Disjoint isomorphic balanced clique subdivisions

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A B S T R A C T

A classical result, due to Bollobás and Thomason, and independently Komlós and Szemerédi, states that there is a constant C such that every graph with average degree at least Ck^2 has a subdivision of K_k, the complete graph on k vertices. We study two directions extending this result.

- Verstraëte conjectured that a quadratic bound guarantees in fact two vertex-disjoint isomorphic copies of a K_k-subdivision.
- Thomassen conjectured that for each $k \in \mathbb{N}$ there is some $d = d(k)$ such that every graph with average degree at least d contains a balanced subdivision of K_k. Recently, Liu and Montgomery confirmed Thomassen's conjecture, but the optimal bound on $d(k)$ remains open.

In this paper, we show that a quadratic lower bound on average degree suffices to force a balanced K_k-subdivision. This gives the right order of magnitude of the optimal $d(k)$ needed in Thomassen's conjecture. Since a balanced K_{mk}-

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subdivision trivially contains m vertex-disjoint isomorphic K_k-subdivisions, this also confirms Verstraëte's conjecture in a strong sense.

1. Introduction

A *subdivision* of a graph H is obtained by replacing each edge of H by a path, so that the new paths are internally vertex disjoint. This notion has played a central role in topological graph theory since the seminal result of Kuratowski in 1930 that a graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$. In this paper, we are specifically interested in the optimal average degree for forcing particular subdivisions of a clique.

One of the first results in this direction was proved by Mader [17], who showed that there is some $d = d(k)$ such that every graph with average degree at least d contains a subdivision of the complete graph K_k. After some further results by Mader [19], it was proved by Bollobás and Thomason [3], and independently by Komlós and Szemerédi [11], that we may take $d(k) = O(k^2)$. This is optimal: e.g., the complete balanced bipartite graph on $k^2/4$ vertices contains no subdivision of K_ℓ for $\ell \geq k$; indeed such a subdivision would require at least $\ell + \binom{\ell}{2} - (\frac{\ell}{2})^2 > \frac{k^2}{4}$ vertices, as at most $(\frac{\ell}{2})^2$ pairs of vertices can be embedded as adjacent pairs and each non-adjacent pair would require at least one additional vertex.

Twenty years ago, a strengthening of the result of Bollobás-Thomason and Komlós-Szemerédi was conjectured by Verstraëte [24], who believed that the quadratic bound $O(k^2)$ suffices also to guarantee a pair of disjoint isomorphic subdivisions of K_k. This can be seen as a natural generalisation of the problem of finding disjoint cycles of the same length in a given graph. Such problem has received considerable attention since the work of Corradi and Hajnal [4], who showed that for any positive integer k, any graph of order at least $3k$ and minimum degree at least $2k$ contains k disjoint cycles.

A different direction of extension was proposed by Thomassen [21] (see also [22,23]), who conjectured that, for each $k \in \mathbb{N}$, there is some $d = d(k)$ such that every graph with average degree at least d contains a balanced subdivision of K_k. Here, a subdivision is *balanced* if every added path is of the same length. Very recently, Liu and Montgomery [14] confirmed this conjecture, but it remains to determine optimal bounds for $d(k)$. Clearly, we have by $d(k) = \Omega(k^2)$ by the same complete bipartite graph example above. Very recently, Wang [25] proved that, for all sufficiently large d, every n-vertex graph with average degree at least d contains a balanced subdivision of K_r, where $r = \Omega(d^{1/2}/\log^{10} n)$.

Our main result simultaneously settles the conjecture of Verstraëte and gives optimal bounds for $d(k)$ in Thomassen's conjecture. It shows that the quadratic bound is optimal

in forcing balanced disjoint isomorphic clique subdivisions. Indeed, simply notice that a balanced K_{tk}-subdivision contains t disjoint isomorphic balanced K_k-subdivisions.

Theorem 1.1. *Every graph with average degree d contains a balanced subdivision of a complete graph of order $\Omega(\sqrt{d})$.*

Our approach uses a version of expander called sublinear expanders. For more recent applications of the theory of sublinear expanders, we refer the interested readers to [5–10,13–15].

Organisation   The rest of the paper is organised as follows. Preliminaries are given in Section 2. In Section 2.1 we reduce to proving Theorem 1.1 in the case when G is a bipartite expander graph. Then in Section 2.2, we further reduce to proving Theorem 1.1 when $n/K \geq d \geq \log^{800}(n)$ for some large constant K (stated as Theorem 2.8). Section 3 contains a proof sketch of Theorem 2.8. In Section 4, we demonstrate the existence of a certain structure, called a *unit*, in an expander graph. In Section 5, we use these units to build a certain absorbing structure introduced by Liu and Montgomery [14] called an *adjuster*, which will allow us to 'adjust' the length of a path between two points under certain conditions. In Section 6, we will use these adjusters to prove Theorem 2.8. Section 7 contains concluding remarks.

2. Definitions and some auxiliary results

Notation   For sets X and Y, we define $X \times Y := \{(x, y) : x \in X, y \in Y\}$. We often omit brackets when writing small sets, for example, abbreviating $\{x\}$ and $\{x, y\}$ to x and xy, respectively. For $\ell \in \mathbb{N}$, we define $[\ell] := \{1, 2, \ldots, \ell\}$. We omit floor and ceiling signs when they are not essential, that is, we treat large numbers as integers. We sometimes write $(a, b, c, d)_X = (a', b', c', d')$ meaning that we choose constants a, b, c, d in the statement of Result X to be a', b', c', d'.

Let G be a graph. We will denote the set of vertices of G by $V(G)$ and the set of edges by $E(G)$, and define $|G| := |V(G)|$ and $e(G) := |E(G)|$. Let $A \subseteq V(G)$. We define $G[A]$ to be the *subgraph of G induced by A* with vertex set A and edge set $\{xy \in E(G) : x, y \in A\}$. Further, we define $G - A := G[V(G) \setminus A]$ to be the graph with vertex set $V(G) \setminus A$ and edge set $E(G) \setminus \{vw \in E(G) : v \in A \text{ or } w \in A\}$. We define the *external neighbourhood of A in G* to be

$$N_G(A) := \{w \in V(G) \setminus A : \exists\, v \in A \text{ such that } vw \in E(G)\}.$$

For $k \in \mathbb{N}$ we define the *ball of radius k around A in G*, denoted by $B_G^k(A)$, to be the set of vertices with graph distance at most k from a vertex in A. For $v \in V(G)$, we define the *degree of v in G* to be $d_G(v) := |N_G(v)|$. Let

$$d(G) := \frac{1}{|G|} \sum_{v \in V(G)} d_G(v)$$

denote the *average degree* of G. For $A, B \subseteq V(G)$, we define $N_G(A, B) := N_G(A) \cap B$. We define an A, B-*path in* G to be a path which has one endpoint in A and the other endpoint in B and has no other vertices in A or B. For a subgraph $F \subseteq G$, we define $G \setminus F$ to be the graph with vertex set $V(G)$ and edge set $E(G) \setminus E(F)$. The *length* of a path P is $e(P) = |P| - 1$, the number of edges in it. We say a collection of paths $\mathcal{P}$ is *internally vertex disjoint* if for each pair of paths $P_1, P_2 \in \mathcal{P}$ the set of internal vertices of P_1 is disjoint from $V(P_2)$; in other words, if a vertex belongs to two different paths of $\mathcal{P}$ then it is an endpoint in both. We note that sometimes, when it is clear from context, we drop the subscript G from the above nomenclature. For $h \in \mathbb{N}$, we define an h-*star* to be the graph on $h + 1$ vertices where one vertex has degree h and all other vertices have degree one.

For $\ell, t \in \mathbb{N}$, we write $\mathsf{TK}_t^{(\ell)}$ for a balanced K_t-subdivision in which each edge is replaced by a path with ℓ internal vertices (i.e., a path of length $\ell + 1$).

