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New upper bound for lattice covering by spheres

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Abstract

We show that there exists a lattice covering of $\mathbb{R}^n$ by Euclidean spheres of equal radius with density $O(n \ln^{\beta} n)$ as $n \to \infty$, where

$$\beta := \frac{1}{2} \log_2 \left(\frac{8\pi e}{3\sqrt{3}} \right) = 1.85837 \ldots .$$

This improves upon the previously best known upper bound by Rogers from 1959 of $O(n \ln^{\alpha} n)$, where $\alpha := \frac{1}{2} \log_2(2\pi e) = 2.0471 \ldots$.

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1 | INTRODUCTION

Given n, we would like to cover the entire space $\mathbb{R}^n$ by placing spheres[†] of the same radius r at each element of a lattice Λ, that is, we require that

$$\Lambda + B_r^n = \mathbb{R}^n, \tag{1}$$

where $B_r^n := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 \leqslant r\}$ denotes the Euclidean sphere of radius r in $\mathbb{R}^n$ centred at the origin and $X + Y := \{\mathbf{x} + \mathbf{y} : \mathbf{x} \in X \text{ and } \mathbf{y} \in Y\}$ denotes the *sum* of two sets $X, Y \subseteq \mathbb{R}^n$. We call any such pair (Λ, B_r^n) a *(sphere) lattice covering* of $\mathbb{R}^n$ and define its *density* as

$$\Theta(\Lambda, B_r^n) := \frac{\mathrm{vol}(B_r^n)}{|\det(\Lambda)|},$$

[†] Throughout this work, we adopt the convention that 'sphere' means a closed Euclidean ball.

where $\mathrm{vol}(B_r^n)$ denotes the volume of B_r^n and $\det(\Lambda)$ is the *determinant* of Λ which can be defined as

$$\det(\Lambda) := \det[\mathbf{b}_1, \dots, \mathbf{b}_n],$$

the determinant of the matrix made of some (equivalently, any) linearly independent vectors[†] $\mathbf{b}_1, \dots, \mathbf{b}_n \in \mathbb{R}^n$ that *generate* the lattice Λ, that is, satisfy

$$\Lambda = \{\lambda_1 \mathbf{b}_1 + \cdots + \lambda_n \mathbf{b}_n : \lambda_i \in \mathbb{Z} \text{ for } i \in [n]\}, \tag{2}$$

where $[n] := \{1, \dots, n\}$. The *covering density* of Λ is then defined as

$$\Theta(\Lambda) := \min_{r \geqslant 0} \left\{\Theta(\Lambda, B_r^n) : \mathbb{R}^n = \Lambda + B_r^n\right\}.$$

The classical *lattice covering problem*, a central topic in the combinatorial geometry (see, e.g., books [7, 24]), asks for the *optimal lattice covering density* in dimension n, defined as

$$\Theta_n := \inf\{\Theta(\Lambda) : \Lambda \subseteq \mathbb{R}^n \text{ is a lattice}\}.$$

Determining Θ_n seems a very difficult problem, with exact values known only for $n \leqslant 5$ (see [2, 4, 11, 16, 19, 25]) and with many questions (such as, for example, whether the Leech lattice is optimal) being still open. Various lower and upper bounds for Θ_n were obtained in a large number of works, starting with the classical papers [3, 8, 9, 14, 22] from the 1950s; we refer the reader to the papers [13, 26] that contain overviews of more recent results.

More generally, for any convex body $K \subseteq \mathbb{R}^n$, one can similarly define the optimal lattice covering density $\Theta_{n,K}$ of K (see, e.g., [24] for details). Improving upon Rogers' [22] upper bound $\Theta_{n,K} = O\left(n^{\log_2 \ln n + O(1)}\right)$ from 1959, a recent breakthrough by Ordentlich–Regev–Weiss [20] shows that $\Theta_{n,K} = O(n^2)$ holds universally for all convex bodies $K \subseteq \mathbb{R}^n$. For convex bodies $K \subseteq \mathbb{R}^n$ with "rich" family of reflection symmetries, the bound was earlier improved by Gritzmann [17] to $\Theta_{n,K} = O(n \ln^{1+\log_2 \mathrm{e}} n)$.

However, in perhaps the most fundamental case when K is the sphere, the above bounds do not improve upon Rogers' other result from [22] that $\Theta_n = O\left(n \ln^\alpha n\right)$, where $\alpha := \frac{1}{2}\log_2(2\pi \mathrm{e}) = 2.0471\dots$.

In this work, we establish the following upper bound for Θ_n, improving upon the above-mentioned bound of Rogers [22].

Theorem 1.1. *There exists a constant C such that for every integer $n \geqslant 1$, it holds that*

$$\Theta_n \leqslant C n \ln^\beta n, \quad \textit{where} \quad \beta := \frac{1}{2}\log_2\left(\frac{8\pi \mathrm{e}}{3\sqrt{3}}\right) = 1.85837\dots .$$

Let us remark that the factor n in Theorem 1.1 is necessary, as shown by Coxeter–Few–Rogers [8] who proved that $\Theta_n \geqslant \left(\mathrm{e}^{-3/2} + o(1)\right) n$, improving upon earlier results of Bambah–Davenport [3] and Erdős–Rogers [14].

[†] Unless otherwise specified, all vectors in this work are considered column vectors.

