertices in A_1. Let $x \in A_1$ be such that $d_B(x) = \max_{u \in A_1} d_B(u)$. Then $e(B[A_1]) \leq (|W_1|+1) \cdot d_B(x)$. On the other hand, there are at least $|W_1| \cdot \gamma n$ edges in M incident to W_1. So, $e(M) \geq |W_1| \cdot \gamma n$. By our assumption, $e(B[A_1]) \geq e(B)/p$. So,

$$|W_1| \cdot \gamma n \leq e(M) \leq e(B) \leq e(B[A_1]) \cdot p \leq (|W_1|+1) \cdot d_B(x) \cdot p.$$

Let $\alpha=\gamma/2p$. We get

$$d_B(x)\geq \frac{|W_1|}{|W_1|+1}\cdot\frac{\gamma n}{p}\geq\frac{\gamma n}{2p}=\alpha n.$$

In other words, $|N_L(x)\cap A_1|\geq \alpha n$, where $N_L(x)$ denotes the set of neighbors of x in the graph L. If for some i, $|N_L(x)\cap A_i|<|N_L(x)\cap A_1|$, then by moving x from A_1 to A_i we would strictly increase $e(T')$, contradicting our choice of T. Hence, for each $i=1,\ldots,p$, $|N_L(x)\cap A_i|\geq \alpha n$. For each $i=1,\ldots,p$, let C_i be a subset of $A_i\cap N_L(x)$ of size $N=\lfloor \alpha n\rfloor$.

Note that $L(C_1,\ldots,C_p)$ cannot contain a copy of $T_{(m+7)p,p}$. Indeed, since x is connected by an L-edge to every vertex of $C_1\cup\cdots\cup C_p$, we can swap x with some vertex of $T_{(m+7)p,p}$ in a way that contradicts Claim 1. Applying Lemma 5, with $G=L(C_1,\ldots,C_p)$, $K=T(C_1,\ldots,C_p)\subseteq T$, $N=\lfloor\alpha n\rfloor$, $S=\emptyset$, we conclude that

$$|E(T(C_1,\ldots,C_p))\setminus E(L(C_1,\ldots,C_p))|\geq \frac{1}{2}\left(\frac{N}{m+7}\right)^2>\varepsilon n^2,$$

which contradicts (9). This completes the proof of Theorem 4. ■

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