2.1. Robust Komlós-Szemerédi expansion

We use the following notion of expansion introduced by Haslegrave, Kim and Liu [9], which is essentially a robust form of the sublinear expansion property introduced by Komlós and Szemerédi in [11,12]. Informally speaking, this property states that even after removing a relatively small set of edges we can still guarantee sublinear expansion properties.

Definition 2.1. Let $\varepsilon_1 > 0$ and $k \in \mathbb{N}$. A graph G is an (ε_1, k)-*robust-expander* if for all $X \subseteq V(G)$ with $k/2 \le |X| \le |G|/2$, and any subgraph $F \subseteq G$ with $e(F) \le d(G) \cdot \varepsilon(|X|) \cdot |X|$, we have

$$|N_{G \setminus F}(X)| \ge \varepsilon(|X|) \cdot |X|,$$

where

$$\varepsilon(x) = \varepsilon(x, \varepsilon_1, k) := \begin{cases} 0 & \text{if } x < k/5, \\ \varepsilon_1 / \log^2(15x/k) & \text{if } x \ge k/5. \end{cases} \tag{1}$$

Observe that $\varepsilon(x, \varepsilon_1, k)$ decreases for $x \ge k/2$, but $\varepsilon(x, \varepsilon_1, k) \cdot x$ increases for $x \ge \frac{e^2 k}{15}$ (in particular, for $x \ge k/2$).

Importantly, Komlós and Szemerédi [12] proved that every graph G contains an expander subgraph with comparable average degree to G, and Haslegrave, Kim and Liu [9, Lemma 3.2] proved the analogous result for robust-expanders. From now on we will refer to a robust-expander as an expander.

The following is a direct consequence of [9, Lemma 3.2] and the well known facts that every graph G has a bipartite subgraph H with $d(H) \geq d(G)/2$ and that every graph H contains a subgraph of minimum degree at least $d(H)/2$.

Lemma 2.2. *Let $\varepsilon_1 \leq 1/1000$, $\varepsilon_2 < 1/2$ and $d > 0$. Every graph G with $d(G) \geq d$ has a bipartite $(\varepsilon_1, \varepsilon_2 d)$-expander subgraph H with $\delta(H) \geq d/8$.*

Note that the expansion subgraph $H \subseteq G$ found in Lemma 2.2 may be significantly smaller than $|G|$. Indeed, G could be disjoint union of many copies of H.

A common use of the expansion condition is to connect two vertex sets together by a short path, which, as the following result shows, we can do even after removing a relatively smaller set of vertices. We will use the following version, which is a slight variation of Lemma 3.4 in [14].

Lemma 2.3. *For each $0 < \varepsilon_1, \varepsilon_2 < 1$, there exists $d_0 = d_0(\varepsilon_1, \varepsilon_2)$ such that the following holds for each $n \geq d \geq d_0$ and $x \geq 1$. Let G be an n-vertex $(\varepsilon_1, \varepsilon_2 d)$-expander with $\delta(G) \geq d - 1$. Let $A, B \subseteq V(G)$ with $|A|, |B| \geq x$, and let $W \subseteq V(G) \setminus (A \cup B)$ satisfy*

$$|W| \leq \frac{\varepsilon_1 x}{4 \log^2(\frac{15n}{\varepsilon_2 d})}.$$

Then, there is an A, B-path in $G - W$ with length at most $\frac{100}{\varepsilon_1} \log^3 \frac{15n}{\varepsilon_2 d}$.

Proof. We first prove the following claim.

Claim 2.4. *Set $m_0 := \frac{40}{\varepsilon_1} \log^3 \frac{15n}{\varepsilon_2 d}$. Let $C \subseteq V(G - W)$ with $|C| \geq \varepsilon_2 d/2$. Let $\varepsilon(y) = \varepsilon(y, \varepsilon_1, \varepsilon_2 d)$ be the function defined in (1). If $|W| \leq \varepsilon(|C|) \cdot |C|/4$, then $|B_{G-W}^{m_0}(C)| > n/2$.*

Proof of claim. For each $i \in \mathbb{N}$, denote $C_i := B_{G-W}^i(C)$ and set $C_0 := C$. Using the expansion property of G (with F as the empty graph), our assumption that $|W| \leq \varepsilon(|C|) \cdot |C|/4$ and the fact that $\varepsilon(y) \cdot y$ increases for $y \geq \varepsilon_2 d/2$, we have for any $i \leq m_0 + 1$ that $|C_{i-1}| > n/2$ (and so the claim holds by $|C_{m_0}| \geq |C_{i-1}|$), or

$$|N_G(C_{i-1})| \geq \varepsilon(|C_{i-1}|) \cdot |C_{i-1}| \geq \varepsilon(|C|) \cdot |C| \geq 4 \cdot |W|.$$

Assuming that we are not done yet, we have

$$|N_{G-W}(C_{i-1})| \geq |N_G(C_{i-1})| - |W| \geq \frac{1}{2} |N_G(C_{i-1})|.$$

Now, using the above two inequalities and that $\varepsilon(y)$ decreases for $y \geq \varepsilon_2 d/2$, we bound

$$|N_{G-W}(C_{i-1})| \geq \frac{1}{2} \varepsilon(|C_{i-1}|) \cdot |C_{i-1}| \geq \frac{1}{2} \varepsilon(n) \cdot |C_{i-1}|.$$

Then, since $|C| \geq \varepsilon_2 d/2$ we have

$$|C_{m_0}| \geq \left(1 + \frac{1}{2}\, \varepsilon(n)\right)^{m_0} \cdot \frac{\varepsilon_2 d}{2}.$$

Finally, we can solve $\left(1 + \frac{1}{2}\, \varepsilon(n)\right)^r \cdot \varepsilon_2 d/2 > n/2$ for r, and using the inequality $\log(1+x) \geq (1 + \frac{1}{x})^{-1}$, see that the desired inequality holds for $r = m_0$. $\square$

We have two cases. If $x \geq \varepsilon_2 d/2$, we have

$$\varepsilon(x) \cdot \frac{x}{4} = \frac{\varepsilon_1 x}{4 \log^2\left(\frac{15x}{\varepsilon_2 d}\right)} \geq |W|.$$

Then by Claim 2.4 applied to A and B, we have that $|B^{m_0}_{G-W}(A)|, |B^{m_0}_{G-W}(B)| > n/2$, which implies that there exists an A, B-path in $G - W$ with length at most

$$2m_0 \leq \frac{100}{\varepsilon_1} \log^3 \frac{15n}{\varepsilon_2 d},$$

as desired.

If $x < \varepsilon_2 d/2$, take vertices $a \in V(A)$ and $b \in V(B)$ and let $N_{G-W}(a) =: A'$ and $N_{G-W}(b) =: B'$. Observe that since

$$|W| \leq \frac{\varepsilon_1 \varepsilon_2 d}{4 \log^2\left(\frac{15n}{\varepsilon_2 d}\right)} \leq \varepsilon_2 d/4$$

and $d(a), d(b) \geq d-1$, we have that $|A'|, |B'| \geq \varepsilon_2 d/2 > x$. Hence we have $\varepsilon(|A'|) \cdot |A'|/4 \geq |W|$ and $\varepsilon(|B'|) \cdot |B'|/4 \geq |W|$. Then by Claim 2.4 applied to A' and B', we have that $|B^{m_0}_{G-W}(A')| > n/2$ and $|B^{m_0}_{G-W}(B')| > n/2$, which implies that there exists an A, B-path in $G - W$ with length at most

$$2m_0 + 2 \leq \frac{100}{\varepsilon_1} \log^3 \frac{15n}{\varepsilon_2 d},$$

as desired. $\square$

2.2. Reducing Theorem 1.1

Using Lemma 2.2, we can assume that G is a bipartite expander graph. If G is very dense, a classic result of Alon, Krivelevich and Sudakov [1, Theorem 6.1] provides a balanced 1-subdivision of a clique on $\Omega(\sqrt{|G|})$ vertices.

Theorem 2.5. *Let $\alpha > 0$. If G is a graph with n vertices and average degree αn, then G contains a copy of $\mathsf{TK}_r^{(1)}$ where $r := \alpha n^{1/2}/2$.*

For sparse expanders, the following result of Wang [25, Lemma 1.3] provides a balanced clique subdivision of size linear in its average degree.