Another obstacle to improving the upper bound of Θ_n is that, even when the condition that $\Lambda \subseteq \mathbb{R}^n$ is a lattice is removed and arbitrary sphere coverings of $\mathbb{R}^n$ are allowed, the best known asymptotic upper bound still has order $n \ln n$ (see, e.g., [5, 6, 12, 15, 21, 23]).

Theorem 1.1 follows relatively quickly from the following more general theorem, which provides a general strategy for proving upper bounds on Θ_n. To state the result, we first introduce some necessary definitions.

Given a point $\mathbf{x} \in \mathbb{R}^n$ and n linearly independent vectors $\mathbf{b}_1, \dots, \mathbf{b}_n \in \mathbb{R}^n$, the *parallelepiped*

$$\mathrm{P} = \mathrm{P}_{\mathbf{x}}(\mathbf{b}_1, \dots, \mathbf{b}_n) \tag{3}$$

starting at $\mathbf{x} \in \mathbb{R}^n$ and generated by $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is defined as the convex hull of

$$V_{\mathbf{x}}(\mathbf{b}_1, \dots, \mathbf{b}_n) := \{\mathbf{x} + \lambda_1 \mathbf{b}_1 + \cdots + \lambda_n \mathbf{b}_n : \lambda_i \in \{0, 1\} \text{ for } i \in [n]\}.$$

Trivially, $V_{\mathbf{x}}(\mathbf{b}_1, \dots, \mathbf{b}_n)$ is exactly the set of the vertices of the polytope P and we will refer to this set as $V(\mathrm{P})$. We say that a parallelepiped $\mathrm{P} \subseteq \mathbb{R}^n$ is a Λ-*parallelepiped* if $V(\mathrm{P}) \subseteq \Lambda$. If, in addition, $\mathrm{vol}(\mathrm{P}) = |\det(\Lambda)|$, then P is called a *fundamental parallelepiped* of Λ. For example, any set of vectors that generates Λ as in (2) produces a fundamental parallelepiped.

The following concept will be crucial for our result.

Definition 1.2 (Robust lattice covering). Let $d \geqslant 1$ be an integer and $r \geqslant 0$ be a real number. A lattice covering (Λ, B_r^d) of $\mathbb{R}^d$ is *robust* if every closed ball of radius r in $\mathbb{R}^d$ contains a fundamental parallelepiped of Λ.

Extending the definition of Θ_n, we define the *optimal robust lattice covering density* of $\mathbb{R}^n$ as

$$\tilde{\Theta}_n := \inf\{\Theta(\Lambda, B_r^n) : (\Lambda, B_r^n) \text{ is a robust lattice covering of } \mathbb{R}^n\}.$$

For every integer $d \geqslant 1$, define

$$\nu_d := \mathrm{vol}\left(B_{\sqrt{d}}^d\right) = \frac{(\pi d)^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2} + 1\right)},$$

where Γ denotes the gamma function.

The following result provides an asymptotic upper bound for Θ_n in terms of $\tilde{\Theta}_d$.

Theorem 1.3. *For every integer $d \geqslant 1$, there exists a constant $C_{1.3} = C_{1.3}(d)$ such that for $n \geqslant d$,*

$$\Theta_n \leqslant C_{1.3} n \ln^{\gamma} n, \quad \textit{where} \quad \gamma = \gamma_d := \frac{1}{2} \log_2(2\pi \mathrm{e}) - \frac{1}{d} \log_2 \left(\nu_d / \tilde{\Theta}_d\right).$$

It is straightforward to verify that $\left(\mathbb{Z}^d, B_{\sqrt{d}}^d\right)$ is a robust lattice covering of $\mathbb{R}^d$ with density ν_d for any $d \geqslant 1$. Hence, $\tilde{\Theta}_d \leqslant \nu_d$.

Thus that $\gamma_d \leqslant \frac{1}{2} \log_2(2\pi \mathrm{e})$, which recovers Rogers' bound by Theorem 1.3 (applied with any chosen $d \geqslant 1$).

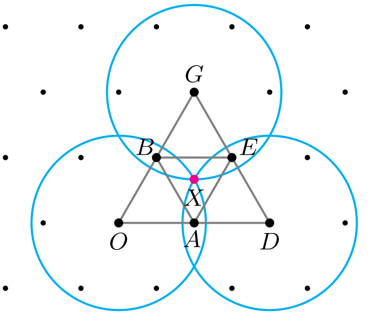

FIGURE 1 The lattice generated by $\mathbf{v}_1 = (1,0)^t$ and $\mathbf{v}_2 = \left(1/2, \sqrt{3}/2\right)^t$, three balls of radius $2/\sqrt{3}$ centred at the origin O, $D = 2\mathbf{v}_1$ and $G = 2\mathbf{v}_2$, and three different fundamental parallelepipeds OAEB, DABE, GBAE. The point X is the centre of the triangle ABE.

Theorem 1.1 follows immediately from Theorem 1.3 and the following upper bound for $\tilde{\Theta}_2$.

