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Article

New Lower Bound on Ball Packing Density in High-Dimensional Hyperbolic Spaces

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We present a new lower bound on the Bowen–Radin maximal density of radius-R ball packings in the m-dimensional hyperbolic space, improving on the basic covering bound by factor $\Omega(m(R+\ln m))$ as m tends to infinity. This is done by applying the recent theorem of Campos, Jenssen, Michelen, and Sahasrabudhe on independent sets in graphs with sparse neighbourhoods.

1 Introduction

Let $R>0$ be a positive real and let (V,d) be a metric space endowed with a Borel non-zero measure μ. An R-*packing* in V is a subset $X\subseteq V$ such that any two distinct points of X are at distance at least $2R$. If the measure μ is *finite* (that is, $\mu(V)<\infty$), then we define the *density* of an R-packing X by

$$D_R(X) := \frac{\mu\left(B_R(X)\right)}{\mu(V)},$$

where we denote

$$B_R(X) := \{y\in V \mid \exists\, x\in X,\ d(x,y)\leqslant R\}.$$

In other words, $D_R(X)$ is the fraction of the measure space V covered by closed radius-R balls around the points of X. The problem of determining or estimating the R-*packing density* $D_R(V)$ of a given metric space V (i.e., the supremum of the densities of R-packings in V) is the archetypical problem of coding theory which also has a large number of applications to other fields.

One important case is when $V=\mathbb{R}^m$, endowed with the Euclidean distance and the Lebesgue (uniform) measure. Since the measure is not finite here, one defines the R-*packing density* $D(\mathbb{R}^m)$ as the limit of the packing densities of growing cubes that exhaust $\mathbb{R}^m$:

$$D(\mathbb{R}^m) := \lim_{n\to\infty} D_R\left([-n,n]^m\right). \tag{1}$$

It is easy to see that the limit exists and, by the scaling properties of the Lebesgue measure, does not depend on the radius R.

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It is clear that the packing density of the real line $D(\mathbb{R}^1) = 1$. The packing density of $\mathbb{R}^m$ for $m = 2, 3, 8, 24$ was determined respectively by Thue [30]; Hales [21]; Viazovska [34]; and Cohn, Kumar, Miller, Radchenko, and Viazovska [13]. The value of $D(\mathbb{R}^m)$ is still unknown for any other m, although the bounds for $m = 4$ (from [11, 25]) are rather close to each other. We refer the reader to Cohn [10] for the table of the known bounds for $m \leqslant 48$.

When $m \to \infty$, the best known upper and lower bounds on $D(\mathbb{R}^m)$ are exponentially far apart. The best known upper bound $D(\mathbb{R}^m) \leqslant 2^{-(0.5990\ldots+o(1))\,m}$ is due to Kabatjanskiĭ and Levenšteĭn [24]; see also [12, 14].

The trivial *covering lower bound* $D(\mathbb{R}^m) \geqslant 2^{-m}$ (take a maximal packing and observe that balls of doubled radius cover the whole space) was improved by a factor of $\Omega(m)$ by Rogers [29]. After a sequence of improvements in the constant factor [1, 15, 31, 33], Venkatesh [33] was the first to beat the $\Omega(\frac{m}{2^m})$ lower bound by showing that $D(\mathbb{R}^m) = \Omega(\frac{m \ln\ln m}{2^m})$ for some infinite (sparse) sequence of dimensions m. In a recent breakthrough, Campos, Jenssen, Michelen and Sahasrabudhe [8] proved that $D(\mathbb{R}^m) \geqslant (\frac{1}{2} - o(1))\,\frac{m \ln m}{2^m}$. This was done by discretising the problem and then applying their new result (which is stated as Theorem 12 here) on the existence of a large independent set in graphs with sparse neighbourhoods.

The aim of this paper is to observe that the method from [8] also applies to ball packing in hyperbolic spaces. Let $\mathbb{H}^m$ denote the m-dimensional hyperbolic space, equipped with the standard metric d_m and the standard (isometry invariant) measure μ_m. (See Section 2 for all formal definitions.)

The study of packings in $\mathbb{H}^m$ was held back for a long while by the absence of a good definition of the maximum packing density; see, for example, the discussion in [18, pp. 831–834]. Analogues of (1) lack many desired properties because growing polygons or balls in $\mathbb{H}^m$ have most of their mass near their boundary. So some "local" approaches were considered; see [20, Chapter 11] for a survey of known results from this point of view. However, even here one has to be careful in view of the following striking example of Böröczky [2] (also described on page 833 of [18]). Namely, Böröczky showed that there is an R-packing $X \subseteq \mathbb{H}^2$ in the hyperbolic plane and two tilings $\mathcal{T}_1$ and $\mathcal{T}_2$ of $\mathbb{H}^2$ by single polygonal tiles T_1 and T_2 respectively such that each tile of $\mathcal{T}_i$ contains exactly one radius-R disk centred at a point of X and is disjoint from all other such disks; at the same time, T_1 and T_2 have different areas. Thus, if we measure the "density" of this packing X using the fraction occupied by the corresponding disks within each tile of $\mathcal{T}_1$ (resp. $\mathcal{T}_2$), then we get two different numbers.

A solution was proposed by Bowen and Radin [5, Definition 5] who presented a new notion of density where, roughly speaking, one considers probability distributions on R-packings in $\mathbb{H}^m$ that are invariant under isometries and maximises the probability that a fixed point is covered by a ball. This parameter has many nice properties and is the one that will be used in this paper as the definition of the *R-packing density* of $\mathbb{H}^m$ and thus denoted by $D_R(\mathbb{H}^m)$. We refer the reader to Section 3 for the formal definition of $D_R(\mathbb{H}^m)$ and to [6, 27] for further discussions and motivation behind it.

For dimension $m = 2$ and countably many radii R, Bowen and Radin [5, Theorem 2] were able to determine $D_R(\mathbb{H}^2)$. Namely, for every integer $n > 6$, if we tile $\mathbb{H}^2$ by (equilateral) triangles having all three angles $2\pi/n$ and take a "uniform shift" of the set of the triangles' vertices then this distribution attains $D_R(\mathbb{H}^2)$, where $R = R_n$ is the half of the side length of the triangles. As far as we know, these are the only pairs (m, R) with $m \geqslant 2$ for which the exact value of $D_R(\mathbb{H}^m)$ is known.

Let us discuss the known upper bounds on $D_R(\mathbb{H}^m)$ for $m \to \infty$. Before Bowen and Radin's paper [5], Fejes Tóth [19] (for $m = 2$), Böröczky and Florian [4] (for $m = 3$), and Böröczky [3] (any m) proved that for every R-packing $X \subseteq \mathbb{H}^m$ and any $x \in X$ the fraction of volume occupied by the *R-ball* $B_R(x) := B_R(\{x\})$ around x inside the Dirichlet–Voronoi cell of $x \in X$ is at most $\delta_m(2R)$, which is defined to be the fraction of volume of a regular simplex of side length $2R$ occupied by (touching) balls of radius R at its vertices. It follows from these results (see Lemma 4 here) that $D_R(\mathbb{H}^m) \leqslant \delta_m(2R)$. More recently, Cohn and Zhao [14, Theorem 4.1] showed that, independently of $R > 0$, the Bowen–Radin density satisfies

$$D_R(\mathbb{H}^m) \leqslant \inf_{\pi/3 \leqslant \theta \leqslant \pi} \sin^{m-1}(\theta/2)\, A(m, \theta), \tag{2}$$

where $A(m, \theta)$ be the maximum number of unit vectors in the Euclidean space $\mathbb{R}^m$ such that the scalar product of every two is at most $\cos\theta$; in other words, $A(m, \theta)$ is the maximum size of a spherical code with minimum angle θ. In particular, the method of Kabatjanskiĭ and Levenšteĭn [24] (that also applies to spherical codes) gives that

$$D_R(\mathbb{H}^m) \leqslant 2^{-(0.5990\ldots+o(1))\,m}; \tag{3}$$

see [14, Corollary 4.2]. For large m, this bound is smaller than $\delta_m(0) := \lim_{R\to 0} \delta_m(R)$, which can be shown to be the volume ratio coming from $m+1$ touching unit balls in $\mathbb{R}^m$. Note that Marshall [26, Theorem 2] proved that, for all large m, the function $\delta_m(R)$ is strictly increasing in R (and thus $\delta_m(0) \leqslant \delta_m(R)$ for any $R > 0$ for such m).