Theorem 2.6. *There exists $\varepsilon_1 > 0$ such that for any $0 < \varepsilon_2 < 1/5$ and $s \geq 20$, there exist $d_0 = d_0(\varepsilon_1, \varepsilon_2, s)$ and some constant $t > 0$ such that the following holds for each $n \geq d \geq d_0$ and $d < \log^s n$. Suppose that G is an n-vertex bipartite $(\varepsilon_1, \varepsilon_2 d)$-expander graph with $\delta(G) \geq d$. Then G contains a copy of $\mathsf{TK}_{td}^{(\ell)}$ for some $\ell \in \mathbb{N}$.*

By Theorems 2.5 and 2.6, to prove Theorem 1.1 it suffices to prove the following:

Theorem 2.7. *For each $0 < \varepsilon_1, \varepsilon_2 < 1$, the following holds for all sufficiently large $K = K(\varepsilon_1, \varepsilon_2)$. Let G be an n-vertex bipartite $(\varepsilon_1, \varepsilon_2 d)$-expander with $\delta(G) \geq d$, $n \geq Kd$ and $d \geq \log^{800} n$. Then G contains a copy of $\mathsf{TK}_{\sqrt{d}}^{(\ell)}$ for some $\ell \in \mathbb{N}$.*

For brevity, throughout this paper we set

$$m := \left\lfloor \log^4 \frac{n}{d} \right\rfloor.$$

Actually, we shall prove a stronger version of Theorem 2.7:

Theorem 2.8. *For each $0 < \varepsilon_1, \varepsilon_2 < 1$, the following holds for all sufficiently large $K = K(\varepsilon_1, \varepsilon_2)$. Let G be an n-vertex bipartite $(\varepsilon_1, \varepsilon_2 d)$-expander with $\delta(G) \geq d$, $n \geq Kd$ and $d \geq \log^{800} n$. Then G contains a copy of $\mathsf{TK}_{\sqrt{dm}}^{(\ell)}$ for $\ell = 1$ or $\ell = 80m^3$.*

Note that $m \geq \lfloor \log^4 K \rfloor$, so we can choose K sufficiently large to ensure that m is large enough for all later statements and proofs. Moreover, since $n/d \geq K$, taking K sufficiently large we can assume that $n \geq dm^{200}$.

3. Proof sketch of Theorem 2.8

Assume that the graph G contains no $\mathsf{TK}_{\sqrt{dm}}^{(1)}$. Thus we have to find a copy of $\mathsf{TK}_{\sqrt{dm}}^{(\ell)}$, where $\ell := 80m^3$. One immediate obstruction to a naive greedy construction is that the desired subdivision has $\binom{\sqrt{dm}}{2} \ell \geq dm$ vertices which is much larger than our lower bound d on the minimal degree $\delta(G)$. That is, if we were to construct a copy of $\mathsf{TK}_{\sqrt{dm}}^{(\ell)}$ one path at a time, we may arrive at a point when all unused vertices in G have neighbours only internal to the previously constructed paths. This would be overcome if there existed sufficiently many vertices of degree $\Omega\left(\binom{\sqrt{dm}}{2}\ell\right)$, but we cannot guarantee this.

However, using the expansion property of G (in particular, Lemma 2.3) we can find $\sqrt{dm}$ rooted trees, called *units*, each containing $\Theta(dm^{28})$ leaves, all at the same distance from the root (see Definition 4.1 and Fig. 1). Also, each constructed unit will have very small *interior* (that is, the set of its non-leaf vertices), namely of size at most $O(\sqrt{dm^8})$, and any two distinct units will have disjoint interiors. Thus we would like to find, for

every pair of units, a path between their *boundaries* (that is, their sets of leaves) so that the paths extended to the roots all have length ℓ and are internally disjoint.

For this step, we create certain structures called *adjusters*, introduced by Liu and Montgomery in [14]. While the adjusters were constructed in sparse expanders in [14], the bulk of the work in our paper is to construct adjusters in dense expanders whose average degree could be a polynomial of the order of the expander. Roughly speaking, a (j, k)-*adjuster* consists of two units rooted at x and y together with a collection of (not necessarily internally vertex disjoint) x, y-paths that have lengths $\ell', \ell' + 2, \ell' + 4, \ldots, \ell' + 2k$ for some $\ell' \leq j$. We call $\ell' + 2k$ the *length* of the adjuster. We first construct an $(O(m), 1)$-adjuster (Lemma 5.3). We then chain together $(O(m), 1)$-adjusters to form an $(O(m^3), \Omega(m^2))$-adjuster whose length is contained in some fixed interval of length $o(m^2)$ (Lemma 5.4). Such an $(O(m^3), \Omega(m^2))$-adjuster can then be used along with Lemma 2.3 to construct a path between the boundaries of any two given units that has the desired length and also avoids any given relatively small set W because when finding such paths, one may need to avoid certain vertices, e.g. previously used vertices (Lemma 6.1). Namely, we connect the two unit boundaries to the two opposite ends of the adjuster via short paths and then choose a path of the appropriate length inside the adjuster.

The proof is completed by connecting each pair of roots of the $\sqrt{d}m$ constructed units, one by one in some order, by a path of length ℓ through a new $(O(m^3), \Omega(m^2))$-adjuster for each pair as above (satisfying appropriate disjointness conditions in each of these steps). Of course, there are a number of technical issues to take care of such as, for example, making sure that a bulk of each unit remains available throughout the whole procedure.

4. Building units

Definition 4.1. An (h_0, h_1, ℓ)-*unit* F consists of a *core* vertex v and h_0 pairwise vertex disjoint h_1-stars S_{u_i} in $F - v$, with centres u_i respectively, $i \in [h_0]$, along with v, u_i-paths P_i, which are internally vertex disjoint from each other and $\cup_{i \in [h_0]} V(S_{u_i})$. Furthermore, all paths P_i are of length exactly s, for some $s < \ell$. We call the union of all vertices in the paths $\mathrm{int}(F) := \cup_{i \in [h_0]} V(P_i)$ the *interior of* F, and $\mathfrak{bd}(F) := V(F) \setminus \mathrm{int}(F)$ the *boundary of* F. We say that two units are *disjoint* if their interiors are vertex-disjoint.

See Fig. 1 for an illustration of a unit. We now show the existence of a unit in a dense graph after removing a relatively small set of vertices.

Lemma 4.2. *For each* $0 < \varepsilon_1, \varepsilon_2 < 1$, *the following holds for all sufficiently large* $K = K(\varepsilon_1, \varepsilon_2)$. *Let* G *be an* n-*vertex bipartite* $(\varepsilon_1, \varepsilon_2 d)$-*expander with* $\delta(G) \geq d$, $n \geq Kd$ *and* $d \geq m^{200}$. *Then, given any* $W \subseteq V(G)$ *with* $|W| \leq dm^{30}$, $G - W$ *contains a* $(4\sqrt{d}m^6, \sqrt{d}m^{22}, 10m)$-*unit.*
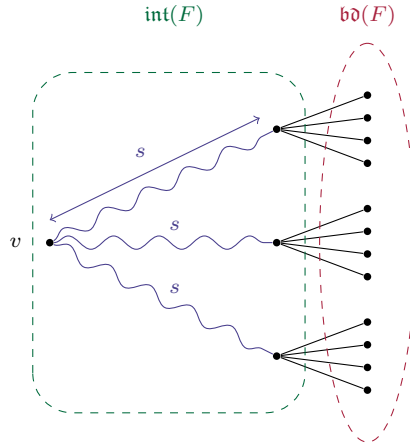
Fig. 1. A $(3, 4, \ell)$-unit F.

Proof. We will first construct $m^{40} + \sqrt{d}m^{49}$ disjoint stars, m^{40} of which will have d/m^5 leaves each and $\sqrt{d}m^{49}$ of which will have $\sqrt{d}m^{24}$ leaves each. We will then show that some collection of stars can be connected together (possibly losing some leaves of the stars in the process) in order to create a $(4\sqrt{d}m^6, \sqrt{d}m^{22}, 10m)$-unit.