Lemma 1.4. *There exists a robust lattice covering of* $\mathbb{R}^2$ *with density* $8\pi/(3\sqrt{3})$. *In particular,*

$$\tilde{\Theta}_2 \leqslant \frac{8\pi}{3\sqrt{3}}.$$

Proof of Lemma 1.4. Let

$$\mathbf{v}_1 := (1,0)^t, \quad \mathbf{v_2} := \left(1/2, \sqrt{3}/2\right)^t, \quad \text{and} \quad r := 2/\sqrt{3},$$

where $\mathbf{v}^t$ denotes the transposition of a vector $\mathbf{v}$. Let $\Lambda \subseteq \mathbb{R}^2$ denote the lattice generated by $\{\mathbf{v}_1, \mathbf{v}_2\}$ (see Figure 1). We claim that $\left(\Lambda, B_r^2\right)$ is a robust lattice covering of $\mathbb{R}^2$. By definition, it amounts to showing that for every point $\mathbf{w} \in \mathbb{R}^2$, the sphere $B_r^2(\mathbf{w})$ of radius r centred at $\mathbf{w}$ contains a fundamental parallelepiped of Λ. By symmetry, it suffices to prove this statement for all points $\mathbf{w}$ contained in the equilateral triangle $\triangle_{\mathrm{ABE}}$ shown in Figure 1.

Let X denote the centre of $\triangle_{\mathrm{ABE}}$. It is easy to see that

- if $\mathbf{w} \in \triangle_{\mathrm{AXB}}$, then the ball $B_r^2(\mathbf{w})$ contains the fundamental parallelepiped $\square_{\mathrm{OAEB}}$;
- if $\mathbf{w} \in \triangle_{\mathrm{AXE}}$, then the ball $B_r^2(\mathbf{w})$ contains the fundamental parallelepiped $\square_{\mathrm{DABE}}$;
- if $\mathbf{w} \in \triangle_{\mathrm{BXE}}$, then the ball $B_r^2(\mathbf{w})$ contains the fundamental parallelepiped $\square_{\mathrm{GBAE}}$.

Therefore, $\left(\Lambda, B_r^2\right)$ is a robust lattice covering of $\mathbb{R}^2$. The covering density of (Λ, B_r^2) is

$$\frac{\mathrm{vol}(B_r^2)}{\det(\Lambda)} = \frac{\left(2/\sqrt{3}\right)^2 \pi}{\sqrt{3}/2} = \frac{8\pi}{3\sqrt{3}},$$

which completes the proof of Lemma 1.4. □

In the next section, we present the proof of Theorem 1.3, assuming a key lemma (Lemma 2.3) whose proof is deferred to Section 3. We include some concluding remarks in Section 4.

2 | PROOF OF THEOREM 1.3

In this section, we present the proof of Theorem 1.3. We begin by listing some auxiliary results from Rogers' earlier work [22].

Given a lattice $\Lambda \subseteq \mathbb{R}^n$ and a measurable set $K \subseteq \mathbb{R}^n$, let $\bar{\rho}(\Lambda + K)$ denote the density of the points in $\mathbb{R}^n$ that are not covered by the (periodic) set $\Lambda + K$.

Lemma 2.1 [22, Lemma 2]. *There exist constants $N_{2.1}$ and $C_{2.1}$ such that the following holds for every $n \geqslant N_{2.1}$. For every convex body $K \subseteq \mathbb{R}^n$, there exists a lattice $\Lambda \subseteq \mathbb{R}^n$ with* $\det(\Lambda) = \mathrm{vol}(K)/\eta_n$, *where $\eta_n := \frac{n}{4}\ln\left(\frac{27}{16}\right) - 3\ln n$, such that*

$$\bar{\rho}(\Lambda + K) \leqslant C_{2.1} n^3 \left(\frac{16}{27}\right)^{n/4}. \tag{4}$$

Lemma 2.2 [22, Lemma 4]. *Let $K \subseteq \mathbb{R}^n$ be a convex body and $\Lambda \subseteq \mathbb{R}^n$ be a lattice. Suppose that $\bar{\rho}(\Lambda + K) \leqslant (n^n + 1)^{-1}$. Then, $(\Lambda, (1 + 1/n)K)$ is a lattice covering of $\mathbb{R}^n$, that is, $\Lambda + (1 + 1/n)K = \mathbb{R}^n$.*

The following lemma, which extends [22, Lemma 3], will be crucial for our proof. Due to its technical complexity, we postpone its proof to Section 3.

Lemma 2.3. *For every integer $d \geqslant 1$, there is a constant $C_{2.3} = C_{2.3}(d)$ such that, for any $n \geqslant 1$, if $K \subseteq \mathbb{R}^n$ is a measurable set and $\Lambda \subseteq \mathbb{R}^n$ is a lattice, then there is a lattice $\tilde{\Lambda} \subseteq \mathbb{R}^{n+d}$ with* $\det(\tilde{\Lambda}) = \det(\Lambda)$ *satisfying*

$$\bar{\rho}\left(\tilde{\Lambda} + \tilde{K}\right) \leqslant C_{2.3}(\bar{\rho}(\Lambda + K))^{2^d},$$

where $\tilde{K} \subseteq \mathbb{R}^{n+d}$ denotes the Cartesian product of K and the d-dimensional sphere of volume $\tilde{\Theta}_d$.

We will also use the following simple fact.

Fact 2.4. *Suppose that $n, d, k \geqslant 1$ are integers satisfying $1 \leqslant kd \leqslant n$. Then*

$$K_{k,d} := B^{n-kd}_{\sqrt{n-kd}} \times \underbrace{B^d_{\sqrt{d}} \times \cdots \times B^d_{\sqrt{d}}}_{k \text{ times}}$$

is a subset of $B^n_{\sqrt{n}}$ and

$$\mathrm{vol}(K_{k,d}) = \frac{\nu_{n-kd} \cdot \nu_d^k}{\nu_n} \cdot \mathrm{vol}\left(B^n_{\sqrt{n}}\right).$$

Now, we present the proof of Theorem 1.3.