The covering principle (that if X is a maximal R-packing then $2R$-balls around points of X cover the whole space) translates with some work (see Lemma 8) into the basic lower bound

$$D_R(\mathbb{H}^m) \geqslant L_R(\mathbb{H}^m) := \frac{\mu_m(B_R)}{\mu_m(B_{2R})}, \tag{4}$$

where $\mu_m(B_r)$ denotes the volume of some (equivalently, any) r-ball in $\mathbb{H}^m$.

There are various constructions of packings in small dimensions (see, e.g., [20, Chapter 11] for references) and it is plausible that their "locally" measured densities translate into lower bounds on the Bowen–Radin density. However, we are not aware of any improvements to (4) for large m apart that, as far as we can see, the method of Jenssen, Joos, and Perkins [22, 23], which is based on the hard-sphere model of statistical physics, can be applied to improve the bound in (4) by factor $\Omega(m)$. Anyway, this is superseded by the main result of this paper as follows.

Theorem 1. For every $\varepsilon > 0$ there is m_0 such that for any $m \geqslant m_0$ it holds for every $R \in (0, \infty)$ that

$$D_R(\mathbb{H}^m) \geqslant (1-\varepsilon)\, m \ln\left(\sqrt{m}\cosh(2R)\right) \frac{\mu(B_R)}{\mu(B_{2R})},$$

where $\ln$ is the natural logarithm and $\cosh$ is the hyperbolic cosine.

Note that $\ln(\cosh(2R))$ is at least $2R-1$, so Theorem 1 improves the bound in (4) by factor at least $\Omega(m(R+\ln m))$. We prove Theorem 1 by reducing the lower bound problem to finding a packing in some finite-volume space (namely, the quotient of $\mathbb{H}^m$ by a large-girth lattice) and then (as it was done in [8]) discretising the problem by taking a Poisson point process.

Organisation of the paper. We recall some notions related to $\mathbb{H}^m$ in Section 2 and define the Bowen-Radin density in Section 3. We describe the method from [5] of lower bounding $D_R(\mathbb{H}^m)$ via R-packings in some finite-volume space M in Section 4. Various estimates are collected in Section 5. Finally, Theorem 1 is proved in Section 6. Since m and R will be reserved, respectively, for the dimension of the studied hyperbolic space and the packing radius, we may omit them from our notation if the meaning is clear.

2 Hyperbolic Space

This section contains just a bare minimum of material sufficient to formally define the hyperbolic space $\mathbb{H}^m$ and some needed related notions. For a detailed introduction to hyperbolic spaces, see, for example, Bridson and Haefliger [7, Section 2] or Ratcliffe [28].

One representation of $\mathbb{H}^m$ (for more details, see, e.g., [28, Chapter 3]) is to take the bilinear form

$$\langle u, v\rangle_{m,1} := -u_{m+1}v_{m+1} + \sum_{i=1}^{m} u_i v_i, \quad u, v \in \mathbb{R}^{m+1}, \tag{5}$$

identify $\mathbb{H}^m$ with the upper sheet of the hyperboloid

$$\mathcal{H} := \{u \in \mathbb{R}^{m+1} \mid \langle u, u\rangle_{m,1} = -1\}$$

(namely, the sheet where $u_{m+1} > 0$), and define the metric d_m by $\cosh d_m(u, v) = -\langle u, v\rangle_{m,1}$ for $u, v \in \mathbb{H}^m$. The group of isometries of $\mathbb{H}^m$ can be identified with $O(m, 1)_0$, the group of $(m+1) \times (m+1)$-matrices A which leave the bilinear form in (5) invariant and do not swap the two sheets of $\mathcal{H}$; see [7, Theorem 2.24] or [28, Theorem 3.2.3].

Let $\mathcal{G}$ be the group of orientation-preserving isometries of $\mathbb{H}^m$; it corresponds to the subgroup $SO(m, 1)_0$ of index 2 in $O(m, 1)_0$, which consists of matrices with determinant 1. It is a topological group with the topology inherited from $O(m, 1)_0 \subseteq \mathbb{R}^{(m+1)^2}$.

We fix one point of $\mathbb{H}^m$, say $(0,\dots,0,1) \in \mathbb{R}^{m+1}$ in the above representation, which we will denote by $\mathcal{O}$ and refer to as the *origin*.

The space $\mathbb{H}^m$ is equipped with a Borel isometry-invariant measure μ_m whose push-forward under the projection on the first m coordinates of $\mathbb{R}^{m+1}$ has density $(1+(x_1^2+\dots+x_m^2))^{-1/2}$ with respect to the Lebesgue measure on $\mathbb{R}^m$. The group $\mathcal{G}$ is unimodular and locally compact, so there is a Haar measure $\mu_{\mathcal{G}}$ on $\mathcal{G}$ (which is both right- and left-invariant). It is a standard result (see, e.g., [28, Lemma 4 of Section 11.6]) that, by scaling $\mu_{\mathcal{G}}$, we can assume that the projection map $\pi_{\mathcal{O}} : (\mathcal{G}, \mu_{\mathcal{G}}) \to (\mathbb{H}^m, \mu_m)$, where $\gamma \mapsto \gamma.\mathcal{O}$, is measure-preserving.

We consider the natural (left) action $\mathcal{G} \curvearrowright \mathbb{H}^m$. For $\gamma \in \mathcal{G}$ and $A \subseteq \mathbb{H}^m$, we denote $\gamma.A := \{\gamma.x \mid x \in A\}$. For $x \in \mathbb{H}^m$, let $\Sigma_x := \{\gamma \in \mathcal{G} \mid \gamma.x = x\}$ be the *stabiliser* of x. In particular, we have the natural homeomorphism of topological spaces $\mathcal{G}/\Sigma_{\mathcal{O}} \cong \mathbb{H}^m$.

3 Definition of the Bowen–Radin Density

In this section, we give the definition of the packing density $D_R(\mathbb{H}^m)$ introduced by Bowen and Radin [5]. We also state some results from [5], occasionally providing more details (usually when these were implicitly assumed but not stated in [5]).

Let $R > 0$ be a positive real. Let S_R consist of R-packings $X \subseteq \mathbb{H}^m$ that are *relatively dense*, that is, for every $x \in \mathbb{H}^m$, the (closed radius-$2R$) ball $B_{2R}(x)$ around x contains at least one element of X. This is slightly weaker than the notion of a maximal packing (namely, a relatively dense packing can have the distance from some x to X exactly $2R$ and thus not be a maximal one).