Claim 4.3. Let $X \subseteq V(G)$. If $|X| \leq dm^{74}$, then there exists a vertex $v \in G - X$ such that $d_{G-X}(v) \geq d/m^5$.

Proof of claim. Suppose not. Then $\Delta(G - X) < d/m^5$. Recall that, since $n \geq Kd$, we must have $n \geq dm^{200}$. Thus, since $|X| \leq dm^{74}$, we can take a subset of vertices $U \subseteq V(G) \setminus X$ with $|U| = dm^{76}$. Define F be the graph on $V(G) \setminus X$ with edge set

$$E(F) := \{uv \in E(G) : u \in U, v \in V(G) \setminus X\}$$

and no isolated vertices. Then

$$e(F) \leq \Delta(G - X) \cdot |U| \leq \frac{d\,|U|}{m^5} \leq d(G) \cdot \varepsilon(|U|, \varepsilon_1, \varepsilon_2 d) \cdot |U|.$$

Observe that $N_{G \setminus F}(U) \subseteq X$. Hence $|N_{G \setminus F}(U)| \leq |X| \leq dm^{74}$. But, as G is an expander, we have that $|N_{G \setminus F}(U)| \geq \frac{1}{2}|U|\,\varepsilon(|U|, \varepsilon_1, \varepsilon_2 d) \geq dm^{75}$, which is a contradiction. □

Now, iteratively apply Claim 4.3 to $G - W$ a total of m^{40} times, at each iteration i, $1 \leq i \leq m^{40}$, removing a star S_i with centre u_i and d/m^5 leaves from the current graph and adding it to a set S^1. Let $V(S^1)$ be the set of vertices in the stars in S^1 and observe that all stars in S^1 are vertex disjoint. Observe that one can do this because throughout the process $|V(S^1)| \leq dm^{35}$ and thus $|W \cup V(S^1)| \leq 2dm^{40} \leq dm^{73}$ also.

Next, we iteratively apply Claim 4.3 to $G - (W \cup S^1)$ a further $\sqrt{d}m^{49}$ times, at each iteration i, $m^{40} + 1 \leq i \leq m^{40} + \sqrt{d}m^{49}$, removing a star S_i with centre u_i and

$\sqrt{d}m^{24}$ leaves from the current graph and adding it to a set S^2. Let $V(S^2)$ be the set of vertices in the stars in S^2 and observe that all stars in S^2 are vertex disjoint. Observe that one can do this since $d \geq m^{200}$ implies $\sqrt{d}m^{24} \leq d/m^5$, and throughout the process $|V(S^2)| \leq dm^{73}$ and thus $|W \cup V(S^1) \cup V(S^2)| \leq dm^{74}$ also.

So now we have in $G - W$ a set S^1 of m^{40} stars each with d/m^5 leaves and a set S^2 of $\sqrt{d}m^{49}$ stars each with $\sqrt{d}m^{24}$ leaves, such that all stars in $S^1 \cup S^2$ are vertex disjoint from each other. Let $\ell := m^{40} + \sqrt{d}m^{49}$. Take $I \subseteq [m^{40}] \times [\ell]$ to be a maximal subset for which there are paths $P_{(i,j)}$, $(i,j) \in I$, in $G - W$ such that the following holds.

- For each $(i,j) \in I$, $P_{(i,j)}$ is a u_i, u_j-path with length at most $2m$ which is disjoint from $\{u_k : k \in [\ell] \setminus \{i,j\}\}$.
- The paths are internally vertex disjoint.

Suppose there is some $i \in [m^{40}]$ and $J \subseteq [\ell]$ with $(i,j) \in I$, for each $j \in J$, with $|J| = \sqrt{d}m^8$. Let $U := \cup_{j \in J} V(P_{(i,j)})$, and note that $|U| \leq 2 \cdot \sqrt{d}m^9$. Hence, for any $j \in J$,

$$|V(S_j) \setminus U| \geq \sqrt{d}m^{24}/2 \geq \sqrt{d}m^{23}.$$

By pigeonhole, we can pick a subset $J' \subseteq J$ such that all paths $P_{(i,j)}$ with $j \in J'$ have the same length, and $|J'| \geq |J|/2m \geq 4\sqrt{d}m^6$. Taking stars $S_{j'} \subseteq S_j - U$, $j \in J'$, with $\sqrt{d}m^{22}$ leaves, and the paths $P_{(i,j)}$, $j \in J'$, we get a $(4\sqrt{d}m^6, \sqrt{d}m^{22}, 10m)$-unit.

Suppose then that there are no such $i \in [m^{40}]$ and $J \subseteq [\ell]$. Let $J_1 := \{u_1, \ldots, u_{m^{40}}\}$, and let $J_2 \subseteq [\ell]$ be a maximal set such that there is no $j_1 \in J_1$ and $j_2 \in J_2$ with $(j_1, j_2) \in I$. Then, $|J_2| \geq \ell - |J_1| \cdot \sqrt{d}m^8 \geq \sqrt{d}m^{24}$. Let

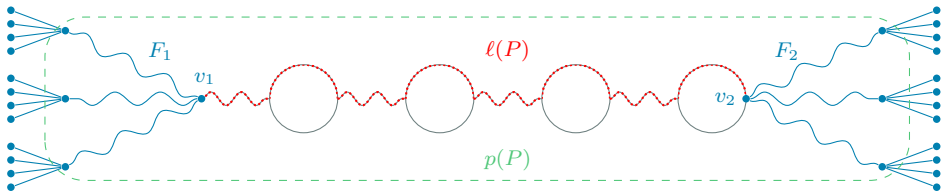$$U := \bigcup_{i \in J_1, \{i,j\} \in I} V(P_{(i,j)} \setminus \{u_i, u_j\}).$$

Using that $d \geq m^{200}$ we have that $|U| \leq m^{40} \cdot \sqrt{d}m^8 \cdot 2m \leq dm^{30}$, and hence $|W \cup U| \leq 2dm^{30}$. Also, observe that

$$|\cup_{i \in J_1} (V(S_i) \setminus U)| \geq m^{40} \cdot d/m^5 - dm^{30} \geq dm^{34} \geq \log^3\left(\frac{n}{d}\right)|W \cup U|$$

and

$$|\cup_{i \in J_2} (V(S_i) \setminus U)| \geq \sqrt{d}m^{49} \cdot \sqrt{d}m^{24} - dm^{30} \geq dm^{71} \geq \log^3\left(\frac{n}{d}\right)|W \cup U|.$$

Hence, by Lemma 2.3, and that $n \geq Kd$ with K sufficiently large, we can find a path connecting $\cup_{i \in J_1}(V(S_i) \setminus U)$ and $\cup_{i \in J_2}(V(S_i) \setminus U)$ which avoids $W \cup U$ and has length at most $2m$. This contradicts the maximality of I. $\square$

Fig. 2. An example of an adjuster P we shall build, in which A is a string of even cycles, joined by paths connecting at almost antipodal vertices of the cycles. The length $\ell(P)$ is found by traversing all the longest paths between the almost antipodal vertices on the cycles.

5. Building adjusters

In order to ensure the clique subdivision we construct is balanced, we utilise special structures called *adjusters*, introduced by Liu and Montgomery in [14].

Definition 5.1. An (ℓ, k)-*adjuster* $P = (v_1, v_2, F_1, F_2, z, A, \mathcal{P})$ in a graph G consists of two vertices v_1, v_2 that are the core vertices of two $(z\sqrt{d}m^6, \sqrt{d}m^{22}, 10m)$-units F_1 and F_2, respectively, where $|\text{int}(F_1) \cap \text{int}(F_2)| = \varnothing$, a real number $z \in [1, 3]$, a vertex set $A \subseteq V(G)$ of size at most 2ℓ disjoint from $\cup_{i \in [2]} V(F_i)$, and a collection $\mathcal{P}$ of $k+1$ v_1, v_2-paths in $G[A \cup \{v_1, v_2\}]$ of lengths $\ell', \ell' + 2, \ell' + 4, \ldots, \ell' + 2k$ for some $\ell' \le \ell$. Further, define $\ell(P) := \ell' + 2k$ and call it the *length of P*. We also define the *perimeter* $p(P) := A \cup \text{int}(F_1) \cup \text{int}(F_2)$.

