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Bounds on the Generalised Acyclic Chromatic Numbers of Bounded Degree Graphs

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Abstract. We give upper bounds for the generalised acyclic chromatic number and generalised acyclic edge chromatic number of graphs with maximum degree d, as a function of d. We also produce examples of graphs where these bounds are of the correct order.

1. Introduction

The acyclic chromatic number $A(G)$ of a graph G is the minimum number of colours required to properly colour the vertices of the graph such that every cycle has more than 2 colours. (A proper colouring is one in which no two adjacent vertices receive the same colour.) The acyclic chromatic number was introduced by Grünbaum [7] in the context of planar graphs. This led to many results, see for example [5, 1, 2]. Similarly, the acyclic edge chromatic number $A'(G)$ of a graph G is the minimum number of colours required to properly colour the edges of the graph such that every cycle has more than 2 colours. (Here a colouring is proper if no two adjacent edges receive the same colour.)

A generalisation of the acyclic edge chromatic number was introduced by Gerke, Greenhill and Wormald in [6]. Let $r \geq 3$ be a positive integer. Given a graph G, let $A'_r(G)$ be the minimum number of colours required to colour the edges of the graph such that every k-cycle has at least $\min\{k, r\}$ colours, for all $k \geq 3$. So $A'(G) = A'_3(G)$. The quantity $A'_r(G)$ is called the *r-acyclic edge chromatic number* of G. Replacing the word "edges" with "vertices" gives the definition of the *r-acyclic chromatic number $A_r(G)$* of G. We also say that a proper vertex (respectively, edge) colouring is *r-acyclic* if each k-cycle has at least $\min\{k, r\}$ colours, for $k \geq 3$.

Clearly a proper r-acyclic colouring of the line graph $L(G)$ of a graph G gives a proper r-acyclic edge colouring of G. Hence

$$A'_r(G) \leq A_r(L(G)). \tag{1}$$

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The converse of this does not hold, however. To see this consider $r = 4$ and the graph G on vertex set $\{1, 2, 3, 4\}$ with four edges $\{1, 2\}$, $\{2, 3\}$, $\{2, 4\}$, $\{3, 4\}$, where edges $\{1, 2\}$ and $\{3, 4\}$ are coloured red, edge $\{2, 3\}$ is coloured green and $\{2, 4\}$ is coloured blue.

Let d be a positive integer. We wish to consider the generalised acyclic chromatic number and generalised acyclic edge chromatic number of graphs with maximum degree d, as a function of d. To that end, let $A_r(d)$, respectively, $A'_r(d)$, be the maximum of $A_r(G)$, respectively, $A'_r(G)$, over all graphs G with maximum degree d. It is easy to show that $A_r(d)$ and $A'_r(d)$ are bounded, see e.g. (2). Throughout the paper, asymptotic notation such as $O(\cdot)$ and $\Omega(\cdot)$ refers to the limit $d \to \infty$. We also use the phrase "r-acyclic chromatic numbers" of a graph G to refer informally to the two quantities $A_r(G)$ and $A'_r(G)$.

Trivially, a graph with maximum degree 1 has no cycles at all, so $A_r(1) = 2$ and $A'_r(1) = 1$. Similarly, a graph of maximum degree 2 is a union of paths and cycles, so $A_r(2) = A'_r(2) = r$ for $r \geq 3$. For the rest of the paper we assume that $d \geq 3$.

Alon, McDiarmid and Reed [1] gave upper and lower bounds for the 3-acyclic chromatic number. They proved that for some constants $C, c > 0$,

$$\frac{c\, d^{4/3}}{(\log d)^{1/3}} \leq A_3(d) \leq C d^{4/3}.$$

For 3-acyclic edge colourings, clearly at least d colours are required for graphs with maximum degree d. It is well-known that for d-regular graphs at least $d + 1$ colours are required. Alon, McDiarmid and Reed [1] proved that

$$A'_3(d) \leq 64d.$$

This shows that the acyclic edge chromatic number grows linearly in d. Molloy and Reed [8] used a simpler argument to improve this bound to $A'_3(d) \leq 16d$ for $d \geq 3$. These upper bounds were obtained by applying the Lovász Local Lemma.

In this paper we prove the following.

Theorem 1. *Let $r \geq 4$ be fixed. There exist positive constants c, C, c', C' such that*

$$cd^{\lfloor r/2 \rfloor} \leq A_r(d) \leq Cd^{\lfloor r/2 \rfloor}, \quad c'd^{\lfloor r/2 \rfloor} \leq A'_r(d) \leq C'd^{\lfloor r/2 \rfloor}$$

for $d \geq 3$.

The upper bounds are proved in Section 2 using the Local Lemma, while the lower bounds are given in Section 3 by considering the Hamming cube graphs. Finally, we show that some well-known algebraic constructions produce families of graphs which, unlike the Hamming cubes, have girth at least r but still provide fairly good lower bounds for the r-acyclic chromatic numbers when $r = 6, 8, 12$.