Bowen and Radin [5] considered the following metric d_R on S_R:

$$d_R(X,Y) := \sup_{r\in[1,\infty)} \frac{1}{r}\, h\big(B_r(\mathcal{O})\cap X, B_r(\mathcal{O})\cap Y\big), \quad X, Y \in S_R, \tag{6}$$

where

$$h(A,B) := \max\left\{ \sup_{a\in A}\inf_{b\in B} d_m(a,b),\ \sup_{b\in B}\inf_{a\in A} d_m(a,b) \right\}, \quad A, B \subseteq \mathbb{H}^m,$$

is the usual *Hausdorff distance* on the subsets of $\mathbb{H}^m$. As noted in [5, p. 25], the metric space (S_R, d_R) is compact. Let $\mathcal{M}(R)$ be the set of Borel probability measures on (S_R, d_R).

The group $\mathcal{G}$ naturally acts on S_R and the corresponding map $\mathcal{G} \times S_R \to S_R$ is continuous. A measure $\mu \in \mathcal{M}(R)$ is called $\mathcal{G}$*-invariant* if for every $\gamma \in \mathcal{G}$ and every Borel set $E \subseteq S_R$ we have $\mu(\gamma.E) = \mu(E)$. Let $\mathcal{M}_I(R)$ consist of all $\mathcal{G}$-invariant measures in $\mathcal{M}(R)$. Let $\mathcal{M}_I^{\mathbf{e}}(R)$ consist of those measures $\mu \in \mathcal{M}_I(R)$ that are *ergodic*, that is, are extreme points of the convex set $\mathcal{M}_I(R)$.

For $p \in \mathbb{H}^m$, define

$$F_p(X) := \begin{cases} 1, & \text{if } d_m(p,X) \leqslant R, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{for } X \in S_R,$$

to be the indicator function that R-balls around X cover p.

Lemma 2. For every $p \in \mathbb{H}^m$, the function F_p is a Borel function on (S_R, d_R).

Proof. Let us show that $F_p^{-1}(1) = \{X \in S_R \mid F_p(X) = 1\}$ is a closed subset of (S_R, d_R). Take any sequence $(X_n)_{n=1}^{\infty}$ in $F_p^{-1}(1)$ convergent to some $X \in S_R$. Each X_n has a point x_n in $B_R(p)$. By the compactness of the closed ball $B_R(p)$, we can pass to a subsequence of n so that x_n converges to some $x \in B_R(p)$ as $n \to \infty$. Since X has at most one point in $B_{R/2}(x)$, it follows from the definition of the Hausdorff distance by taking any $r > d_m(x, \mathcal{O})$ in (6) that $x \in X$. Thus, $F_p(X) = 1$ and the set $F_p^{-1}(1)$ is closed (and thus Borel).

Since the function F_p is $\{0,1\}$-valued and the pre-image $F_p^{-1}(1)$ is closed, the function F_p is Borel. ■

Thus, for any $\mu \in \mathcal{M}_I(R)$, we can define the *average density*

$$D(\mu) := \int_{S_R} F_{\mathcal{O}}(X) d\mu(X). \tag{7}$$

By the $\mathcal{G}$-invariance of μ and the transitivity of $\mathcal{G} \curvearrowright \mathbb{H}^m$, we could have taken any other point of $\mathbb{H}^m$ instead of the origin in this definition, without changing the value.

Following Bowen and Radin [5, Definition 5], we define the *R-packing density of* $\mathbb{H}^m$ as

$$D_R(\mathbb{H}^m) := \sup_{\mu \in \mathcal{M}_I^{\mathbf{e}}(R)} D(\mu). \tag{8}$$

We refer the reader to [5] and also to [6, 27] for discussion and further properties of this parameter. For example, [5, Theorem 1] implies that the supremum in (8) is attained by some measure $\mu \in \mathcal{M}_I^{\mathbf{e}}(R)$.

Let us show that the definition in (8) is not affected if we take the supremum over invariant (not necessarily ergodic) measures.

Lemma 3. For every $R \in (0, \infty)$ and $m \in \mathbb{N}$, we have that $D_R(\mathbb{H}^m) = \sup_{\mu \in \mathcal{M}_I(R)} D(\mu)$.

Proof. Trivially, $D_R(\mathbb{H}^m) \leqslant \sup_{\mu \in \mathcal{M}_I(R)} D(\mu)$, so let us show the converse inequality.

The ergodic invariant decomposition theorem of Varadarajan [32, Theorem 4.2] (see also Farrell [16, Theorem 5] for a similar result) applies to Borel actions of locally compact groups on standard Borel spaces. When applied to the action $\mathcal{G} \curvearrowright S_R$, it gives a Borel map $\beta : S_R \to \mathcal{M}_I^{\mathbf{e}}(R)$ which is constant on every orbit of $\mathcal{G}$ such that every measure ν in the image of β assigns value 1 to the pre-image $\beta^{-1}(\nu) \subseteq S_R$ and every $\mu \in \mathcal{M}_I(R)$ satisfies

$$\mu(A) = \int_{S_R} \beta(X)(A) d\mu(X), \quad \text{for every Borel } A \subseteq S_R. \tag{9}$$

Take any $\mu \in \mathcal{M}_I(R)$ and let A be $\{X \in S_R \mid \mathcal{O} \in B_R(X)\} = F_{\mathcal{O}}^{-1}(1)$. By Lemma 2, A is a Borel subset of S_R. Note that $\mu(A) = D(\mu)$. Thus, by (9) applied to μ and A, $D(\mu)$ is some average of $D(\nu)$ over ergodic invariant measures $\nu = \beta(X)$ for $X \in S_R$. Thus, there is $\nu \in \mathcal{M}_I^{\mathbf{e}}(R)$ with $D(\nu) \geqslant D(\mu)$ and we have $D_R(\mathbb{H}^m) \geqslant D(\mu)$. Since $\mu \in \mathcal{M}_I(R)$ was arbitrary, the desired inequality follows. ∎

Also, it is plausible that one can replace S_R by the space of arbitrary (not necessarily relatively dense) R-packings by completing a random R-packing into a relatively dense one in a measurable and invariant way. However, this would require some extra work (and the definition of the distance in (6) would need some tweaking) so we stay with the definitions from [5].

At this point, let us observe that the classical "local" upper bounds from [3, 4, 19] also apply to $D_R(\mathbb{H}^m)$.

Lemma 4. For any real $R \geqslant 0$ and integer m, we have $D_R(\mathbb{H}^m) \leqslant \delta_m(2R)$.

Proof. Take any $\mu \in \mathcal{M}_I^{\mathbf{e}}(R)$. Bowen and Radin [5, Proposition 3] showed that $D(\mu)$ can be computed as the expectation over a random packing X distributed according to μ of the ratio of volume of $B_R(\mathcal{O})$ to the volume of the Dirichlet–Voronoi cell of X containing the origin $\mathcal{O} \in \mathbb{H}^m$ in its interior. (Note that, by the invariance of μ, the probability that $\mathcal{O}$ lies on the boundary of a Dirichlet–Voronoi cell of X is zero.)

By the results from [3, 4, 19] mentioned in the Introduction, the integrated function is always at most $\delta_m(2R)$, so its average is at most $\delta_m(2R)$, as desired. ∎

4 Lower Bounds via a Finite-Volume Space

Here we describe the method, implicit in [5], of proving lower bounds on $D_R(\mathbb{H}^m)$ coming from ball packings in some finite-volume space. We use (as in [17]) the following definition of a lattice which is stronger than the standard one. Namely, let us call a subgroup $\mathcal{L} \subseteq \mathcal{G}$ a *(uniform torsion-free) lattice* if (i) there is $g > 0$ such that for every $x \in \mathbb{H}^m$ and every non-identity $\gamma \in \mathcal{L}$ we have $d_m(x, \gamma.x) \geqslant g$ and

(ii) the quotient space $M := \mathbb{H}^m/\mathcal{L}$ is compact. Denote the supremum of g that work in (i) as $g(\mathcal{L})$ and call it the *girth* of $\mathcal{L}$. The topological space M comes with the natural metric

$$d_M(x, y) := d_m(\pi_M^{-1}(x), \pi_M^{-1}(y)), \quad x, y \in M,$$

where $\pi_M : \mathbb{H}^m \to M$ is the projection. Note that π_M is a *local isometry*, that is, there is $r > 0$ such that for every $x \in \mathbb{H}^m$ the map π_M gives an isometry between $B_r(x)$ and the ball of radius r around $\pi_M(x)$ in d_M.