We refer the Reader to Fig. 2 for an illustration. Note that the condition $\ell' \le \ell$ from the definition of an (ℓ, k)-adjuster P is equivalent to $\ell(P) \le \ell + 2k$ (and this will be the form in which we will be verifying it). We say that an adjuster P is 'in a set X' to mean that $p(P)$ is a subset of X. Also, we may just write $P = (v_1, v_2, F_1, F_2, z, A)$ when the collection $\mathcal{P}$ of paths is understood. We may additionally omit z if the size of the units is not important.

5.1. Asymmetric bipartite parts

To construct robustly many adjusters in our expander, we need the following result showing that a dense asymmetric bipartite subgraph is enough for finding a 1-subdivision of large clique.

Proposition 5.2. *Let $d \ge 40$ and suppose that a graph G contains disjoint vertex sets U and W such that every vertex in U has at least d neighbours in W. Let $\ell := \min\{\lceil d\sqrt{|U|}/8|W| \rceil, d/2\}$ and suppose $\ell \ge 20$. Then, G contains a copy of $\mathsf{TK}_\ell^{(1)}$.*

Proof. If $|U| > 16|W|^2$, then replace U by a subset of size exactly $16|W|^2$; the new $\ell = \lceil d/2 \rceil$ is still at least 20.

Let $p := \sqrt{|U|}/4|W| \le 1$, so that $\lceil pd/2 \rceil = \ell$, and hence $pd/12 \ge 2$. Let $W' \subseteq W$ be a subset formed by choosing each element of W independently at random with probability

p. For $u \in U$, set $X_u := d_G(u, W') = \sum_{w \in N(u,W)} \mathbb{1}_{\{w \in W'\}}$, which is a sum of independent Bernoulli random variables of parameter p with expectation $\mathbb{E}[X_u] \geq pd$. Then, using a standard lower-tail Chernoff bound (e.g. [20, Theorem 4.5]) gives that, for each $u \in U$,

$$\mathbb{P}(X_u < pd/2) \leq \exp(-pd/12) \leq 1/4,$$

using that $pd/12 \geq 2$. Therefore, letting $U' := \{u \in U : X_u \geq pd/2\}$, it is clear that we have $\mathbb{E}\left[|U'|\right] = |U| \cdot \mathbb{P}(X_u \geq pd/2) \geq 3|U|/4$. The last inequality (combined with $|U'| \leq |U|$) gives that

$$\mathbb{P}(|U'| < |U|/4) \leq 1/2.$$

Now, we have $|W| \geq d$, and hence $\mathbb{E}\left[|W'|\right] = p|W| \geq 24$. Thus, using an upper-tail Chernoff bound (e.g. [20, Theorem 4.4]), we have that

$$\mathbb{P}(|W'| > 2p|W|) \leq \exp(-p|W|/3) \leq 1/4.$$

And so we have that $\mathbb{P}(|W'| > 2p|W|) + \mathbb{P}(|U'| < |U|/4) < 1$. Therefore, there is some choice of W' for which $|U'| \geq |U|/4$ and $|W'| \leq 2p|W|$.

Take a maximal set of pairs $I \subseteq \binom{W'}{2}$ for which there is a set of distinct vertices $v_{\{x,y\}}$ in U', $\{x,y\} \in I$, such that $x, y \in N(v_{\{x,y\}})$ for each $\{x,y\} \in I$.

Noting that $16p^2|W|^2 = |U|$, we have

$$|U'| \geq |U|/4 = 4p^2|W|^2 \geq |W'|^2 > \binom{|W'|}{2}.$$

Thus, there is some $u \in U' \setminus \{v_{\{x,y\}} : \{x,y\} \in I\}$. Let $A := N(u, W')$. Then $|A| \geq \lceil pd/2 \rceil = \ell$. Moreover, $\binom{A}{2} \subseteq I$, by the maximality of I. Note that $A \cup \{v_{\{x,y\}} : \{x,y\} \in \binom{A}{2}\}$ is the vertex set of a copy of $\mathsf{TK}_\ell^{(1)}$ in G with core vertices those in A and edge set $\{xv_{\{x,y\}}, yv_{\{x,y\}} : \{x,y\} \in \binom{A}{2}\}$. $\quad\square$

5.2. Constructing an adjuster

Lemma 5.3. *For each $0 < \varepsilon_1, \varepsilon_2 < 1$, the following holds for all sufficiently large $K = K(\varepsilon_1, \varepsilon_2)$. For $d > 0$, let G be an n-vertex bipartite $(\varepsilon_1, \varepsilon_2 d)$-expander with $\delta(G) \geq d$ and $n \geq Kd$. Suppose $d \geq m^{200}$ and G contains no copy of $\mathsf{TK}_{\sqrt{dm}}^{(1)}$. Let $W \subseteq V(G)$ satisfy $|W| \leq dm^{11}$. Then, there exists a $(50m, 1)$-adjuster $P = (u_1, u_2, F_1, F_2, 3, A, \mathcal{P})$ in $G - W$. Moreover, $|A| \leq 2\ell(P)$.*

Proof. By Lemma 4.2, since $|W| \leq dm^{11} \leq dm^{30}$, we can find a $(3\sqrt{d}m^6 + 2, \sqrt{d}m^{22}, 10m)$-unit F_1 in $G - W$. Now, set $W_0 := W \cup V(F_1)$ and use again Lemma 4.2 to find a $(3\sqrt{d}m^6 + 2, \sqrt{d}m^{22}, 10m)$-unit F_2 in $G - W_0$. Repeat the process one more time to find a $(4\sqrt{d}m^6, \sqrt{d}m^{22}, 10m)$-unit F_3 in $G - W_1$, with $W_1 := W_0 \cup V(F_2)$. This

can be done, since $|W_0| \leq |W| + |V(F_1)| \leq dm^{11} + 40dm^{29} \leq dm^{30}$ and, similarly, $|W_1| \leq dm^{11} + 80dm^{29} \leq dm^{30}$.

Set $W' := W \cup \mathrm{int}(F_1) \cup \mathrm{int}(F_2) \cup \mathrm{int}(F_3)$ and denote the core vertices of the units F_1, F_2, F_3 by v_1, v_2, v_3 respectively.

Note that $|W'| \leq 2dm^{11}$. As $|\mathfrak{bo}(F_1)|, |\mathfrak{bo}(F_2)| \geq dm^{28}$, and recalling that $n \geq Kd$ for sufficiently large K implies $n \geq dm^{200}$, iteratively applying Lemma 2.3 we can find in $G - W'$ a collection of dm^{26} pairwise vertex disjoint $\mathfrak{bo}(F_1), \mathfrak{bo}(F_2)$-paths $\mathcal{P}'$, each of length at most m. By averaging, there exists a subcollection $\mathcal{P} \subseteq \mathcal{P}'$ of dm^{25} paths of equal length. Let $B := V(\mathcal{P}) \cap \mathfrak{bo}(F_1)$ be the set of endpoints of $\mathcal{P}$ in $\mathfrak{bo}(F_1)$, so that $|B| = dm^{25}$.