Proof of Theorem 1.3. Given $d \geqslant 1$, let $C_{2.3} = C_{2.3}(d)$ be the constant given by Lemma 2.3.

Let

$$C := 2\mathrm{e}(2\pi\mathrm{e})^{5d/2}/5.$$

Let n be a sufficiently large integer. Fix an integer k satisfying

$$\frac{1}{d}\log_2 \ln n + 4 \leqslant k \leqslant \frac{1}{d}\log_2 \ln n + 5. \tag{5}$$

Let

$$\eta := \frac{n-kd}{4}\ln\left(\frac{27}{16}\right) - 3\ln(n-kd) < \frac{n}{5}. \tag{6}$$

We aim to show that there exists a lattice covering (Λ, B^n) of $\mathbb{R}^n$ with density at most

$$Cn\left(\tilde{\Theta}_d/\nu_d\right)^{\frac{1}{d}\log_2 \ln n}(2\pi\mathrm{e})^{\frac{1}{2}\log_2 \ln n},$$

where $n \geqslant C'$ and C' is a constant in terms of d. Then, we can take $C_{1.3} = \max\{C, (2C')^{C'}\}$ as $\Theta_n \leqslant (2n)^n$ holds trivially.

Let $K_0 \subseteq \mathbb{R}^{n-kd}$ be a sphere with volume η at the origin. Let $r \in \mathbb{R}$ be such that $\mathrm{vol}(B_r^d) = \tilde{\Theta}_d$. Since $\left(\mathbb{Z}^d, B^d_{\sqrt{d}}\right)$ is a robust lattice covering of $\mathbb{R}^d$ with density ν_d for any $d \geqslant 1$, we have that

$$\tilde{\Theta}_d \leqslant \nu_d.$$

For $i \in [k]$, define $K_i := K_{i-1} \times B_r^d$. Note that for $i \in [0, k]$,

$$\mathrm{vol}(K_i) = \eta\tilde{\Theta}_d^i.$$

By Fact 2.4, there exists an n-dimensional ball $B \subseteq \mathbb{R}^n$ such that, after some linear transformation T (scaling the radii of the balls K_0 and B_r^d), the set K_k is contained in B, and

$$\mathrm{vol}(B) = \mathrm{vol}(T(K_k)) \cdot \frac{\nu_n}{\nu_{n-kd} \cdot \nu_d^k} = |\det(T)| \cdot \eta\left(\frac{\tilde{\Theta}_d}{\nu_d}\right)^k \frac{\nu_n}{\nu_{n-kd}}. \tag{7}$$

Using the estimate $\Gamma(1+x) = (1+o(1))\sqrt{2\pi x}\,(x/\mathrm{e})^x$ as $x \to \infty$ (see, e.g., [10]), we obtain

$$\nu_n = \frac{(\pi n)^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} = (1+o(1))\frac{(2\pi\mathrm{e})^{\frac{n}{2}}}{\sqrt{\pi n}}.$$

It follows from (7), together with the assumption that n is sufficiently large, that

$$\begin{aligned}\mathrm{vol}(B) &= (1+o(1))|\det(T)| \cdot \eta\left(\frac{\tilde{\Theta}_d}{\nu_d}\right)^k \frac{(2\pi\mathrm{e})^{\frac{n}{2}}\sqrt{\pi(n-kd)}}{(2\pi\mathrm{e})^{\frac{n-kd}{2}}\sqrt{\pi n}} \\ &\leqslant 2\,|\det(T)| \cdot \eta\left(\frac{\tilde{\Theta}_d}{\nu_d}\right)^k (2\pi\mathrm{e})^{\frac{kd}{2}}.\end{aligned} \tag{8}$$

Let $C_{2.1}$ be the constant given by Lemma 2.1. Applying Lemma 2.1 to K_0, we obtain a lattice $\Lambda_0 \subseteq \mathbb{R}^{n-kd}$ with $\det(\Lambda_0) = \mathrm{vol}(K_0)/\eta = 1$ such that $\bar{\rho}(\Lambda_0 + K_0) \leqslant \delta_0$, that is, the set of points in $\mathbb{R}^{n-kd}$ not covered by $K_0 + \Lambda_0$ has density at most δ_0, where

$$\delta_0 := C_{2.1}(n-kd)^3(16/27)^{\frac{n-kd}{4}} \leqslant C_{2.1}n^3(16/27)^{\frac{n}{5}}.$$

By applying Lemma 2.3 iteratively k times, we obtain lattices $\Lambda_1, \dots, \Lambda_k$ such that, for each $i \in [k]$, the following properties hold:

- the lattice $\Lambda_i \subseteq \mathbb{R}^{n-kd+id}$ satisfies $\det(\Lambda_i) = 1$, and
- $\bar{\rho}(\Lambda_i + K_i) \leqslant \delta_i := C_{2.3}\, \delta_{i-1}^{2^d}$.