Let us show that, in fact, we can take the local isometry radius r to be $\frac{1}{4}g(\mathcal{L})$. Take any $x \in \mathbb{H}^m$. To show the surjectivity on radius-r balls, take any $w \in M$ with $d_M(w, \pi_M(x)) \leqslant r$. Since $\pi_M^{-1}(\pi_M(x)) = \mathcal{L}.x$ and $\mathcal{L}$ acts by isometries, we have $d(\pi_M^{-1}(w), x) = d_m(\pi_M^{-1}(w), \mathcal{L}.x) \leqslant r$. Since $\pi_M^{-1}(w)$ is closed and, say, $B_{2r}(x)$ is compact, we have that $\pi_M^{-1}(w) \cap B_r(x)$ is non-empty. Thus, π_M is a surjection of $B_r(x)$ onto the ball of radius r around $\pi_M(x)$. To show that π_M preserves distances on $B_r(x)$, take any $y, z \in B_r(x)$. Clearly, $d_m(y, z) \geqslant d_M(\pi_M(y), \pi_M(z))$. For the converse inequality, we have to show that $d_m(y', z') \geqslant d_m(y, z)$ for any $y', z' \in \mathbb{H}^m$ with $\pi_M(y') = \pi_M(y)$ and $\pi_M(z') = \pi_M(z)$. By applying an element of $\mathcal{L}$ to y' and z', we can assume that $z' = z$. If $y = y'$ then there is nothing to do; otherwise $d_m(y, y') \geqslant g(\mathcal{L})$ and thus $d_m(y', z) \geqslant d_m(y', y) - d_m(y, z) \geqslant g - 2r \geqslant 2r \geqslant d_m(y, z)$, as required.

Since $\mathbb{H}^m$ has constant curvature -1, the same applies to M; so M is a hyperbolic manifold (compact, without boundary). Because the measure μ_m on $\mathbb{H}^m$ can be defined via the element of length (see, for example, [28, Section 3.4]), this definition carries over to M giving that there is r (which we can take again to be $\frac{1}{4}g(\mathcal{L})$) such that for every $x \in \mathbb{H}^m$ the restriction of π_M to $B_r(x)$ is measure-preserving. By the compactness of M, it can be covered by finitely many balls of radius $\frac{1}{4}g(\mathcal{L}) > 0$. Each of these balls has finite volume (since finite-radius balls in $\mathbb{H}^m$ have finite volume) so $\mu_M(M) < \infty$.

Lemma 5. Let a lattice $\mathcal{L} \subseteq \mathcal{G}$ have girth $g \geqslant 8R$. Define $M := \mathbb{H}^m/\mathcal{L}$ and let $\pi_M : \mathbb{H}^m \to M$ be the projection. Let $R > 0$ and let Y be an R-packing in (M, d_M). Define $X := \pi_M^{-1}(Y) \subseteq \mathbb{H}^m$. Then the following statements hold:

1) The set X is an R-packing in $(\mathbb{H}^m, d_m)$.
2) If the packing Y is relatively dense in (M, d_M) then X is relatively dense in $(\mathbb{H}^m, d_m)$.

Proof. Let us show the first claim that X is a packing. Take any distinct $x, x' \in X$ and let $y := \pi_M(x)$ and $y' := \pi_M(x')$. If $y \neq y'$, then $d_M(y, y') \geqslant 2R$ and, by the definition of d_M, we have $d_m(x, x') \geqslant 2R$. If $y = y'$ then, since π_M maps $B_{g/4}(x)$ isometrically to the ball of radius $g/4 \geqslant 2R$ around y in M and this ball contains $\pi_M(x') = y$, we have $d_m(x, x') > 2R$, giving that X is an R-packing, as desired.

Let us show that the packing $X \subseteq \mathbb{H}^m$ is relatively dense if Y is. Take any $x \in \mathbb{H}^m$. Since the packing Y is relatively dense, we have that $d_M(y, Y) \leqslant 2R$, where $y := \pi_M(x)$. As $\mathcal{L}$ acts by isometries and X is invariant under $\mathcal{L}$, we have

$$d_m(x, X) = d_m(\mathcal{L}.x, X) = d_m(\pi_M^{-1}(y), \pi_M^{-1}(Y)) = d_M(y, Y) \leqslant 2R.$$

By the compactness of, say, $B_{3R}(x)$, we have that $B_{2R}(x) \cap X \neq \emptyset$. Because $x \in \mathbb{H}^m$ was arbitrary, the packing X is indeed relatively dense. ■

Lemma 6. If $\mathcal{L} \subseteq \mathcal{G}$ is a lattice of girth at least $8R$ and $M = \mathbb{H}^m/\mathcal{L}$, then

$$D_R(\mathbb{H}^m) \geqslant D_R(M). \tag{10}$$

Proof. Take any relatively dense R-packing Y in M and let $X := \pi_M^{-1}(Y)$. Then X is a relatively dense R-packing in $\mathbb{H}^m$ by Lemma 5. Moreover, it is invariant under the action of the lattice $\mathcal{L} \curvearrowright \mathbb{H}^m$.

Let $\Gamma_X := \{\gamma \in \mathcal{G} \mid \gamma.X = X\}$ be the subgroup of $\mathcal{G}$ that fixes X. Clearly, $\mathcal{L} \subseteq \Gamma_X$.

Define the probability measure μ_X on S_R by

$$\mu_X(E) := \mu_{\mathcal{G}}(\{\gamma \in \mathcal{G} \mid \gamma.X \in E\}), \quad \text{for Borel } E \subseteq S_R.$$

Informally speaking, we take the translate of the given R-packing X by a random element of $\mathcal{G}$.

This measure is $\mathcal{G}$-invariant. Indeed, for every $\gamma' \in \mathcal{G}$ and $E \subseteq S_R$, we have by the invariance of $\mu_{\mathcal{G}}$ that

$$\begin{aligned}\mu_X(\gamma'.E) &= \mu_{\mathcal{G}}(\{\gamma \in \mathcal{G} \mid \gamma.X \in \gamma'.E\}) \\ &= \mu_{\mathcal{G}}(\gamma'.\{\gamma'' \in \mathcal{G} \mid \gamma''.X \in E\}) \\ &= \mu_{\mathcal{G}}(\{\gamma'' \in \mathcal{G} \mid \gamma''.X \in E\}) \;=\; \mu_X(E).\end{aligned}$$

By Lemma 3 we have that $D_R(\mathbb{H}^m) \geqslant D(\mu_X)$.