Suppose first that there is some vertex $w \notin W'$ with two neighbours w_1, w_2 in B. Let $P_1, P_2 \in \mathcal{P}$ be the paths that w_i is an endvertex of P_i for each $i \in [2]$ (see Fig. 3). Denote the paths in F_1 joining v_1 with w_1 and w_2 by Q_1 and Q_2, respectively. Denote the paths in F_2 joining $P_1 \cap \mathfrak{bo}(F_2)$ and $P_2 \cap \mathfrak{bo}(F_2)$ with v_2 as R_1 and R_2, respectively. (Note that these paths may meet before reaching the core vertex.) By symmetry, we can assume that $w \notin P_2$. Then, $v_1 Q_1 P_1 R_1 v_2$ and $v_1 Q_1 w w_2 P_2 R_2 v_2$ are two v_1, v_2-paths whose lengths differ by two and are at most

$$e(Q_1) + 2 + e(P_2) + e(R_1) \leq 50m + 2.$$

Let F_1' be the $(3\sqrt{d}m^6, \sqrt{d}m^{22}, 10m)$-unit with core vertex v_1 constructed from F_1 by removing the paths Q_1 and Q_2 and the leaves of the stars incident to Q_1 and Q_2. Similarly, let F_2' be the $(3\sqrt{d}m^6, \sqrt{d}m^{22}, 10m)$-unit with core vertex v_2 constructed from F_2 by removing the paths R_1 and R_2 and the leaves of the stars incident to R_1 and R_2. If Q_1 and Q_2 (resp. R_1 and R_2) only differ by an edge, then remove in addition an arbitrary path and its leaves from F_1 (resp. F_2) to construct F_1' (resp. F_2'). Let

$$A' := (V(P_1) \cup V(P_2) \cup \{w\} \cup V(Q_1) \cup V(R_1) \cup V(R_2)) \setminus \{v_1, v_2\}.$$

Note that $|A'| \leq 2|P_2| + 1 + 30m \leq 2 \cdot 50m$. Therefore, $(v_1, v_2, F_1', F_2', 3, A')$ is a $(50m, 1)$-adjuster in $G - W$. Moreover,

$$|A'| \leq |P_1| + |P_2| + 1 + |Q_1| - 2 + |R_1 \cup R_2| - 3 \leq 2(|Q_1| + |P_2| + |R_1| - 1) = 2\ell(P),$$

so the additional property also holds.

Suppose then there is no such vertex. Let B_0 be the set of vertices in B with at least $d/2$ neighbours in W'. As G is $\mathsf{TK}_{\sqrt{d}m}^{(1)}$-free and $20 \leq \sqrt{d}m \leq d/2$, we have by Proposition 5.2 that

$$\sqrt{d}m \geq \frac{(d/2)\sqrt{|B_0|}}{8|W'|} \geq \frac{\sqrt{|B_0|}}{32m^{11}},$$

and hence $|B_0| \leq 2^{10}dm^{24} \leq |B|/3$. Let $B' := B \setminus B_0$, so that $|B'| \geq 2|B|/3$ and each vertex in B' has at most $d/2$ neighbours in W'.
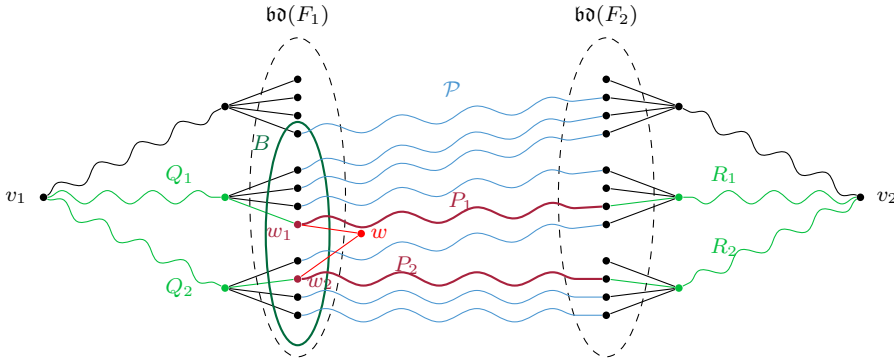
Fig. 3. An illustration of the first case in the proof of Lemma 5.3.

Now, remove from B' at most $3\sqrt{d}m^6 + 2$ vertices to ensure for each $x \in \mathrm{int}(F_1)$ either $|N_G(x) \cap B'| = 0$ or $|N_G(x) \cap B'| \geq 2$. The new set B' satisfies $|B'| \geq |B|/2$. For each $v \in B'$, let $P_v \in \mathcal{P}$ be the path with v as an endvertex. For each $v \in B'$, remove any edges between v and $V(P_v)$ in G, excluding the edge emanating from v in P_v. Call the resulting graph G'. Note that we have removed at most $|B'|m$ edges. There are thus at least $|B'| \cdot (d/2) - |B'|m \geq |B'|d/8$ edges from B' to $V(G) \setminus (B' \cup W')$ in G'. (Note that there are no edges inside B' as our graph is bipartite.) Now, by construction of G' and that no vertex $w \notin W'$ has at least two neighbours in B' in G', we get $|N_{G'}(B', V(G) \setminus W')| \geq |B'|d/8 \geq |B|d/16 = d^2m^{25}/16$.

Let $C := N_{G'}(B', V(G) \setminus W')$. Let $W'' := W' \cup (\cup_{v \in B'} V(P_v))$, noting at this point that $|W''| \leq 2dm^{11} + dm^{25} \cdot m \leq 2dm^{26}$. Next, we apply Lemma 2.3 to connect C and $\mathfrak{bd}(F_3)$, the former of size at least $d^2m^{25}/16$ and the latter of size at least $2dm^{28}$, with a path P of length at most m avoiding W'', as $d \geq m^{200}$. Let $u \in N_{G'}(B')$ be the endvertex of P in C, w_1 be a neighbour of u in B' and x be the neighbour of w_1 in $\mathrm{int}(F_1)$. Let Q_3 be the path in F_3 between v_3 and $P \cap \mathfrak{bd}(F_3)$. Fix another path $P' \in \mathcal{P}$ which has an endvertex w_2 in $(B' \cap N(x)) \setminus \{w_1\}$. That is, $P' = P_{w_2}$. As P avoids W'', P is disjoint from P_{w_1} and P_{w_2}. Let Q_4, Q_5 be the paths in F_2 joining P_{w_1} and P_{w_2}, respectively, with v_2. Then $v_3Q_3Pw_1P_{w_1}Q_4v_2$ and $v_3Q_3Pw_1xw_2P_{w_2}Q_5v_2$ are two v_3, v_2-paths whose lengths differ by two and are at most

$$e(Q_3) + e(P) + 3 + e(P_{w_2}) + e(Q_5) \leq 20m + |P| + |P_{w_2}| + 5 \leq 50m + 2.$$

Observe that $|(V(P_{w_1}) \cup V(P_{w_2})) \cap \mathfrak{bd}(F_3)| \leq 2m \leq \sqrt{d}m^5$. Hence, we can construct an $(3\sqrt{d}m^6, \sqrt{d}m^{22}, 10m)$-unit F_3', with core vertex v_3, from F_3 by removing all $\{v_3\}, (V(P_{w_1}) \cup V(P_{w_2})) \cap \mathfrak{bd}(F_3)$-paths (in F_3) and leaves of the stars incident to these paths, as well as removing the path Q_3 and the leaves of the star incident to Q_1. Let F_2' be the $(3\sqrt{d}m^6, \sqrt{d}m^{22}, 10m)$-unit with core vertex v_2 constructed from F_2 by removing the paths Q_4 and Q_5 and the leaves of the stars incident to Q_4 and Q_5. (If Q_4 and Q_5 only differ by an edge, then remove in addition an arbitrary path and its leaves from F_2 to construct F_2'.) Let
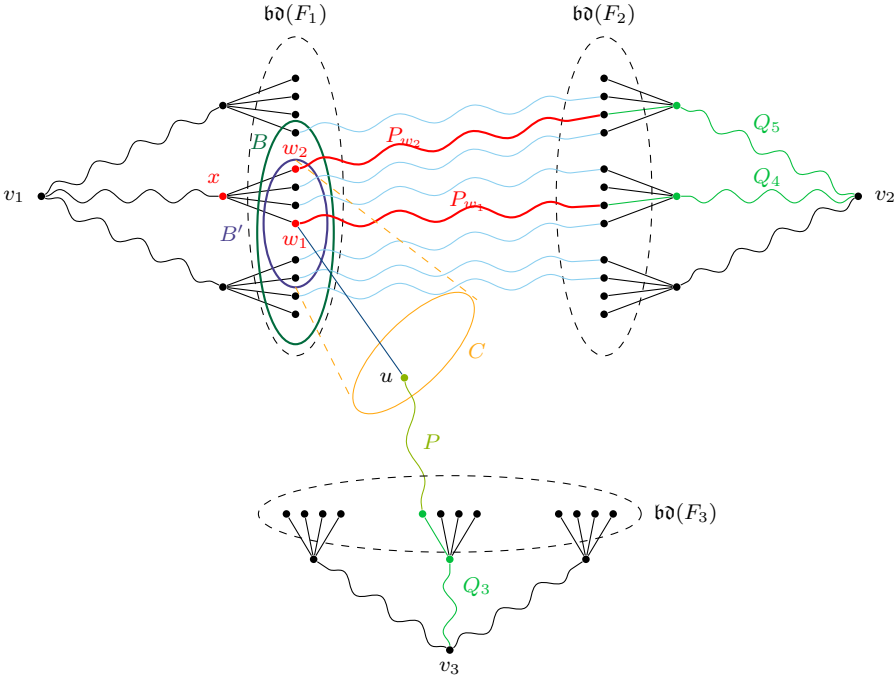
Fig. 4. An illustration of the second case in the proof of Lemma 5.3.