As we know that $A'_3(d)$ grows linearly in d, the bounds of Theorem 1 for $A'_r(d)$ also hold when $r = 3$. But the upper bound given by that theorem for $A_r(d)$ certainly does not hold when $r = 3$, considering Alon, McDiarmid and Reed's lower bound of $\Omega(d^{4/3}/(\log d)^{1/3})$. Indeed the true situation for 3-acyclic vertex colourings is not clear. That is, it is not known whether $O\left(d^{4/3}/(\log d)^{1/3}\right)$ colours suffice for a 3-acyclic (vertex) colouring of graphs of maximum degree d, or whether $\Omega(d^{4/3})$ colours are necessary (or something in between).

Studying random graphs is a way of finding the "typical" value of graph parameters such as r-acyclic chromatic numbers. Nešetřil and Wormald [9] proved that with probability which tends to 1 as $n \to \infty$, a uniformly chosen d-regular graph on n vertices satisfies $A'_3(G) \leq d + 1$. Using a similar approach (but with more technical arguments required), Gerke, Greenhill and Wormald [6] showed that with probability which tends to 1 as $n \to \infty$, a uniformly chosen d-regular graph on n vertices has $A'_r(G) \leq (r-2)d$, for fixed $d \geq 2$ and $r \geq 4$. They also gave a simple proof of the fact that $(r-1)d/2 < A'_r(G)$ for any d-regular graph G. Therefore almost all d-regular graphs have r-acyclic edge chromatic number which is linear in r and d. This says that d-regular graphs requiring $\Omega(d^{\lfloor r/2 \rfloor})$ colours for an r-acyclic edge colouring are vanishingly rare.

We close this section with a couple of standard definitions that we will need. Let G be a graph. The length of a path or cycle is the number of edges it contains. Two vertices of G are said to be at distance ℓ apart in G if the shortest path between them in G has length ℓ. Two edges e and f of G are said to be at distance ℓ apart in G if ℓ is the minimum, over the four pairs of endpoints $(x, y) \in e \times f$, of the distance between x and y in G.

2. Upper Bounds

It is easy to see that in any graph of maximum degree d the number of vertices at distance at most $r-1$ from any one vertex is at most $d\left((d-1)^{r-1}-1\right)/(d-2)$. A simple argument involving greedy colourings shows that

$$A_r(d) \leq \frac{d\left((d-1)^{r-1}-1\right)}{d-2} + 1, \qquad A'_r(d) \leq \frac{2(d-1)\left((d-1)^{r-1}-1\right)}{d-2} + 1 \quad (2)$$

for $r \geq 3$ and $d \geq 3$. In this section we improve these bounds by applying the Lovász Local Lemma, motivated by the proof given by Molloy and Reed [8] when $r = 3$ (see also [1, Proposition 2.2]). We will use the Local Lemma in the form given in [8], which we repeat here for ease of reference.

The Local Lemma *Suppose that* $\mathcal{A} = \{\mathcal{A}_1, \ldots, \mathcal{A}_n\}$ *is a set of random events such that each* $\mathcal{A}_i$ *is mutually independent of* $\mathcal{A} \setminus (\{\mathcal{A}_i\} \cup \mathcal{D}_i)$, *for some* $\mathcal{D}_i \subseteq \mathcal{A}$. *Suppose further that we have* $x_1, \ldots, x_n \in [0, 1)$ *such that*

$$\mathbf{P}(\mathcal{A}_i) \leq x_i \prod_{\mathcal{A}_j \in \mathcal{D}_i} (1 - x_j) \quad (3)$$

for $1 \leq i \leq n$. *Then* $\mathbf{P}(\overline{\mathcal{A}_1} \wedge \overline{\mathcal{A}_2} \wedge \cdots \wedge \overline{\mathcal{A}_n}) > 0$.

As a first application we give an upper bound on $A_r(d)$ when $r \geq 3$. In the results of this section we do not make a serious attempt to minimise the constant implicit in the upper bounds. However we do seek arguments which hold for all $d \geq 3$ and $r \geq 3$. The constant implicit in the $O(\cdot)$ notation is $\lambda(r)$, as defined in the proof.

Lemma 1. *Let $r \geq 3$ be fixed. Then for $d \geq 3$,*

$$A_r(d) = O\left(d^{\lceil r/2 \rceil}\right).$$

Proof. Given $d \geq 3$, let G be a graph with maximum degree d. Let $\ell = \lceil r/2 \rceil$. Note that $\ell \geq 2$. Set $k = \lambda d^\ell$ for some constant $\lambda = \lambda(r) \in \mathbb{Z}^+$ to be fixed later.

Colour the vertices of G independently and uniformly using k colours. For two vertices u, v at distance at most ℓ apart, a type 1 event $\mathcal{A}_{uv}$ says that u and v receive the same colour. The probability of this event is $p_1 = 1/k$. For a cycle C of length $r + i$, $i \geq 2$, the type i event $\mathcal{A}_C$ says that C receives at most $r - 1$ colours. The probability of this event is at most $((r-1)/k)^{i+1}\binom{r+i}{i+1}$. To see this, suppose that there are at most $r - 1$ colours in an $(r+i)$-cycle C. Then there exists a set $A \subset C$ of $i + 1$ vertices such that the colour of any vertex of A is equal to some colour in $C \setminus A$. There are $\binom{r+i}{i+1}$ ways to choose A. The vertices in $C \setminus A$ may be coloured arbitrarily, and every vertex of A must be coloured with one of colours appearing on the vertices of $C \setminus A$, of which there are at most $r - 1$. Hence the probability of a type i event is bounded above by

$$p_i = (r/k)^{i+1}\binom{r+i}{i+1}.$$

(We use r/k rather than $(r-1)/k$ simply for convenience.)