By [5, Proposition 1], the density $D(\mu_X)$ is equal to the relative volume taken by the R-balls around X inside any fundamental domain of $\Gamma_X \curvearrowright \mathbb{H}^m$. The lattice $\mathcal{L}$, as a subgroup of Γ_X, has finite index (which can be upper bounded by $k!$ where k is the maximum size of an R-packing in M, where k in turn can be upper bounded by $\mu_M(M)/\mu_m(B_R) < \infty$). Thus, a fundamental domain $\mathcal{F} \subseteq \mathbb{H}^m$ of $\mathcal{L} \curvearrowright \mathbb{H}^m$ can be obtained by taking the union of finitely many (pairwise disjoint) translates of a fundamental domain of $\Gamma_X \curvearrowright \mathbb{H}^m$ (one per each coset of $\Gamma_X/\mathcal{L}$). Since each of the latter translates has the same occupied ratio (by [5, Proposition 1]), this ratio is the same as that for their union $\mathcal{F}$. Note that the restriction of π_M to the fundamental domain $\mathcal{F}$ is a measure-preserving map between $(\mathcal{F}, \mu_m)$ and (M, μ_M). By the girth assumption on $\mathcal{L}$, the R-balls around points of the $\mathcal{L}$-periodic tiling $X \subseteq \mathbb{H}^m$ occupy the same fraction of volume of $\mathcal{F}$ as the R-balls around points of Y in M.

Thus, $D(\mu_X)$ is equal to the radius-R density of Y in M, implying the lemma. ■

Lemma 7. For every $m \in \mathbb{N}$ and $r \in (0, \infty)$, there is a lattice $\mathcal{L}$ of isometries of $\mathbb{H}^m$ of girth at least r.

Proof. This is Theorem 4.1 in [17] that contains a detailed proof. (The authors of [17] wrote that this fact had been known for a long time but they could not find a suitable reference.) ■

Recall that we defined $L_R(\mathbb{H}^m) := \mu(B_R)/\mu(B_{2R})$ to be the ratio of the volumes of balls of radius R and $2R$ in $\mathbb{H}^m$. We can now argue that this gives a lower bound on the Bowen–Radin packing density, just to show that this natural lower bound also holds in this framework.

Lemma 8. For every $m \in \mathbb{N}$ and $R > 0$, we have $D_R(\mathbb{H}^m) \geqslant L_R(\mathbb{H}^m)$.

Proof. Take any lattice $\mathcal{L} \subseteq \mathcal{G}$ of girth at least $8R$, which exists by Lemma 7. Take a maximal packing Y in $M := \mathbb{H}^m/\mathcal{L}$. By the maximality of Y, the $2R$-balls around points of Y cover the whole space M. By the girth assumption, for any $r \leqslant 2R$, the volume of any r-ball in M is the same as the volume of an r-ball in $\mathbb{H}^m$. Thus, $D_R(M) \geqslant L_R(\mathbb{H}^m)$. Now the lemma follows from (10). ■

5 Various Estimates

We will use various facts about hyperbolic functions $\cosh x := (e^x + e^{-x})/2$, $\sinh x := (e^x - e^{-x})/2$, $\tanh x := \frac{\sinh x}{\cosh x}$ and $\coth x := 1/\tanh x$. First, we have $\cosh^2 x = 1 + \sinh^2 x$. The following formulas for angle doubling are easy to check directly:

$$\cosh(2x) = 2\cosh^2 x - 1 \quad \text{and} \quad \sinh(2x) = 2\sinh x \cosh x.$$

Also, we will use the monotonicity of the above hyperbolic functions on $[0, \infty)$, approximations $\tanh x, \sinh x = (1 + o(1))x$ for $x \to 0$ and the following inequalities that are routine to check:

$$\tanh x < 1, \quad 2\sinh x \leqslant \sinh(2x), \quad \tanh(2x) \leqslant 2\tanh x, \quad \text{for } x \in [0, \infty). \tag{11}$$

Furthermore, we will use the following formula (see, e.g., [28, Theorem 5.3.5] for a proof).

Lemma 9 (The First Law of Cosines). If α, β, γ are the angles of a hyperbolic triangle and a, b, c are the lengths of the opposite sides, then

$$\cos\gamma = \frac{\cosh a \cosh b - \cosh c}{\sinh a \sinh b}. \tag{12}$$

Lemma 10. Let $x, u \in \mathbb{H}^m$ be two distinct points and let $\tau := d_m(x, u) > 0$. Let $r > 0$ satisfy $\tau < 2r$. Then the intersection $B_r(x) \cap B_r(u)$ is contained in (and thus has volume at most as that of) a hyperbolic ball of radius $\sigma = \sigma(\tau, r)$, where $\sigma > 0$ is defined by

$$\sinh^2(\sigma) = \sinh^2(r) - \cosh^2(r) \tanh^2(\tau/2). \tag{13}$$

Proof. Let w be the point on the geodesic line through x and u such that $d_m(x, w) = d_m(w, u)$. We will show that $B_\sigma(w)$ works in the lemma. Let $y \in B_r(x) \cap B_r(u)$ be any point at distance r_1 from x and r_2 from u for some $r_1, r_2 \leqslant r$. By the convexity of $B_r(x) \cap B_r(u)$, we can assume that y is not equal to x nor u, that is, that $r_1, r_2 > 0$. Consider the hyperbolic triangle with the vertices x, y, u and let α be the angle at the vertex u in this triangle. Let $\rho = d_m(y, w)$ be the distance between y and w.

Applying Lemma 9 to the hyperbolic triangle xyu and the hyperbolic triangle ywu, we obtain

$$\cos\alpha = \frac{\cosh\tau \cosh r_2 - \cosh r_1}{\sinh\tau \sinh r_2} \quad \text{and} \quad \cos\alpha = \frac{\cosh(\tau/2)\cosh r_2 - \cosh\rho}{\sinh(\tau/2)\sinh r_2}.$$

As these two expressions have the equal value, using the equalities $\sinh\tau = 2\sinh(\tau/2)\cosh(\tau/2)$ and $\cosh\tau = 2\cosh^2(\tau/2) - 1$, one can obtain

$$\begin{aligned}\cosh\rho &= \cosh(\tau/2)\cos r_2 - \frac{1}{2\cosh(\tau/2)}\left(\cosh\tau\cosh r_2 - \cosh r_1\right)\\ &= \cosh(\tau/2)\cosh r_2 - \frac{(2\cosh^2(\tau/2) - 1)\cosh r_2 - \cosh r_1}{2\cosh(\tau/2)}\\ &= \frac{\cosh r_2 + \cosh r_1}{2\cosh(\tau/2)} \leqslant \frac{\cosh r}{\cosh(\tau/2)}.\end{aligned}$$

The final inequality holds as $\cosh x$ is an increasing function of $x \geqslant 0$ and $r_1, r_2 \leqslant r$. This yields

$$\begin{aligned}\sinh^2(\rho) = \cosh^2(\rho) - 1 &\leqslant \frac{\cosh^2(r)}{\cosh^2(\tau/2)} - 1 = \cosh^2(r) - 1 - \frac{\cosh^2(r)(\cosh^2(\tau/2) - 1)}{\cosh^2(\tau/2)}\\ &= \sinh^2(r) - \cosh^2(r)\tanh^2(\tau/2).\end{aligned}$$

As $\sinh^2(x)$ is an increasing function on $[0, \infty)$, this shows that ρ is at most σ. This proves that any point $y \in B_r(x) \cap B_r(u)$ has distance at most σ from w, as desired. ■

The volume of an r-ball in $\mathbb{H}^m$ is given by the following formula (see, e.g., [9, Equation (III.4.1)]):

$$\mu(B_r) = \mathrm{vol}(S^{m-1}) \int_0^r \sinh^{m-1}\eta \, \mathrm{d}\eta,$$

where $\mathrm{vol}(S^{m-1})$ denotes the $(m-1)$-dimensional (surface) measure of the unit sphere in the Euclidean space $\mathbb{R}^m$. Recall that

$$\mathrm{vol}(S^{m-1}) = \frac{2\pi^{m/2}}{\Gamma(m/2)} = \mathrm{e}^{-(1/2+o_m(1))m\ln m}.$$