$$A^* := (V(P) \cup V(P_{w_1}) \cup \{x\} \cup V(P_{w_2}) \cup V(Q_3) \cup V(Q_4) \cup V(Q_5)) \setminus \{v_2, v_3\}.$$

We clearly have $|A^*| \leq 2 \cdot 50m$. Therefore,

$$P^* = (v_3, v_2, F_3', F_2', 3, A^*)$$

is a $(50m, 1)$-adjuster in $G-W$ (see Fig. 4). Also, it is routine to check that $|A^*| \leq 2\ell(P^*)$, finishing the proof of the lemma. □

5.3. Constructing larger adjusters

Lemma 5.4. *For each $0 < \varepsilon_1, \varepsilon_2 < 1$, the following holds for all sufficiently large $K = K(\varepsilon_1, \varepsilon_2)$. For $d > 0$, let G be an n-vertex bipartite $(\varepsilon_1, \varepsilon_2 d)$-expander with $\delta(G) \geq d$ and $n \geq Kd$. Suppose $d \geq m^{200}$ and G contains no copy of $\mathsf{TK}^{(1)}_{\sqrt{d}m}$. Let $W \subseteq V(G)$ satisfy $|W| \leq dm^{10}$. Then, there exists an $(100m^3, m^2)$-adjuster $P = (v_1, v_2, F_1, F_2, 2, A)$ in $G - W$ such that $80m^3 \leq \ell(P) \leq 80m^3 + 80m$.*

Proof. Inductively on $i = 1, 2, \ldots$ we will construct, in the graph $G - W$, an $(80mi, i)$-adjuster $P_i = (v_1^i, v_2^i, F_1^i, F_2^i, z_i, A_i, \mathcal{P}_i)$ with the additional properties that $z_i \geq 2$ and $|A_i| \leq 2\ell(P_i)$, stopping when its length $\ell(P_i)$ becomes at least $80m^3$ for the first time.

For $i = 1$, we apply Lemma 5.3 in order to get a $(50m, 1)$-adjuster $P_1 = (v_1^1, v_2^1, F_1^1, F_2^1, 3, A_1, \mathcal{P}_1)$ in $G - W$, which can be done since $|W| \leq dm^{10} \leq dm^{11}$.

We then proceed as follows. Suppose that we have an $(80m(i-1), i-1)$-adjuster

$$P_{i-1} = (v_1^{i-1}, v_2^{i-1}, F_1^{i-1}, F_2^{i-1}, z_{i-1}, A_{i-1}, \mathcal{P}_{i-1})$$

as above, whose length $\ell(P_i)$ is still less than $80m^3$. Define $W_i := W \cup p(P_{i-1})$. Then

$$|W_i| = |W \cup A_{i-1} \cup \text{int}(F_1^{i-1}) \cup \text{int}(F_2^{i-1})| \leq dm^{10} + 2 \cdot 80m(i-1) + 6\sqrt{d}m^6 \cdot 10m + 2 \leq dm^{11},$$

since $i < \ell(P_i) < 80m^3$. We then apply Lemma 5.3 to get a $(50m, 1)$-adjuster $\widetilde{P} = (w_1, w_2, \widetilde{F}_1, \widetilde{F}_2, 3, \widetilde{A})$ in $G - W_i$. Define $W_i' := W_i \cup p(\widetilde{P})$. As above, we have that $|W_i'| \leq dm^{11}$ also.

Let $X := \mathfrak{bd}(\widetilde{F}_1)$ and $Y := \mathfrak{bd}(F_2^{i-1}) \setminus p(\widetilde{P})$. Apply Lemma 2.3 with $(A, B, x)_{2.3} = (X, Y, dm^{28})$ in order to find an X, Y-path of length at most m in $G - W_i'$. Thus there exists a v_2^{i-1}, w_1-path Q of length at most $10m + m + 10m < 22m$. Consider paths between v_1^{i-1} and w_2 obtained by first taking a v_1^{i-1}, v_2^{i-1}-path from the adjuster P_{i-1}, followed by Q, followed by a w_1, w_2-path from the adjuster $\widetilde{P}$. Clearly, we can choose $i + 1$ of these paths so that their lengths form an arithmetic progression with difference 2 and are all at most $\ell(P_{i-1}) + e(Q) + \ell(\widetilde{P})$. Let A_i be the set of the vertices used by these paths, except for their endpoints v_1^{i-1} and w_2. The longest among these paths has length

$$\ell_i := \ell(P_{i-1}) + e(Q) + \ell(\widetilde{P}) \leq \ell(P_{i-1}) + 80m, \tag{2}$$

where by induction the right-hand side is at most $(80m(i-1)+2(i-1))+80m \leq 80mi+2i$. Again by induction, we have

$$|A_i| \leq |A_{i-1}| + |Q| + |\widetilde{A}| \leq 2 \cdot 80m(i-1) + 22m + 2 \cdot 50m \leq 2 \cdot 80mi.$$

Truncate our units F_1^{i-1} and $\widetilde{F}_2$ in order to ensure that A_i is disjoint from $V(F_1^{i-1}) \cup V(\widetilde{F}_2)$. Since $i < 80m^3$ and $n/d \geq K$, this requires removing in total at most $|A_i| \leq 2 \cdot 80mi \leq m^5$ starsfrom each original unit.

It follows that, for some $z_i \in [2, 3]$, $P_i := (v_1^{i-1}, w_2, F_1^{i-1}, \widetilde{F}_2^i, z_i, A_i)$ is a $(80mi, i)$-adjuster with $\ell(P_i) = \ell_i$. Furthermore, by induction and our choice of $\widetilde{P}$, we have $|A_i| \leq 2\ell(P_{i-1}) + |e(Q)| + 2\ell(\widetilde{P}) \leq 2\ell(P_i)$, as claimed.