In particular,

$$\delta_k = C_{2.3}\, \delta_{k-1}^{2^d} = C_{2.3}^{1+2^d} \delta_{k-2}^{2^{2d}} = \cdots = C_{2.3}^{1+2^d+\cdots+2^{(k-1)d}} \delta_0^{2^{kd}}$$
$$= C_{2.3}^{\frac{2^{kd}-1}{2^d-1}} \delta_0^{2^{kd}} = C_{2.3}^{\frac{-1}{2^d-1}} \left(C_{2.3}^{\frac{1}{2^d-1}} \delta_0 \right)^{2^{kd}}.$$

Since $C_{2.1}, C_{2.3}$, and d are fixed, we can choose n sufficiently large so that

$$n \geqslant \max\left\{ C_{2.3}^{\frac{-1}{2^d-1}},\ C_{2.1} C_{2.3}^{\frac{1}{2^d-1}} \right\}, \text{ and } 4\ln n - \frac{n}{5}\ln\frac{27}{16} < 0.$$

Combining it with the assumption $kd \geqslant \log_2 \ln n + 4$, we obtain

$$\ln \delta_k \leqslant \ln C_{2.3}^{\frac{-1}{2^d-1}} + 2^{kd} \ln\left(C_{2.3}^{\frac{1}{2^d-1}} \delta_0 \right) \leqslant \ln n + 2^{kd}\left(\ln n^4 + \ln\left(\frac{16}{27}\right)^{n/5} \right)$$
$$\leqslant \ln n + 16 \ln n \left(4\ln n - \frac{n}{5}\ln\frac{27}{16} \right)$$
$$= -\left(\frac{16}{5}\ln\frac{27}{16} \right) n \ln n + 64 \ln^2 n + \ln n,$$

which is smaller than $-\ln(n^n+1) = -(1+o(1))n\ln n$ as n is sufficiently large. Thus,

$$\delta_k \leqslant (n^n+1)^{-1}.$$

So, it follows from Lemma 2.2 that $\Lambda_k + (1+1/n)K_k = \mathbb{R}^n$. Since $T(K_k) \subseteq B$, we obtain

$$\left(\Lambda_k, (1+1/n)T^{-1}(B)\right) = \mathbb{R}^n,$$

which implies that $\left(T(\Lambda_k), (1+1/n)B\right)$ forms a (sphere lattice) covering of $\mathbb{R}^n$.

It remains to show that the density of $\left(T(\Lambda_k), (1+1/n)B\right)$ gives the desired upper bound. Indeed, by (5), (6), and (8), we have

$$\Theta(T(\Lambda_k), (1+1/n)B) = \frac{\mathrm{vol}((1+1/n)B)}{|\det(T(\Lambda_k))|} = \left(1+\frac{1}{n}\right)^n \frac{\mathrm{vol}(B)}{|\det(T)||\det(\Lambda_k)|}$$
$$\leqslant 2e\eta \left(\frac{\tilde{\Theta}_d}{\nu_d}\right)^k (2\pi e)^{\frac{kd}{2}} \leqslant Cn \left(\frac{\tilde{\Theta}_d}{\nu_d}\right)^{\frac{1}{d}\log_2 \ln n} (2\pi e)^{\frac{1}{2}\log_2 \ln n},$$

as claimed. This completes the proof of Theorem 1.3. □

3 | PROOF OF LEMMA 2.3

In this section, we present the proof of Lemma 2.3, starting with a few preliminary lemmas.

We use $\operatorname{dist}(\mathbf{x},\mathbf{y}) := \|\mathbf{x}-\mathbf{y}\|_2$ denote the Euclidean distance between two points $\mathbf{x},\mathbf{y} \in \mathbb{R}^d$. Given a set $S \subseteq \mathbb{R}^d$ and a point $\mathbf{x} \in \mathbb{R}^d$, we define

$$\operatorname{dist}(\mathbf{x}, S) := \inf\{\operatorname{dist}(\mathbf{x},\mathbf{y}) : \mathbf{y} \in S\}.$$

In the following lemma, we make no attempt to optimise $C_{3.1}$ as a function of d.

Lemma 3.1. *Let $d \geqslant 1$ be an integer and $D \in [0, \nu_d]$ be a real number. There exists a constant $C_{3.1}$ depending only on d such that, for every robust lattice covering (Λ, B_r^d) of $\mathbb{R}^d$ with density D, the number of fundamental parallelepipeds contained in B_{2r}^d is at most $C_{3.1}$.*

Proof of Lemma 3.1. By scaling Λ and r if necessary, we may assume that $|\det(\Lambda)| = 1$. Consequently, $\operatorname{vol}(B_r^d) = D$. Since $D \leqslant \nu_d$, it follows that $r \leqslant \sqrt{d}$. First, we show that the number of lattice points contained in B_{2r}^d is bounded.

Claim 3.2. There exists a constant C in terms of d such that we have

$$\left|\Lambda \cap B_{2r}^d\right| \leqslant C.$$

Proof of Claim 3.2. For $1 \leqslant i \leqslant d$, let

$$\lambda_i := \min\{\lambda \in \mathbb{R}_{\geqslant 0}, \lambda B_{2r}^d \text{ contains } i \text{ linearly independent lattice points of } \Lambda\}.$$

be the ith successive minimum of B_{2r}^d with respect to Λ. Since B_r^d contains a fundamental parallelepiped, we derive that $\lambda_d \leqslant 1/2$. By Minkowski's second theorem and the monotonicity, we have

$$\lambda_1 \cdot 2^{-d+1} \geqslant \lambda_1\lambda_2 \cdots \lambda_d \geqslant \frac{2^d}{d!}\det(\Lambda)/\operatorname{vol}(B_{2r}^d).$$

It follows from $r \leqslant \sqrt{d}$ that there exists a constant C' in terms of d such that $\lambda_1 \geqslant C'$. By, for example, [18, Theorem 1.5], there exists a constant C in terms of d such that $\left|\Lambda \cap B_{2r}^d\right| \leqslant C$. □

Note that each d-dimensional (fundamental) parallelepiped has 2^d vertices, so it follows from Claim 3.2 that the number of fundamental parallelepipeds contained in B_{2r}^d is at most C^{2^d}, which completes the proof of Lemma 3.1. □

Let $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$ denote the n-dimensional torus. The following two lemmas routinely follow from standard results. For completeness, we include their proofs.