Moreover, we have for $r \to \infty$ and $m \geqslant 3$ that

$$\begin{aligned}\int_0^r \sinh^{m-1}\eta \, \mathrm{d}\eta &= (1 + O(\mathrm{e}^{-r(m-1)/2})) \int_{r/2}^r \left(\frac{\mathrm{e}^\eta - \mathrm{e}^{-\eta}}{2}\right)^{m-1} \mathrm{d}\eta\\ &= (1 + O(m\mathrm{e}^{-r})) \int_{r/2}^r \frac{\mathrm{e}^{(m-1)\eta}}{2^{m-1}} \mathrm{d}\eta = (1 + O(m\mathrm{e}^{-r})) \frac{\mathrm{e}^{(m-1)r}}{(m-1)2^{m-1}}.\end{aligned}$$

Using these, for $r \to \infty$ and any $m \geqslant 3$,

$$\mu(B_r) = (1 + O(me^{-r}))\,\frac{e^{(m-1)r}}{(m-1)2^{m-1}}\,\mathrm{vol}(S^{m-1}) = e^{(m-1)r-(1/2+o_r(1))m\ln m}. \tag{14}$$

We will also need the following bound on the ratio between $\mu(B_r)$ and $\mu(B_R)$ from [14, Lemma 4.6]:

Lemma 11. For $0 < r < R$ and the balls B_r, B_R of radii r, R in $\mathbb{H}^m$, we have

$$\left(\frac{\sinh r}{\sinh R}\right)^m \leqslant \frac{\mu(B_r)}{\mu(B_R)} \leqslant \left(\frac{\sinh r}{\sinh R}\right)^{m-1}.$$

Let us estimate the ratio $L_R(\mathbb{H}^m) = \mu(B_R)/\mu(B_{2R})$. Using the first equality in (14), we have for R satisfying $me^{-R} \to 0$ that

$$L_R(\mathbb{H}^m) \quad = \quad \frac{(1 + O(me^{-R}))\,e^{R(m-1)}}{(1 + O(me^{-2R}))\,e^{2R(m-1)}} = (1 + O(me^{-R}))\,e^{-R(m-1)}.$$

(If $me^{-R} \neq o(1)$ then we have to be more careful with the lower order terms.)

It is conceivable that, for any fixed $m \geqslant 2$, the function $L_R(\mathbb{H}^m)$ is monotone decreasing for $R \in (0, \infty)$, so it is never larger than the lower bound 2^{-m} from the Euclidean case (which is the limit as $R \to 0$).

Note that Lemmas 9–11 also hold in $M = \mathbb{H}^m/\mathcal{L}$ as long as all involved points are are within some ball in M of radius at most $g(\mathcal{L})/4$.

6 Proof of Theorem 1

Recall that Theorem 1 states that, for every $\varepsilon > 0$ and large m, we have

$$D_R(\mathbb{H}^m) \geqslant (1-\varepsilon)m\ln(\sqrt{m}\cosh(2R))\,\frac{\mu(B_R)}{\mu(B_{2R})}. \tag{15}$$

In order to prove this, we will use the following result of Campos, Jenssen, Michelen and Sahasrabudhe [8] on independent sets in a graph G with small *maximum codegree* $\Delta_2(G)$, which is the maximum over distinct $x, y \in V(G)$ of the number of common neighbours of x and y.

Theorem 12. Let G be a graph on n vertices such that its maximum degree $\Delta(G) \leqslant \Delta$ and its maximum codegree $\Delta_2(G) \leqslant 2^{-7}\Delta(\ln\Delta)^{-7}$. Then the independence number $\alpha(G)$ satisfies

$$\alpha(G) \geqslant (1 - o(1))\,\frac{n\ln\Delta}{\Delta},$$

where $o(1)$ tends to 0 as $\Delta \to \infty$.

Our proof of Theorem 1 goes as follows. Take any small constant $\varepsilon > 0$ and assume that $m \geqslant m_0(\varepsilon)$ is sufficiently large. Then take any number $R > 0$. Take a lattice $\mathcal{L} \subseteq \mathcal{G}$ of girth at least $16R$, which exists by Lemma 7. Let $M := \mathbb{H}^m/\mathcal{L}$ with measure $\mu = \mu_M$ and distance d_M. Consider $X \sim \mathrm{Po}_\lambda(M)$, that is, the Poisson process $X \subseteq M$ with intensity $\lambda\,\mu$ for some choice of $\lambda \in (0, \infty)$. Once we obtain X, we will delete some bad points from X to obtain Y, and consider G_Y, the graph with the vertex set Y and the edge set

$$E(G_Y) := \{\{x, y\} : x, y \in Y \text{ and } 0 < d_M(x, y) \leqslant 2R\}. \tag{16}$$

Based on our choice of λ and the removal process, the graph G_Y will satisfy the condition in Theorem 12. Thus Theorem 12 yields an independent set in G_Y, which corresponds to an R-packing in M. This gives a lower bound on $D_R(M)$ which is also a lower bound on $D_R(\mathbb{H}^m)$ by Lemma 6.

Let us give the details of the proof (for finding Y after $\mathcal{L}$ and M have been defined). We will use asymptotic notation, like $O(1)$, with respect to $m \to \infty$ for constants that can be chosen independently of R.

Given m and R, consider the following function of $x \in (0, R]$:

$$\gamma(x) := \begin{cases} m \cdot \tanh^2(x/2) - 50 \tanh^2(2R) \cdot \left(\ln(m) + \ln\ln\left(\frac{\sinh(2R)}{\sinh x}\right)\right), & \text{if } R < m, \\ \cosh^2(x/2) - m \ln(\cosh(2R)), & \text{if } R \geqslant m. \end{cases} \tag{17}$$

Note that, by the monotonicity of $\sinh$, the double logarithm in (17) is well-defined. We define $\tau = \tau(m, R)$ to be the number between 0 and R satisfying $\gamma(\tau) = 0$.

Let us show that such τ uniquely exists (if m is sufficiently large). Note that $\gamma(x)$ is a strictly increasing and continuous function of $x \in (0, R]$. So it is enough to show that it assumes both positive and negative values. If $R \geqslant m$ then we have rather generously that $\gamma(R) \geqslant e^R/4 - 2Rm > 0$ and $\gamma(\ln m) \leqslant m - m^2 < 0$, as desired. Moreover, we have $\tau > \ln m$ in this case. So let us consider the case $R < m$. Using $0 < \tanh(2R) < 4\tanh(R/2)$, we obtain that

$$\gamma(R) \geqslant \tanh^2(R/2)\left(m - O(\ln m)\right) > 0.$$

On the other hand, if we let $x \to 0$ then $\gamma(x)$ tends to $-\infty$. Thus, $\tau \in (0, R]$ satisfying $\gamma(\tau) = 0$ exists and is unique.

Claim 12.1. If $R < m$, then the following holds:

$$\frac{\sinh(2R)}{\sinh \tau} = \Theta\left(\cosh(2R)\sqrt{\frac{m}{\ln(m)}}\right).$$

Proof. Observe that $\tau = o(1)$ for otherwise $R \geqslant \tau \geqslant \Omega(1)$ and $\gamma(\tau) = \Omega(m)$ cannot be 0.

Let $\kappa := \ln m + \ln\ln\left(\frac{\sinh(2R)}{\sinh \tau}\right)$. By the monotonicity of $\sinh$ and by $\sinh 2x \geqslant 2\sinh x$ it holds that, for example,

$$\ln\left(\frac{\sinh(2R)}{\sinh x}\right) \geqslant \ln\left(\frac{\sinh(2R)}{\sinh R}\right) \geqslant \ln 2 \geqslant \frac{1}{\sqrt{m}}, \quad \text{for any } 0 < x \leqslant R. \tag{18}$$

Thus, $\kappa \geqslant \frac{1}{2}\ln m$.