Thus we can always proceed until we reach an $(80mi, i)$-adjuster $P_i = (v_1^i, v_2^i, F_1^i, F_2^i, z_i, A_i, \mathcal{P}_i)$ such that, in addition to $z_i \geq 2$ and $|A_i| \leq 2\ell(P_i)$, we have $\ell(P_i) \geq 80m^3$. By (2) we increase the length at each stage by at least 1 and at most $80m$, so we have that $\ell(P_i) \leq 80m^3 + 80m$ and $i \geq 80m^3/80m = m^2$. Moreover, $|A_i|/2$ does not exceed $\ell(P_i) \leq 80m^3 + 80m \leq 100m^3$ so P_i is also a $(100m^3, i)$-adjuster. Finally, by taking only the shortest $m^2 + 1$ paths of $\mathcal{P}_i$ (and trimming the units F_1^i and F_2^i to make z_i exactly 2), we get a $(100m^3, m^2)$-adjuster with all required properties. $\square$

6. Using the adjusters

Lemma 6.1. *For each $0 < \varepsilon_1, \varepsilon_2 < 1$, the following holds for all sufficiently large $K = K(\varepsilon_1, \varepsilon_2)$. For $d > 0$, let G be an n-vertex bipartite $(\varepsilon_1, \varepsilon_2 d)$-expander with $\delta(G) \geq d$ and $n \geq Kd$. Suppose $d \geq m^{200}$ and G contains no copy of $\mathsf{TK}^{(1)}_{\sqrt{d}m}$. Let $W \subseteq V(G)$ be such that $|W| \leq dm^9$. Let F_1, F_2 be two $(\sqrt{d}m^6 + 1, \sqrt{d}m^{22}, 10m)$-units, with core vertices $v_1, v_2 \in V(G) \setminus W$ which lie on the same part of the bipartite graph G. Further, suppose that $|(\mathsf{int}(F_1) \cup \mathsf{int}(F_2)) \cap W| \leq \sqrt{d}m^5$. Then there exists a v_1, v_2-path L of length precisely $80m^3$ in $G - W$.*

Proof. Let $W' := W \cup \mathsf{int}(F_1) \cup \mathsf{int}(F_2)$. As $|W'| \leq dm^{10}/3$, we can apply Lemma 5.4 to find a $(100m^3, m^2)$-adjuster $P = (w_1, w_2, E_1, E_2, 2, A)$ in $G - W'$ such that $80m^3 \leq \ell(P) \leq 80m^3 + 80m$.

For $i = 1, 2$, let X_i be the set of vertices v in $\mathfrak{bo}(F_i)$ such that the path between v_i and v is internally vertex-disjoint from W. There are $\sqrt{d}m^6$ different paths in $\mathsf{int}(F_i)$, thus

$$|X_i| \geq (\sqrt{d}m^6 - \sqrt{d}m^5) \cdot \sqrt{d}m^{22} \geq dm^{15}.$$

Let $Y := \mathfrak{bo}(E_1)$ and $W'' := W' \cup p(P)$. We have $|W''| \leq dm^{10}/2$. Hence, we can apply Lemma 2.3 with $(A, B, x) = (X_1, Y, dm^{12})$ in order to find an X_1, Y-path Q_1 of length at most m connecting $\mathfrak{bo}(F_1)$ and $\mathfrak{bo}(E_1)$ in $G - W''$. Let $W''' := W'' \cup V(Q_1)$ and observe that $|W'''| \leq dm^{10}$. Then apply Lemma 2.3 similarly as before to find a path Q_2 of length at most m connecting $X_2 \subseteq \mathfrak{bo}(F_2)$ and $\mathfrak{bo}(E_2)$ in $G - W'''$.

Let R_1 be the path from v_1 to $Q_1 \cap \mathfrak{bo}(F_1)$, S_1 be the path from $Q_1 \cap \mathfrak{bo}(E_1)$ to w_1, S_2 be the path from w_2 to $Q_2 \cap \mathfrak{bo}(E_2)$ and R_2 be the path from $Q_2 \cap \mathfrak{bo}(F_2)$ to v_2, with all these paths taken inside the respective units. Observe that $|R_i| \leq 10m + 1$ and $|S_i| \leq 10m + 1$ for $i = 1, 2$. Removing the paths R_1 and R_2, and incident leaves, from F_1 and F_2, respectively, we see that

$$P' := (v_1, v_2, F_1, F_2, R_1 \cup Q_1 \cup S_1 \cup A \cup S_2 \cup Q_2 \cup R_2)$$

is a $(101m^3, m^2)$-adjuster in $G - W$. Furthermore, we have

$$80m^3 \leq \ell(P) \leq \ell(P') \leq \ell(P) + 100m \leq 80m^3 + 200m.$$

Since v_1 and v_2 lie on the same part of G, $\ell(P')$ is even. Thus, since P' is a $(101m^3, m^2)$-adjuster and $m^2 \geq 200m$ (as $n \geq Kd$), we can find a v_1, v_2-path L of length precisely $80m^3$ in $G - W$. □

We can now prove our main result, Theorem 2.8.

Proof. Set $t := \sqrt{d}m$. Using Lemma 4.2 iteratively, we choose $2t$ $(\sqrt{d}m^6+1, \sqrt{d}m^{22}, 10m)$-units such that their interiors are pairwise vertex disjoint. This is possible since each interior has at most $\sqrt{d}m^6 \cdot 10m$ vertices, which is smaller than $dm^{30}/(2t)$. Clearly, we can choose some t of these units $F_1, F_2, \cdots, F_t$ such that their core vertices $v_1, v_2, \ldots, v_t$ lie in the same part of the bipartite graph G.

We have to show that G contains a copy of $\mathsf{TK}_t^{(1)}$ or $\mathsf{TK}_t^{(\ell)}$, where $\ell := 80m^3$. To this end, assume G is $\mathsf{TK}_t^{(1)}$-free. Let $\mathcal{P} := \{P_1, \ldots, P_S\}$ be a maximal collection of internally vertex disjoint paths such that:

- for each $s \in [S]$, P_s is a v_i, v_j-path of length precisely ℓ for some distinct $i, j \in [t]$;
- if P_s is a v_i, v_j-path, then for every $k \in [t] \setminus \{i, j\}$, P_s is disjoint from $\mathsf{int}(F_k)$;
- for distinct $i, j \in [t]$, there is at most one path in $\mathcal{P}$ with v_i and v_j as end vertices.

If $S = \binom{t}{2}$, then the graph formed by all the paths in $\mathcal{P}$ is our desired copy of $\mathsf{TK}_t^{(\ell)}$. Hence, we may assume that there exist distinct $i, j \in [t]$ such that $\mathcal{P}$ contains no v_i, v_j-path.

Let $W := (\cup_{s \in S} V(P_s) \setminus \{v_i, v_j\}) \cup \{\mathsf{int}(F_q) : q \in [t] \setminus \{i, j\}\}$. Then

$$|W| \leq t^2\ell + t \cdot \sqrt{d}m^6 \cdot 10m \leq dm^9.$$

Furthermore,

$$|\mathsf{int}(F_i) \cap W| = |\mathsf{int}(F_i) \cap (\cup_{s \in S} V(P_s) \setminus \{v_i\})| \leq \sqrt{d}m \cdot 80m^3 \leq \frac{1}{2}\sqrt{d}m^5,$$

and similarly $|\mathsf{int}(F_j) \cap W| \leq \frac{1}{2}\sqrt{d}m^5$. Thus, by Lemma 6.1 there exists a v_i, v_j-path P_{S+1} of length ℓ in $G-W$. Also, since P_{S+1} is in $G-W$, this path is disjoint from $\mathsf{int}(F_k)$ for $k \neq i, j$ and internally disjoint from all paths in $\mathcal{P}$. This contradicts the maximality of $\mathcal{P}$. Thus $S = \binom{t}{2}$ and we are done. $\quad\square$

7. Concluding remarks

While our main result gives the optimal degree bound forcing balanced clique subdivisions, it would be very interesting to consider balanced subdivisions of more general graphs. Let H be a graph with p vertices and q edges. It is known that if G is a graph with average degree at least $88(p + q)$, then G contains a subdivision of H. Let us give a proof of this, for which we need the following definition. A graph is k-*linked* if, for any choices $x_1, ..., x_k, y_1, ..., y_k$ of $2k$ distinct vertices there are vertex disjoint paths $P_1, ..., P_k$ with P_i joinings x_i to y_i, for all $i \in [k]$. We first find a subgraph G' in G which is $22(p + q)$-connected due to a result of Mader [18]. Then G' contains a subgraph G'' which is $(p + q)$-linked due to a result of Bollobás and Thomason [2]. One can then embed an H-subdivision in G'' by taking $\{x_i, y_i\}_{i \in [q]}$ to be the pair of endvertices of edges of H.

Perhaps the following is true.

Problem 7.1. Does there exist a constant C such that for any p-vertex q-edge graph H, if a graph G has average degree at least $C(p+q)$ then G contains a balanced subdivision of H?

Note added before submission While preparing this paper, we learnt that Bingyu Luan, Yantao Tang, Guanghui Wang and Donglei Yang [16] independently proved Theorem 1.1.

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