Suppose that a k-colouring of G exists in which no event of type i occurs, for $i \geq 1$. We claim that that this colouring is r-acyclic. Let C be a cycle of length m. If $m \leq r + 1$, then type 1 events alone imply that any two vertices of C have different colours, since any two vertices of C are at distance at most ℓ apart. If $m \geq r+2$ then the type $(m - r)$ event $\mathcal{A}_C$ shows that C has at least r colours. Hence the colouring is r-acyclic, as claimed.

Now we use the Local Lemma to show that such a colouring exists. Choose $\lambda = \lambda(r)$ large enough so that

$$2^{r+i} r^{i+1}\binom{r+i}{i+1} \leq \lambda^{i+1} \quad \text{for all} \quad i \geq 2. \tag{4}$$

The expression

$$\sigma(i) = \left(2^{r+i} r^{i+1}\binom{r+i}{i+1}\right)^{1/(i+1)}$$

is a decreasing function of i. To see this, note that

$$\begin{aligned}\left(\frac{\sigma(i+1)}{\sigma(i)}\right)^{(i+1)(i+2)} &= \frac{(r+i+1)^{i+1}(r-1)!(i+1)!}{2^{r-1}(i+2)^{i+1}(r+i)!} \\ &= \frac{1}{2^{r-1}} \prod_{j=0}^{i} \frac{(r+i+1)(i+1-j)}{(i+2)(r+i-j)} \\ &< 1\end{aligned}$$

since each term in the product over j is strictly less than 1 when $r \geq 3$. Hence we can choose $\lambda(r) = \lceil 2^{(r+2)/3} r(r+2) \rceil$, say. Clearly the function $\lambda(r)$ is increasing, so $\lambda(r) \geq \lambda(3) = 48$ for $r \geq 3$.

The number of type 1 events involving a given vertex of G is at most

$$d\left(1 + (d-1) + \cdots + (d-1)^{\ell-1}\right) \leq d^\ell$$

since $\ell \geq 2$. So $t_1 = d^\ell$ is an upper bound for the number of type 1 events involving a given vertex. Similarly, $t_i = d^{r+i-1}$ is an upper bound for the number of type i events involving a given vertex, for $i \geq 2$. Define $x_1 = 4p_1$ and $x_i = 2^{r+i} p_i$ for $i \geq 2$. We must show that the conditions (3) of the Local Lemma are satisfied with the weights x_i. Clearly these weights are positive. Now $x_1 = 4p_1 = 4/(\lambda d^\ell) \leq 4/(9\lambda) \leq 1/108$. Certainly $x_1 < 1/100$. Although we only need prove that $x_i < 1$ to apply the Local Lemma, it will be convenient to know that $x_i < 1/100$ for all $i \geq 1$. This follows by choice of λ, since

$$x_i = \frac{2^{r+i} r^{i+1}}{\lambda^{i+1} d^{\ell(i+1)}} \binom{r+i}{i+1} \leq d^{-\ell(i+1)} < 1/100$$

using (4), for $i \geq 2$. The conditions (3) will be satisfied if

$$\begin{aligned} p_1 &\leq x_1 (1 - 4p_1)^{2t_1} \prod_{i \geq 2} (1 - 2^{r+i} p_i)^{2t_i}, \\ p_i &\leq x_i (1 - 4p_1)^{(r+i)t_1} \prod_{j \geq 2} (1 - 2^{r+j} p_j)^{(r+i)t_j}, \quad i \geq 2. \end{aligned}$$

By the choice of weights x_i, both these equations are equivalent to the following:

$$\frac{1}{2} \leq (1 - 4p_1)^{t_1} \prod_{j \geq 2} (1 - 2^{r+j} p_j)^{t_j}.$$

It is easy to check that $1 - x \geq \exp(-1.02x)$ when $0 \leq x < 1/100$. (Let $g(x) = 1 - x - \exp(-1.02x)$. Then the derivative of $g(x)$ is positive for $0 \leq x < 1/100$, and $g(0) = 0$.) Hence it suffices to prove that

$$2^{-1/1.02} \leq \exp\left(-4p_1 t_1 - \sum_{j \geq 2} 2^{r+j} p_j t_j\right). \tag{5}$$