It is enough to show that $\kappa = O(\ln m)$. Indeed, then $c := \frac{\tanh^2(\tau)}{\tanh^2(\tau/2)}$ satisfies $1 \leqslant c \leqslant 4$ and we have

$$\frac{\sinh^2(2R)}{\sinh^2(\tau)} = \frac{\cosh^2(2R)}{\cosh^2(\tau)} \cdot \frac{\tanh^2(2R)}{c\tanh^2(\tau/2)} = \frac{\cosh^2(2R)}{\cosh^2(\tau)} \cdot \frac{m}{50c\kappa} = \frac{\cosh^2(2R)}{\cosh^2(\tau)} \cdot \frac{m}{\Theta(\ln m)},$$

from which the claim follows by $\cosh \tau = 1 + o(1) = \Theta(1)$.

The proof of $\kappa = O(\ln m)$ is routine except we have to be careful to rule out the possibility that τ is extremely small relative to m and R. Suppose on the contrary that $\kappa \neq O(\ln m)$. Hence, $\ln(\frac{\sinh(2R)}{\tau}) = \omega(m)$. Then the identity $\gamma(\tau) = 0$ implies by $\tau = o(1)$ that

$$m\tau^2 = \Theta\left(\frac{\sinh^2(2R)}{1 + \sinh^2(2R)} \ln\ln\left(\frac{\sinh(2R)}{\tau}\right)\right) \tag{19}$$

It follows that $R = o(1)$ as otherwise $\tau \geqslant m^{-1/2+o(1)}$ by (19), contradicting $\ln(\frac{\sinh(2R)}{\tau}) = \omega(m)$. We obtain from (19) that $m = \Theta\left((R/\tau)^2 \ln\ln(R/\tau)\right)$. Since $m \to \infty$, it follows that $R/\tau \to \infty$ and thus $R/\tau = \Theta\left(\sqrt{m/\ln\ln m}\right)$, which again contradicts our assumption $\kappa \neq O(\ln m)$. ■

Now we define the parameters Δ and λ as follows:

$$\Delta := \frac{1}{m^4}\frac{\mu(B_{2R})}{\mu(B_\tau)} \quad \text{and } \lambda := \frac{\Delta}{\mu(B_{2R})}.$$

Recall that $\mu = \mu_M$ denotes the measure on M and that, for every $0 \leqslant r \leqslant 4R$, it assigns the same measure to an r-ball in M as the hyperbolic measure μ_m assigns to an r-ball in $\mathbb{H}^m$ by the girth assumption on $\mathcal{L}$.

Let us show that

$$\ln \Delta = (1 + o(1)) m \ln(\sqrt{m} \cosh(2R)). \tag{20}$$

Indeed, if $R < m$ then (20) is a consequence of Lemma 11 and Claim 12.1, so suppose that $R \geqslant m$. Then we have $\tau > \ln m$ and $\sinh \tau \leqslant \cosh \tau \leqslant 2\cosh^2(\tau/2)$, and thus (17) implies

$$0 < \ln(\sinh \tau) = O(\ln R + \ln m) = o(\ln(\sinh(2R))).$$

Thus, Lemma 11 implies that

$$\begin{aligned} \ln \Delta &= (m + O(1))(\ln(\sinh(2R)) - \ln(\sinh \tau)) + O(\ln m) \\ &= (1 + o(1)) m \ln(\sinh(2R)), \end{aligned}$$

giving (20) since $|\sinh(2R) - \cosh(2R)| \leqslant 1$ and $\ln(\sqrt{m}) = o(\ln(\cosh(2R)))$.

In particular, we have that $\Delta \to \infty$ as $m \to \infty$.

With these choices, we have the following lemma. Recall that G_Y is the graph on Y whose edge set is defined by (16).

Lemma 13. There exists $Y \subseteq M$ such that

$$|Y| \geqslant \left(1 - \frac{1}{m}\right) \frac{\Delta}{\mu(B_{2R})} \mu(M)$$

and the graph G_Y satisfies that

$$\Delta(G_Y) \leqslant \Delta + \Delta^{2/3} \text{ and } \Delta_2(G_Y) \leqslant \Delta (\ln \Delta)^{-10}.$$

With this lemma and Theorem 12, we obtain that there is an R-packing in M of size at least

$$(1 - o(1)) \frac{|Y| \ln\left(\Delta + \Delta^{2/3}\right)}{\Delta + \Delta^{2/3}} \geqslant (1 - 1/m - o(1)) \frac{\mu(M) \ln \Delta}{\mu(B_{2R})}.$$

Thus, (20) implies that, as $m \to \infty$,

$$D_R(\mathbb{H}^m) \geqslant D_R(M) \geqslant (1 - o(1)) \ln \Delta \cdot \frac{\mu(B_R)}{\mu(B_{2R})} \geqslant (1 - \varepsilon) m \ln(\sqrt{m} \cosh(2R)) \frac{\mu(B_R)}{\mu(B_{2R})}.$$

This proves (15) (that is, Theorem 1). Thus, it remains to prove Lemma 13.

Proof of Lemma 13. We follow the argument in Section 2 of [8]. In order to prove Lemma 13, we sample a Poisson point process $X \subseteq M$ with the intensity $\lambda \mu$, and obtain a desired set $Y \subseteq X$ by removing points $x \in X$ which satisfy at least one of the following two conditions:

$$|X \cap B_{2R}(x)| \geqslant \Delta + \Delta^{2/3} \text{ or } \exists y \in X \; |X \cap B_{2R}(x) \cap B_{2R}(y)| \geqslant \Delta (\ln \Delta)^{-10}. \tag{21}$$

We will show that Y has almost the same size as X. The following *identity of Mecke* will be useful: for any bounded measurable set $\Lambda \subseteq M$ and events $(A_x)_{x \in \Lambda}$ we have

$$\mathbb{E}\left|\{x \in X \cap \Lambda : A_x \text{ holds for } X\}\right| = \lambda \int_\Lambda \mathbb{P}[A_x \text{ holds for } X \cup \{x\}] \mathrm{d}\mu(x). \tag{22}$$

Also, the following tail bound for a Poisson random variable Z will be used:

$$\mathbb{P}[Z - \mathbb{E}Z \geqslant t\mathbb{E}Z] \leqslant \exp\left(-\min\{t, t^2\} \frac{\mathbb{E}Z}{3}\right). \tag{23}$$

First, we show the following claim stating that, on average, only a small fraction of $x \in X$ satisfies the first bad condition in (21). ■

Claim 13.1. Let $X \sim \mathrm{Po}_\lambda(M)$. Then,

$$\mathbb{E}\left|\{x \in X : |X \cap B_{2R}(x)| \geqslant \Delta + \Delta^{2/3}\}\right| \leqslant \frac{1}{m^2}\,\mathbb{E}|X|.$$

Proof. Take any $x \in M$. Recall that $|X \cap B_{2R}(x)|$ is a Poisson random variable of mean $\lambda\mu(B_{2R}) = \Delta$. Thus, using (23), we have

$$\mathbb{P}[\,|X \cap B_{2R}(x)| \geqslant \Delta + \Delta^{2/3} - 1\,] \leqslant \exp\left(-\frac{1}{4}\Delta^{1/3}\right) \leqslant m^{-2}.$$

Thus, we have by Mecke's identity that

$$\mathbb{E}\,\left|\{x \in X : |X \cap B_{2R}(x)| \geqslant \Delta + \Delta^{2/3}\}\right| = \lambda \int_M \mathbb{P}[\,|X \cap B_{2R}(x)| \geqslant \Delta + \Delta^{2/3} - 1\,]\mathrm{d}\mu(x) \leqslant \frac{\lambda\,\mu(M)}{m^2},$$

giving the desired by $\mathbb{E}|X| = \lambda\,\mu(M)$. ■

Before considering the second bad condition in (21), we prove the following claim.