Lemma 3.3. *Let $K \subseteq \mathbb{R}^n$ be a measurable set and let $\delta := \bar{\rho}(\mathbb{Z}^n + K)$. Let $\mathbf{y} \in \mathbb{T}^n$ be a point chosen uniformly at random, according to the Lebesgue measure restricted to the cube $[0,1)^n$. Let $\tilde{K} := K \cup$*

$(K + \mathbf{y})$. *Then*

$$\mathbb{E}[\bar{\rho}(\mathbb{Z}^n + \tilde{K})] = \delta^2.$$

Proof of Lemma 3.3. Let $\chi : \mathbb{R}^n \to \{0, 1\}$ be the characteristic function of $\mathbb{Z}^n + K$. Then, χ is periodic with period 1 in each of the coordinates. It follows from $\bar{\rho}\,(\mathbb{Z}^n + K) = \delta$ that

$$\int_{[0,1)^n} (1 - \chi(\mathbf{x}))dx_1 \cdots dx_n = \delta. \tag{9}$$

Suppose that $\mathbf{y} \in \mathbb{T}^n$ is a point chosen uniformly at random according to the Lebesgue measure restricted to the cube $[0, 1)^n$, and $\tilde{K} = K \cup (K + \mathbf{y})$. Using (9), we obtain

$$\begin{aligned}\mathbb{E}[\bar{\rho}(\mathbb{Z}^n + \tilde{K})] &= \int_{[0,1)^n} \left(\int_{[0,1)^n} (1 - \chi(\mathbf{x}))(1 - \chi(\mathbf{x} + \mathbf{y}))dx_1 \cdots dx_n \right) dy_1 \cdots dy_n \\ &= \int_{[0,1)^n} (1 - \chi(\mathbf{x})) \left(\int_{[0,1)^n} (1 - \chi(\mathbf{x} + \mathbf{y}))dy_1 \cdots dy_n \right) dx_1 \cdots dx_n \\ &= \int_{[0,1)^n} (1 - \chi(\mathbf{x})) \left(\int_{[0,1)^n} (1 - \chi(\mathbf{z}))dz_1 \cdots dz_n \right) dx_1 \cdots dx_n = \delta^2,\end{aligned}$$

as desired. □

Lemma 3.4. *Let $n, d \geqslant 1$ be integers. Suppose that $M \in \mathbb{Z}^{d\times d}$ is a matrix with* $|\det(M)| = 1$. *Define the map $\psi : \mathbb{R}^{d\times n} \to \mathbb{R}^{d\times n}$ by $\psi(X) = MX$ for all $X \in \mathbb{R}^{d\times n}$. Then, the map ϕ induced by ψ on $\mathbb{T}^{d\times n}$, that is,*

$$\phi(X) := \psi(X) \bmod \mathbb{Z}^{d\times n} \quad \textit{for every} \quad X \in \mathbb{T}^{d\times n},$$

is bijective and (Lebesgue) measure-preserving.

Proof of Lemma 3.4. Let $M_* = \operatorname{diag}(M, \dots, M) \in \mathbb{Z}^{dn\times dn}$ be the matrix obtained by placing n copies of the matrix M along the diagonal. It is clear that M_* is an integer matrix with $|\det(M_*)| = 1$. By Cramer's rule, the inverse M_*^{-1} of M_* is also an integer matrix with $|\det(M_*^{-1})| = 1$.

Define the map $\psi_* : \mathbb{R}^{dn} \to \mathbb{R}^{dn}$ by $\psi(\mathbf{x}) = M_*\mathbf{x}$ for every $\mathbf{x} \in \mathbb{R}^{dn}$. It is clear that ψ and ψ_* define the same linear map under the identification of $\mathbb{R}^{d\times n}$ with $\mathbb{R}^{dn}$. Let φ be the map induced by ψ_* on $\mathbb{T}^{d\times n}$, that is,

$$\varphi(\mathbf{x}) = \psi_*(\mathbf{x}) \bmod \mathbb{Z}^{dn} \quad \text{for every} \quad \mathbf{x} \in \mathbb{T}^{dn}.$$

Since $|\det(M_*)| = 1$, it follows from standard results in analysis (see, e.g., [1, Lemma 40.4]) that ψ_* is measure-preserving. Thus, if we can show that φ is bijective, it will follow that φ is also measure-preserving.

We begin by proving that φ is injective. Suppose to the contrary that there exist two distinct points $\mathbf{x}, \mathbf{y} \in [0, 1)^{dn}$ such that $\varphi(\mathbf{x}) = \varphi(\mathbf{y})$. Then, we have $\varphi(\mathbf{x}) - \varphi(\mathbf{y}) = \mathbf{0}$, which means that

$$\psi_*(\mathbf{x} - \mathbf{y}) = \psi_*(\mathbf{x}) - \psi_*(\mathbf{y}) \in \mathbb{Z}^{dn}. \tag{10}$$

Since both M_* and M_*^{-1} are integer matrices, the map ψ_* induces a bijection from $\mathbb{Z}^{dn}$ onto itself. Combining it with (10), we conclude that $\mathbf{x} - \mathbf{y} \in \mathbb{Z}^{dn}$, which contradicts the assumption that $\mathbf{x} \neq \mathbf{y}$ and $\mathbf{x}, \mathbf{y} \in [0,1)^{dn}$.