But $4p_1 t_1 = 4/\lambda \leq 1/12$ and, by (4),

$$\begin{aligned} \sum_{j \geq 2} 2^{r+j} p_j t_j &= \sum_{j \geq 2} \frac{2^{r+j} r^{j+1}}{\lambda^{j+1} d^{\ell(j+1)}} \binom{r+j}{j+1} d^{r+j-1} \\ &\leq \sum_{j \geq 2} \frac{d^{r+j-1}}{d^{\ell(j+1)}} \end{aligned}$$

$$
\begin{aligned}
&= d^{r-\ell-1}\left(\frac{1}{d^{\ell-1}-1} - \frac{1}{d^{\ell-1}}\right)\\
&= \frac{d^{r-2\ell}}{d^{\ell-1}-1}\\
&= \begin{cases} (d^{\ell-1}-1)^{-1} & \text{if } r \text{ even,}\\ (d(d^{\ell-1}-1))^{-1} & \text{if } r \text{ odd} \end{cases}\\
&\le (d-1)^{-1}.
\end{aligned}
$$

So the right hand side of (5) is bounded below by $\exp(-1/12 - 1/2)$ since $d \ge 3$, and it is easy to check that this quantity is greater than $2^{-1/1.02}$. Hence (5) holds, so by the Local Lemma there exists a k-colouring of G such that no type i event holds, for $i \ge 1$. This completes the proof. □

When r is even this upper bound has the same order of magnitude as the upper bound claimed in Theorem 1, as a function of d. When r is odd and $r \ge 5$ we need a slightly more subtle argument. Again, the constant implicit in the $O(\cdot)$ notation is $\lambda(r)$, given in the proof.

Lemma 2. *Let $r \ge 5$ be odd. Then for $d \ge 3$,*

$$A_r(d) = O\left(d^{(r-1)/2}\right).$$

Proof. Let G be a graph of maximum degree at most d. Let $\ell = (r-1)/2$ and let $k = \lambda d^{\ell}$, where the constant $\lambda = \lambda(r) \in \mathbb{Z}^+$ will be determined below. Colour the vertices of G independently and uniformly using k colours. For vertices $u, v \in V(G)$ at distance at most ℓ apart, the type 1 event $\mathcal{A}_{u,v}$ says that u, v have the same colour. As before, the probability of this event is $p_1 = 1/k$. For a cycle C of length $r+i-1$, $i \ge 3$, the type i event $\mathcal{A}_C$ says that C receives less than r colours. These events are the same as those defined in Lemma 1 (except that a type i event here corresponds to a type $i-1$ event in Lemma 1, for $i \ge 3$). For this result we will need two new types of events, defined below. (We cannot simply use an event which says that any two vertices at distance $\ell + 1$ have different colours, since the number of these pairs involving a given vertex is $\Omega(d^{\ell+1})$, an order of magnitude larger than k.)

Let $\mathcal{M}$ be the set of ordered pairs (u, v) of vertices of G which are precisely at distance $\ell + 1$ in G, such that there are at least d different paths of length $\ell + 1$ between u and v. (These paths need not be vertex or edge disjoint, but they cannot be identical.) For $(u, v) \in \mathcal{M}$ we introduce the type 0 event $\mathcal{A}_{u,v}$ which says that u and v receive the same colour. The probability of this event is clearly $p_0 = 1/k$.

Next, let $\mathcal{N}$ be the set of all cycles $C \subseteq G$ of length $r + 1$ with the following property: there is a pair u, v of opposite vertices of C such that the distance between u and v in G is also $\ell + 1$ and $(u, v) \notin \mathcal{M}$. (Note that $|C| = 2\ell + 2$ is even, so we can define *opposite* vertices as those at distance $\ell + 1$ with respect to the cycle C.)

For each $C \in \mathcal{N}$ we consider type 2 event $\mathcal{A}_C$ which says that the cycle C receives at most $r - 1$ colours. The probability of this event is at most

$$\binom{r+1}{2}\left(\frac{r-1}{k}\right)^2 < \frac{r^4}{2k^2}.$$

So let $p_2 = r^4/(2k^2)$.

Suppose there exists a k-colouring of G such that none of the events of type $i \geq 0$ occur. We claim that this k-colouring is r-acyclic. To see this, let C be a cycle in G. If $|C| \leq r$, then type 1 events alone guarantee that C is properly coloured. Suppose next that $|C| \geq r + 2$. Then C is properly coloured with at least r colours, since no type i events occur where $i = |C| - r + 1 \geq 3$. It remains to consider cycles C of length $r + 1$. If $C \in \mathcal{N}$ then C must be properly coloured with at least r colours since no type 2 events occur. Hence we may assume that C has length $r + 1$ and $C \notin \mathcal{N}$.

We now show that any two distinct vertices $u, v \in C$ receive different colours. If u, v are at distance at most ℓ in G, then this is true since no type 1 events occur. So suppose that u and v are at distance $\ell + 1$ in G. Then u, v are opposite vertices of C. Since $C \notin \mathcal{N}$, it follows that $u, v \in \mathcal{M}$ and these vertices receive different colours since no type 0 events occur. Therefore C receives at least $r + 1$ colours, showing that the colouring is r-acyclic as claimed.

In the remainder of the proof we apply the Local Lemma to show that such a colouring of G exists.