Claim 13.2. For any $x, y \in M$ at distance $d_M(x, y) \geqslant \tau$, it holds that

$$\lambda\,\mu(B_{2R}(x) \cap B_{2R}(y)) \leqslant \Delta\,(\ln\Delta)^{-15}.$$

Proof. Note that we only have to consider the case where τ is at most $4R$, as otherwise $B_{2R}(x) \cap B_{2R}(y)$ is empty. As the lattice $\mathcal{L}$ has girth at least $16R$, Lemma 10 with $r = 2R$ applies to the points x, y in M (instead of $\mathbb{H}^m$). Hence, we obtain that the following where $\sigma = \sigma(2R, \tau) \in (0, 2R)$ was defined in (13):

$$\begin{aligned}\lambda\,\mu(B_{2R}(x) \cap B_{2R}(y)) &\leqslant \lambda\,\mu(B_\sigma) \leqslant \Delta\,\frac{\mu(B_\sigma)}{\mu(B_{2R})}\\ &\leqslant \Delta\,\frac{\sinh^{m-1}(\sigma)}{\sinh^{m-1}(2R)} = \Delta\,\left(1 - \coth^2(2R)\tanh^2(\tau/2)\right)^{(m-1)/2}.\end{aligned} \tag{24}$$

Here, the penultimate inequality is from Lemma 11.

Recall that τ satisfies $\gamma(\tau) = 0$ where γ was defined in (17). If $R < m$, then the final expression in (24) becomes

$$\Delta\left(1 - 50 \cdot \frac{\ln m + \ln\ln(\frac{\sinh 2R}{\sinh\tau})}{m}\right)^{(m-1)/2} \leqslant \Delta\,\frac{1}{m^{20}} \cdot \ln\left(\frac{\sinh 2R}{\sinh\tau}\right)^{-20} \leqslant \Delta\,(\ln\Delta)^{-15},$$

as we want. Here, the final inequality holds since (20) and Claim 12.1 imply that $\ln\Delta \leqslant m\ln(\frac{\sinh 2R}{\sinh\tau})$. If $R > m$, then as $\coth^2(2R)\tanh^2(\tau/2) \geqslant \tanh^2(\tau/2) = 1 - \frac{1}{\cosh^2(\tau/2)}$, the final term in (24) is bounded from above by

$$\begin{aligned}\Delta\left(\frac{1}{\cosh^2(\tau/2)}\right)^{(m-1)/2} &= \Delta\left(\frac{1}{m\ln(\cosh(2R))}\right)^{(m-1)/2}\\ &\leqslant \Delta\,(\ln\Delta)^{-(m-1)/4} \leqslant \Delta\,(\ln\Delta)^{-15},\end{aligned}$$

where the penultimate inequality follows from (20). This proves the claim. ■

Now we bound the number of points satisfying the second bad condition in (21).

Claim 13.3. Let $X \sim \mathrm{Po}_\lambda(M)$ and put $\eta := (\ln \Delta)^{-10}$. Then we have that

$$\mathbb{E}|\{x \in X : |X \cap B_{2R}(x) \cap B_{2R}(y)| \geqslant \eta\Delta \text{ for some } y \in X\}| \leqslant \frac{1}{2m}\,\mathbb{E}|X|.$$

Proof. Take any $x \in M$. For $y \in M$, let $I_{x,y} := |X \cap B_{2R}(x) \cap B_{2R}(y)|$. Using Markov's inequality, we have

$$\mathbb{P}[\,\exists y \in X : I_{x,y} \geqslant \eta\Delta - 1\,] \leqslant \mathbb{E}\,|B_\tau(x) \cap X| + \mathbb{E}\,|\{y \in X \setminus B_\tau(x) \mid I_{x,y} \geqslant \eta\Delta - 1\}|.$$

We bound each of these two terms.

For the first term, using Lemma 11 and the definition of Δ, we have

$$\mathbb{E}\,|B_\tau(x) \cap X| \leqslant \lambda\,\mu(B_\tau(x)) \leqslant \frac{\Delta\,\mu(B_\tau)}{\mu(B_{2R})} = \frac{1}{m^4}.$$

For the second term, we only need to consider $y \in B_{4R}(x)$ as otherwise $I_{x,y} = 0 < \eta\Delta - 1$. (Note that $\eta\Delta \to \infty$.) Using the identity of Mecke (i.e., (22)) and Markov's inequality, the second term is

$$\begin{aligned}\lambda \int_{M\setminus B_\tau(x)} \mathbb{P}[\,I_{x,y} \geqslant \eta\Delta - 2\,]\mathrm{d}\mu(y) &= \lambda \int_{B_{4R}(x)\setminus B_\tau(x)} \mathbb{P}[\,I_{x,y} \geqslant \eta\Delta - 2\,]\mathrm{d}\mu(y)\\ &\leqslant \lambda\,\mu(B_{4R}) \sup_{y\in B_{4R}(x)\setminus B_\tau(x)} \mathbb{P}[\,I_{x,y} \geqslant \eta\Delta - 2\,].\end{aligned}$$

Using Claim 13.2, we have for y with $d_M(y,x) \geqslant \tau$ that

$$\mathbb{E}I_{x,y} = \lambda\,\mu(B_{2R}(x) \cap B_{2R}(y)) \leqslant \Delta\,(\ln\Delta)^{-15}.$$

As $I_{x,y}$ is a Poisson random variable, we can apply (23) to obtain that, rather roughly,

$$\lambda\,\mu(B_{4R})\,\mathbb{P}[\,I_{x,y} \geqslant \eta\Delta - 2\,] \leqslant \Delta\,\frac{\mu(B_{4R})}{\mu(B_{2R})}\exp\left(-\Delta(\ln\Delta)^{-15}\right) < \frac{1}{m^2}.$$

The last inequality follow by observing by Lemma 11 and (20) that

$$\frac{\mu(B_{4R})}{\mu(B_{2R})} \leqslant \left(\frac{\sinh(4R)}{\sinh(2R)}\right)^m \leqslant e^{2Rm} \leqslant \frac{\exp\left(\Delta(\ln\Delta)^{-15}\right)}{\Delta\,m^2}.$$

Thus, we have shown that, for every $x \in X$,

$$\mathbb{P}[\,\exists y \in X : I_{x,y} \geqslant \eta\Delta - 1\,] \leqslant \frac{1}{m^4} + \frac{1}{m^2} < \frac{1}{2m}.$$

By Mecke's idenity, we conclude that the expected number of $x \in X$ satisfying the second bad condition in (21) is at most $\frac{1}{2m}\lambda\,\mu(M)$, proving the claim. ■

Now we prove Lemma 13. Let X be obtained from the Poisson point process in M with intensity $\lambda\,\mu$. Let $S_1, S_2 \subseteq X$ be the points that satisfy the first and second bad properties in (21). Let $Y := X \setminus (S_1 \cup S_2)$. Then the previous claims imply that

$$\mathbb{E}|Y| \geqslant \mathbb{E}|X| - \mathbb{E}|S_1| - \mathbb{E}|S_2| \geqslant \left(1 - \frac{1}{m^2} - \frac{1}{2m}\right)\mathbb{E}|X| \geqslant \left(1 - \frac{1}{m}\right)\frac{\Delta}{\mu(B_{2R})}\,\mu(M).$$

There is an outcome X such that $|Y|$ is at least its expected value, finishing the proof of Lemma 13.

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