Next, we show that φ is surjective. Take an arbitrary point $\mathbf{y} \in [0,1)^{dn}$. Since M_* is invertible, the inverse $\psi_*^{-1}(\mathbf{y})$ exists. Let $\mathbf{x}$ be the unique point in $[0,1)^{dn}$ such that $\mathbf{x} - \psi_*^{-1}(\mathbf{y}) \in \mathbb{Z}^{dn}$. Then, we have

$$\begin{aligned}\varphi(\mathbf{x}) = \psi_*(\mathbf{x}) \bmod \mathbb{Z}^{dn} &= \psi_*\left(\psi_*^{-1}(\mathbf{y}) + \mathbf{x} - \psi_*^{-1}(\mathbf{y})\right) \bmod \mathbb{Z}^{dn} \\ &= \psi_*\left(\psi_*^{-1}(\mathbf{y})\right) + \psi_*\left(\mathbf{x} - \psi_*^{-1}(\mathbf{y})\right) \bmod \mathbb{Z}^{dn} = \mathbf{y},\end{aligned}$$

where the last equality holds because ψ_* maps $\mathbb{Z}^{dn}$ into $\mathbb{Z}^{dn}$. This proves that φ is surjective, and hence completes the proof of Lemma 3.4. □

We are now ready to prove Lemma 2.3.

Proof of Lemma 2.3. Given a measurable set $K \subseteq \mathbb{R}^n$ and a lattice $\Lambda \subseteq \mathbb{R}^n$, let

$$\delta := \bar{\rho}(\Lambda + K).$$

By applying a linear transformation to Λ if necessary, we may assume that $\Lambda = \mathbb{Z}^n$. Let $\{\mathbf{e}_i : i \in [n]\}$ be the standard basis of $\mathbb{R}^n$.

Let $d \geqslant 1$ be an integer. Let $C_{3.1}$ be the constant given in Lemma 3.1 and define $C_{2.3} := ((C_{3.1} + 1)\, d)^{2^d - 1}$ depending only on d. Fix a robust lattice covering (Λ_d, B_r^d) of $\mathbb{R}^d$ with density D, where D is sufficiently close to $\tilde{\Theta}_d$. We can assume that D is at most ν_d, which is the density attained by $\mathbb{Z}^d$. By scaling Λ_d and r if necessary, we may assume that $|\det(\Lambda_d)| = 1$. Hence, r is such that $\mathrm{vol}(B_r^d) = D$. By increasing the final constant $C_{2.3}$ slightly, it suffices to prove that there exists a lattice $\tilde{\Lambda} \subseteq \mathbb{R}^{n+d}$ with $\det(\tilde{\Lambda}) = \det(\Lambda)$ satisfying

$$\bar{\rho}\left(\tilde{\Lambda} + \tilde{K}\right) \leqslant C_{2.3}(\bar{\rho}(\Lambda + K))^{2^d},$$

where $\tilde{K} \subseteq \mathbb{R}^{n+d}$ denotes the Cartesian product of K and the d-dimensional sphere of volume D (instead of $\tilde{\Theta}_d$).

Let $\mathbf{b}_1, \dots, \mathbf{b}_d \in \mathbb{R}^d$ be linearly independent vectors that generate the lattice Λ_d. For $i \in [n]$, let $\tilde{\mathbf{e}}_i := \binom{\mathbf{e}_i}{\mathbf{0}} \in \mathbb{R}^{n+d}$ be the concatenation of $\mathbf{e}_i \in \mathbb{R}^n$ and $\mathbf{0} \in \mathbb{R}^d$. For each $j \in [d]$, choose a vector $\mathbf{y}_i \in \mathbb{T}^n$ uniformly at random according to the Lebesgue measure restricted to the cube $[0,1)^n$, and let $\tilde{\mathbf{b}}_j := \binom{\mathbf{y}_j}{\mathbf{b}_j} \in \mathbb{R}^{n+d}$ be the concatenation of $\mathbf{y}_i$ and $\mathbf{b}_i$. Define a new (random) lattice

$$\tilde{\Lambda}(\mathbf{y}_1, \dots, \mathbf{y}_d) := \left\{ \sum_{i=1}^{n} \lambda_i \tilde{\mathbf{e}}_i + \sum_{j=1}^{d} \mu_j \tilde{\mathbf{b}}_j : \lambda_i \in \mathbb{Z} \text{ for } i \in [n] \text{ and } \mu_j \in \mathbb{Z} \text{ for } j \in [d] \right\}.$$

Note that

$$|\det(\tilde{\Lambda}(\mathbf{y}_1, \dots, \mathbf{y}_d))| = |\det(\Lambda)| \cdot |\det(\Lambda_d)| = 1,$$

which can be seen by expanding the determinant of the corresponding matrix along the first n columns (with each having only one non-zero entry, namely the diagonal entry 1).