Let u be an arbitrary vertex and define $U = \{v \mid (u, v) \in \mathcal{M}\}$. We want to find an upper bound t_0 on $|U|$, which gives an upper bound for the number of type 0 events involving u. Let P consist of all paths of length $\ell + 1$ starting at u and let Q be the multiset containing the other endpoint of each path in P, counting multiplicities. Then $|Q| \leq d^{\ell+1}$. Note that $U \subset Q$ and that every vertex of U has multiplicity at least d in Q. Hence, $t_0 = d^{\ell}$ is an upper bound for the size of U.

Now we find an upper bound t_2 on the number of cycles in $\mathcal{N}$ containing the vertex u. Let $C \in \mathcal{N}$ be any such cycle. By the definition of $\mathcal{N}$ there are $x, y \in C$ such that x and y are at distance $\ell + 1$ in G and $(x, y) \notin \mathcal{M}$. (It is possible that $u \in \{x, y\}$.) This shows that u lies on a path $L \subset C$ of length $\ell + 1$ with x and y for endpoints. There are at most $(\ell + 2)d^{\ell+1}$ ways to choose L, since there are $\ell + 2$ choices for the position of vertex u on the path and at most $d^{\ell+1}$ ways to complete the paths from u to x and from u to y. The cycle C is obtained from L by adding another path L' of length $\ell + 1$ with endpoints x, y. But, as $(x, y) \notin \mathcal{M}$, there are at most d such paths. Hence, the number of cycles $C \in \mathcal{N}$ through u is at most $t_2 = (\ell + 2)d^{\ell+2}$.

Define $\lambda = \lambda(r) = 2^{(r+3)/2}r(r + 2)$, which is an integer since r is odd. This function dominates the function $\lambda(r)$ used in Lemma 1, so (4) holds. Note that $\lambda(r) \geq 120$ for all $r \geq 3$. To apply the Local Lemma, define the weight $x_i = 2^{m_i} p_i$ for a type i event, where $m_0 = 2$, $m_1 = 2$, $m_2 = r + 1$ and $m_i = r + i - 1$ for $i \geq 3$. (Informally, m_i is the number of vertices "involved" in an event of type i.) Clearly each x_i is positive. Moreover, each $x_i < 1/100$ using (4) as in the proof of Lemma 1. (The calculations for $i = 0, 1$ or $i \geq 3$ are in the proof of Lemma 1 and for $i = 2$,

$$x_2 = \frac{2^r r^4}{\lambda^2 d^{2\ell}} < \frac{1}{8d^{2\ell}} < \frac{1}{100}$$

as required.) Condition (3) from the Local Lemma holds if

$$p_i \leq 2^{m_i} p_i \prod_{j\geq 0} (1 - 2^{m_j} p_j)^{t_j m_i} \quad \text{for} \quad i \geq 0.$$

Rearranging, we find that all these inequalities are equivalent to

$$\frac{1}{2} \leq \prod_{j\geq 0} (1 - 2^{m_j} p_j)^{t_j}.$$

Since $1 - x \geq \exp(-1.02x)$ for $0 \leq x < 1/100$, as in the proof of Lemma 1, it suffices to show that

$$2^{-1.02} \leq \exp(-4p_0t_0 - 4p_1t_1 - 2^{r+1} p_2t_2 - \sum_{j\geq 3} 2^{r+j-1} p_j t_j). \tag{6}$$

But $p_0t_0 = p_1t_1 = 1/\lambda \leq 1/120$ and, as in Lemma 1, $\sum_{j\geq 3} 2^{r+j-1} p_j t_j \leq \left(d(d^2 - 1)\right)^{-1} \leq 1/24$ since $r \geq 5$ is odd and $d \geq 3$. Finally

$$2^{r+1} p_2t_2 = \frac{2^r r^4 (\ell + 2)}{\lambda^2 d^{\ell-2}} \leq \frac{r^2(\ell + 2)}{8(r + 2)^2 d^{\ell-2}} \leq \frac{25}{98}$$

since $r \geq 5$, $\ell \geq 2$ and $d \geq 3$. Therefore the right hand side of (6) is bounded below by $\exp(-1/15 - 25/98 - 1/24)$. It is easy to check that this quantity is greater than $2^{-1/1.02}$, as required. This shows that the Local Lemma applies and a proper k-colouring of G exists such that none of the events of type i hold, for $i \geq 0$. This completes the proof. □

We can obtain the desired upper bounds for r-acyclic edge colourings without much extra work. The constant implicit in the $O(\cdot)$ notation can be calculated if needed.

Corollary 1. *Let r be a fixed integer with $r \geq 4$. Then for $d \geq 3$,*

$$A'_r(d) = O\left(d^{\lfloor r/2 \rfloor}\right).$$

Proof. Suppose G is a graph with maximum degree at most d. Then the line graph $L(G)$ of G has maximum degree at most $2(d - 1)$. Applying Lemma 1 and (1), we find that

$$A'_r(G) \leq A_r(L(G)) \leq A_r(2(d - 1)) = O\left(d^{\lceil r/2 \rceil}\right).$$

Since this is true for all graphs G of maximum degree at most d, it follows that $A'_r(G) = O\left(d^{\lceil r/2 \rceil}\right)$. When r is even, this gives the desired result. When r is odd, use Lemma 2 instead of Lemma 1 to complete the proof. □

Combining the results of this section proves the upper bounds of Theorem 1.

3. Lower Bounds

One way to produce good lower bounds for $A_r(d)$ and $A'_r(d)$ is to use graphs with maximum degree at most d and diameter at most $\lfloor r/2 \rfloor$, with as many vertices as possible under these restrictions. It is well known that the maximum possible number of vertices in such a graph is

$$\frac{d(d-1)^r - 2}{d-2}$$

if $d \geq 3$ (see for example Bollobás [4, Section 10.1]). Graphs which attain these bounds are called *Moore graphs*, but unfortunately there are very few of them. A family of graphs which come within a constant factor of this (as a function of d) are *de Bruijn* graphs, but these are quite difficult to work with. Instead we will work with Hamming cubes which will give lower bounds of the desired order (as a function of d).

For $a, b \in \{1, 2, \ldots, h\}^k$, the *Hamming distance* $H(a, b)$ between a and b is the number of coordinates in which a and b differ. The *Hamming cube* $Q(h, k)$ is a graph with vertex set $\{1, \ldots, h\}^k$ and edges between all pairs of vertices at Hamming distance 1. The Hamming cube is d-regular where $d = k(h-1)$. In this section we will sometimes describe a cycle by listing its vertices in order of traversal (in an arbitrary direction).

Lemma 3. *Suppose $h, k \geq 2$ are integers. For every pair of distinct vertices in $Q(h, k)$ there exists a cycle of length at most $2k$ which contains both vertices.*

Proof. Let a, b be two distinct vertices of $Q(h, k)$. Suppose first that a and b differ in just one coordinate. Without loss of generality $a_1 \neq b_1$. Choose $c \in \{1, \ldots, h\} \setminus \{a_2\}$. Then the 4-cycle $a, b, (b_1, c, a_3, \ldots, a_k), (a_1, c, a_3, \ldots, a_k), a$ contains both a and b, as required.

So suppose that a, b differ in at least 2 coordinates. Without loss of generality, $a = (a_1, a_2, \ldots, a_k)$ and $b = (b_1, b_2 \ldots, b_s, a_{s+1}, \ldots, a_k)$ where $s = H(a, b) \geq 2$ and $b_i \neq a_i$ for $1 \leq i \leq s$. One path of length $H(a, b)$ between a and b is

$$a, \quad (b_1, a_2, \ldots, a_k), \quad (b_1, b_2, a_3, \ldots, a_k), \quad \ldots \quad (b_1, \ldots, b_{s-1}, a_s, \ldots, a_k), \quad b$$

and another is

$$a, \quad (a_1, \ldots, a_{s-1}, b_s, a_{s+1}, \ldots, a_k), \quad (a_1, \ldots, a_{s-2}, b_{s-1}, b_s, a_{s+1}, \ldots, a_k), \\ \ldots \quad (a_1, b_2, \ldots, b_s, a_{s+1}, \ldots, a_k), \quad b.$$

These paths are vertex-disjoint except at their endpoints a and b, and the union of these paths is a cycle containing a and b of length $2H(a, b) \leq 2k$. □

Corollary 2. *Given $r \geq 4$, let $k = \lfloor r/2 \rfloor$ and suppose that $d = k(h-1)$ for some $h \geq 3$. Then*

$$A_r(d) \geq h^k = \left(\frac{d}{\lfloor r/2 \rfloor} + 1 \right)^{\lfloor r/2 \rfloor} = \Omega\left(d^{\lfloor r/2 \rfloor} \right).$$

Proof. The Hamming cube $Q(h,k)$ is d-regular and by Lemma 3, any two distinct vertices lie in a $2k$-cycle. Hence all the vertices of $Q(h,k)$ must receive distinct colours in any r-acyclic colouring of $Q(h,k)$, as $2k \le r$. Thus $A_r(d) \ge A_r(Q(h,k)) \ge h^k$, as claimed. □

This shows that the upper bound on $A_r(d)$ obtained in Lemmas 1, 2 is optimal, as a function of d. A similar argument holds for r-acyclic edge colourings.

Lemma 4. *Suppose $h, k \ge 2$ are integers. For every pair of distinct edges in $Q(h,k)$ there exists a cycle of length at most $2k+2$ which contains both edges.*

Proof. Let $e = \{a, b\}$ and $f = \{c, d\}$ be two distinct edges of $Q(h,k)$. Without loss of generality assume that $a = (a_1, \dots, a_k)$, $b = (b_1, a_2 \dots, a_k)$, where $a_1 \ne b_1$. Also without loss of generality we can assume that c and d differ in coordinate ℓ where $\ell \in \{1, 2\}$. So $c = (c_1, \dots, c_k)$ and $d = (d_1, d_2, c_3, \dots, c_k)$ where $d_1 \ne c_1$ and $d_2 = c_2$ if $\ell = 1$, while $d_1 = c_1$ and $d_2 \ne c_2$ if $\ell = 2$.

If e and f are adjacent, then without loss of generality $a = c$. If $\ell = 1$ then a, b, d, a forms a 3-cycle and if $\ell = 2$ there is a 4-cycle $a, b, (b_1, d_2, a_3, \dots, a_k), d, a$. These cycles contain the edges e and f, as required.

For the remainder of the proof we may assume that e and f are not adjacent. Form a path P from a to c of length $H(a,c)$ by correcting differences in increasing coordinate order. That is, take only the first occurence of each vertex in the sequence

$$a, \quad (c_1, a_2, \dots, a_k), \quad (c_1, c_2, a_3, \dots, a_k), \quad \dots \quad (c_1, \dots, c_{k-1}, a_k), \quad c$$

to obtain P. Also form a path Q from b to d of length $H(b,d)$ by correcting differences in *decreasing* coordinate order. That is, take only the first occurence of each vertex in the sequence

$$b, \quad (b_1, a_2, \dots, a_{k-1}, c_k), \quad \dots \quad (b_1, a_2, c_3, \dots, c_k), \quad (b_1, d_2, c_3, \dots, c_k), \quad d.$$

If $b_1 \ne c_1$ and either $\ell = 1$ or $a_2 \ne d_2$ then it is not difficult to check that all the vertices of $P \cup Q$ are distinct. Then $P \cup Q \cup \{e, f\}$ gives a cycle containing e and f of length $H(a,c) + H(b,d) + 2 \le 2k + 2$, as required.

If $b_1 = c_1$ then the first edge of the path P is e, so P goes from a to c and includes the vertex b and edge e. In this case take the path $\widehat{Q}$ from a to d defined by taking the first occurrence of each vertex in the sequence

$$a, \quad (a_1, \dots, a_{k-1}, c_k), \quad \dots \quad (a_1, a_2, c_3, \dots, c_k), \quad (a_1, d_2, c_3, \dots, c_k), \quad d.$$

Again, it is easy to check that the vertices of $P \cup \widehat{Q}$ are distinct (since in particular $b_1 = c_1$ implies that $a_1 \ne c_1$) and that $P \cup \widehat{Q} \cup \{f\}$ forms a cycle of length at most $2k + 2$ containing the edges e and f.

Finally suppose that $b_1 \ne c_1$, $\ell = 2$ and $a_2 = d_2$. The vertices of $P \cup Q$ are distinct unless $d = (c_1, a_2, \dots, a_k)$ In this case $c = (c_1, c_2, a_3, \dots, a_k)$ where $a_2 \ne c_2$ and $a_1 \ne c_1$ as $a \ne d$. Hence

$$a, \quad b, \quad (b_1, c_2, a_3, \dots, a_k), \quad c, \quad d, \quad a$$

is a 5-cycle containing edges e and f. This completes the proof. □

Corollary 3. *Given $r \geq 6$ let $k = \lfloor (r-2)/2 \rfloor$ and suppose that $d = k(h-1)$ for some $h \geq 2$. Then*

$$A'_r(d) \geq \frac{h^k k(h-1)}{2} = \frac{d}{2}\left(\frac{d}{\lfloor (r-2)/2 \rfloor} + 1\right)^{\lfloor (r-2)/2 \rfloor} = \Omega\left(d^{\lfloor r/2 \rfloor}\right).$$

Proof. The Hamming cube $Q(h,k)$ is d-regular. By Lemma 4, distinct edges of $Q(h,k)$ receive distinct colours in any r-acyclic colouring of $Q(h,k)$, since $2k+2 \leq r$. There are $dh^k/2$ edges in $Q(h,k)$, where $d = k(h-1)$. So $A'_r(d) \geq A'_r(Q(h,k)) \geq dh^k/2$, as stated. □

The results of this section prove the lower bounds of Theorem 1.

4. Lower Bounds Using Graphs with Girth at Least *r*

The Hamming cube graphs give good lower bounds for the r-acyclic vertex, edge chromatic number because we can find a cycle of appropriate length containing any two given vertices, respectively edges. Many of these cycles may have length strictly less than r. However, the restriction imposed on k-cycles when $k < r$ (namely, that they have no repeated colours) is arguably less interesting than that imposed on k-cycles for $k \geq r$ (namely, that they have at least r colours). So we might be interested in graphs which have girth at least r yet still give good lower bounds for the r-acyclic chromatic numbers. We now present some results obtained using algebraic constructions, for $r = 6, 8, 12$. The value of the (vertex) r-acyclic chromatic number $A_r(G)$ obtained for these graphs is a factor of d worse than the bound obtained in Corollary 7 (ignoring constants) while the value of the r-acyclic edge chromatic number $A'_r(G)$ is of the correct order in each case.

Lemma 5. *Suppose that $d = q+1$ where q is a prime or prime power. Then there is a graph G which is d-regular, has girth 6 and satisfies*

$$A_6(G) = 2(d^2 - d + 1), \qquad A'_6(G) = d(d^2 - d + 1).$$

Proof. Take the bipartite graph G whose vertices are the points and lines of the projective plane $\mathrm{PG}(2,q)$, and with edges $\{x, \ell\}$ whenever point x lies on the line ℓ. Let V_1 be the set of points and V_2 the set of lines. Then $|V_1| = |V_2| = q^2+q+1$, and G is $(q+1)$-regular and has girth 6. It suffices to prove that any two distinct edges lie in a 6-cycle. For then certainly any two distinct vertices also lie in a 6-cycle (just consider any two distinct edges whose endpoints include the two given vertices). So in order to get a 6-acyclic vertex (respectively, edge) colouring of G, every vertex (respectively, edge) of G must receive a distinct colour. This leads to the bounds shown.

Let $e = \{v_1, w_1\}$ and $f = \{v_2, w_2\}$ be two edges in G, where $v_1, v_2 \in V_1$ and $w_1, w_2 \in V_2$. We show that e and f lie on a 6-cycle of G. There are a few cases to consider.

Case (i). First suppose that e and f share an endpoint. Without loss of generality suppose $v_1 = v_2 = v$. Then $w_1 \neq w_2$. Let $z_i \in V_1 \setminus \{v_1\}$ be a neighbour of w_i, for $i = 1, 2$. By the girth condition, w_1 is not a neighbour of z_2 and w_2 is not a neighbour of z_1. Then the edges $\{z_1, w_1\}$,e, f,$\{w_2, z_2\}$ together with the 2-path from z_1 to z_2 give the required 6-cycle.

Case (ii). Next suppose that all vertices in e and f are distinct but that an edge joins an endpoint of e to an endpoint of f. So without loss of generality, suppose that $\{v_2, w_1\}$ is an edge of G. (By the girth condition, at most one of $\{v_1, w_2\}$, $\{v_2, w_1\}$ can be edges of G.) Let z be a neighbour of w_2 in $V_1 \setminus \{v_1, v_2\}$. Take the edges e,$\{w_1, v_2\}$, f,$\{w_2, z\}$ together with the 2-path from z to v_1 as the 6-cycle.

Case (iii). Finally suppose that all vertices in e and f are distinct and that there is no edge joining an endpoint of e to an endpoint of f. Take edges e, f together with the 2-path from v_1 to v_2 and the 2-path from w_1 to w_2 give a 6-cycle. □

Lemma 6. *Suppose that $d = q + 1$ where q is a prime or prime power. Then there is a graph G which is d-regular, has girth 8 and satisfies*

$$A_8(G) = 2d(d^2 - 2d + 2), \qquad A'_8(G) = d^2(d^2 - 2d + 2).$$

Proof. Benson [3] shows how to construct a graph G from the points and lines on a non-degenerate quadric in $PG(4, q)$, with the following properties: the graph G is bipartite, with each side of the bipartition having $q^3 + q^2 + q + 1$ vertices, and is $(q + 1)$-regular with girth 8. We show that any two distinct edges lie in an 8-cycle of G. (This is sufficient, by exactly the same argument as in the previous lemma.)

Let $v \in V_1$. There are $q + 1$ vertices at distance 1 from v and $(q + 1)q^2$ at distance 3, and these two subsets of V_2 are disjoint (by the girth condition). Since $|V_2| = q + 1 + (q + 1)q^2$, we see that any $u \in V_2$ is connected to v by exactly one path of length at most 3, which we will call the uv-path.

Let e, f be two edges of G. The cases are as in Lemma 5. We use the notation of that lemma and describe how to produce the 8-cycle.

Case (i). Instead of taking the 2-path from z_1 to z_2, choose $u \in V_2 \setminus \{w_1\}$ such that u is a neighbour of z_1. By the girth condition, $u \neq w_2$. Then take edges $\{u, z_1\}$, $\{z_1, w_1\}$, e, f, $\{w_2, z_2\}$ together with the z_2u path, which must have length 3 by the girth condition. This gives an 8-cycle containing e and f.

Case (ii). Instead of taking the 2-path from z to v_1, choose $u \in V_2 \setminus \{w_1\}$ such that u is a neighbour of v_1. Then $u \neq w_2$, by the girth condition. Take edges $\{u, v_1\}$, e, $\{w_1, v_2\}$, f, $\{w_2, z\}$ together with the zu-path. This gives an 8-cycle.

Case (iii). Take e, f and the v_1w_2-path and the v_2w_1-path. This gives an 8-cycle, by the girth condition. □

Lemma 7. *Suppose that $d = q + 1$ where q is a prime or prime power. Then there is a graph G which is d-regular, has girth 12 and satisfies*

$$A_{12}(G) = 2d(d^4 - 4d^3 + 7d^2 - 6d + 3),$$
$$A'_{12}(G) = d^2(d^4 - 4d^3 + 7d^2 - 6d + 3).$$

Proof. Benson [3] shows how to construct a graph G which is bipartite, with each side of the bipartition having $(q + 1)(q^4 + q^2 + 1)$ vertices, and is $(q + 1)$-regular with girth 12. It follows that for every $x \in V_1$ and $y \in V_2$, where V_1 and V_2 are the two sides of the bipartition, there is a unique path from x to y of length at most 5. Call this the xy-path. We wish to prove that every pair of distinct edges of G lies in a 12-cycle of G. But the argument of Lemma 6 works exactly as is, (where instead of having length 3 in each case, the xy-paths used will have length 5, by the girth condition). This completes the proof. □

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