Recall that $\tilde{K} = K \times B_r^d \subseteq \mathbb{R}^{n+d}$. We will show that, with positive probability, the following event occurs:

$$\bar{\rho}\big(\tilde{\Lambda}(\mathbf{y}_1, \dots, \mathbf{y}_d) + \tilde{K}\big) \leqslant C_{2.3}\delta^{2^d},$$

that is, the set of points $(\mathbf{y}_1, \dots, \mathbf{y}_d) \in \mathbb{T}^{n\times d}$ for which this inequality holds has positive Lebesgue measure. For this we need some further definitions and two auxiliary claims.

Let $B := [\mathbf{b}_1, \dots, \mathbf{b}_d] \in \mathbb{R}^{d\times d}$. Let $\phi_{\mathbf{y}_1,\dots,\mathbf{y}_d} : \Lambda_d \to \mathbb{R}^n$ be the linear map defined by

$$\phi_{\mathbf{y}_1,\dots,\mathbf{y}_d}(\mathbf{z}) = [\mathbf{y}_1, \dots, \mathbf{y}_d]B^{-1}\mathbf{z} \quad \text{for every} \quad \mathbf{z} \in \Lambda_d. \tag{11}$$

In other words, $\phi_{\mathbf{y}_1,\dots,\mathbf{y}_d}$ sends a lattice point $\mathbf{z} \in \Lambda_d$ to $\sum_{j=1}^d \mu_j \mathbf{y}_j \in \mathbb{R}^n$, where $(\mu_1, \dots, \mu_d) \in \mathbb{Z}^d$ is the unique collection of integers such that $\mathbf{z} = \sum_{j=1}^d \mu_j \mathbf{b}_j$. For a set of points $S \subseteq \Lambda_d$, we define

$$\phi_{\mathbf{y}_1,\dots,\mathbf{y}_d}(S) := \{\phi_{\mathbf{y}_1,\dots,\mathbf{y}_d}(x) : x \in S\}.$$

Define

$$\mathcal{P} := \left\{\mathrm{P} \subseteq B_{2r}^d : \mathrm{P} \text{ is a fundamental parallelepiped with } \mathbf{0} \text{ as a vertex}\right\}.$$

Lemma 3.1 implies that $|\mathcal{P}| \leqslant C_{3.1}$.

Let P be a fundamental Λ_d-parallelepiped. For any $\mathbf{y}_1, \dots, \mathbf{y}_d \in \mathbb{T}^n$, we define

$$K_{\mathrm{P}}(\mathbf{y}_1, \dots, \mathbf{y}_d) := K + \phi_{\mathbf{y}_1,\dots,\mathbf{y}_d}(V(\mathrm{P})).$$

In other words, $K_{\mathrm{P}}(\mathbf{y}_1, \dots, \mathbf{y}_d)$ is the union of 2^d translations of K corresponding to the vertices in P. Also, let E_{P} denote the event that

$$\bar{\rho}(\Lambda + K_{\mathrm{P}}(\mathbf{y}_1, \dots, \mathbf{y}_d)) \leqslant C_{2.3}\delta^{2^d}.$$

Our goal is to show that for every $\mathrm{P} \in \mathcal{P}$, the event E_{P} occurs with high probability. Recalling the definition of $\mathrm{P}_{\mathbf{0}}(\mathbf{b}_1, \dots, \mathbf{b}_d)$ (see Equation (3)), we begin with the case of E_{P_*}, where, for convenience, we define

$$\mathrm{P}_* := \mathrm{P}_{\mathbf{0}}(\mathbf{b}_1, \dots, \mathbf{b}_d)$$

Claim 3.5. We have

$$\mathbb{P}\left[E_{\mathrm{P}_*}\right] \geqslant 1 - \frac{1}{C_{3.1} + 1}.$$

Proof of Claim 3.5. Define $K_0(\mathbf{y}_1, \dots, \mathbf{y}_d) := K$, and for each $i \in [d]$, let

$$K_i(\mathbf{y}_1, \dots, \mathbf{y}_d) := K_{i-1}(\mathbf{y}_1, \dots, \mathbf{y}_d) \cup (K_{i-1}(\mathbf{y}_1, \dots, \mathbf{y}_d) + \mathbf{y}_i).$$

It follows from the definition of $\phi_{\mathbf{y}_1,\dots,\mathbf{y}_d}$ that $\phi_{\mathbf{y}_1,\dots,\mathbf{y}_d}(\mathbf{b}_i) = \mathbf{y}_i$ for $i \in [d]$. Therefore,

$$K_{\mathrm{P}_*}(\mathbf{y}_1, \dots, \mathbf{y}_d) = K_d(\mathbf{y}_1, \dots, \mathbf{y}_d).$$

Let

$$C := C_{3.1} + 1.$$

For $i \in [0, d]$, define

$$\boldsymbol{\rho}_i := \bar{\rho}(\Lambda + K_i) \quad \text{and} \quad \delta_i := \frac{(Cd\delta)^{2^i}}{Cd},$$

and let E_i denote the event that $\boldsymbol{\rho}_i \leqslant \delta_i = Cd\delta_{i-1}^2$. By assumption, we have $\boldsymbol{\rho}_0 = \delta = \delta_0$.

Let us prove by induction on $i = 0, \dots, d$ that

$$\mathbb{P}[E_0 \wedge \cdots \wedge E_i] \geqslant \left(1 - \frac{1}{Cd}\right)^i. \tag{12}$$

This is true for $i = 0$ since $\boldsymbol{\rho}_0$ is the constant function δ_0. So suppose that $i \geqslant 1$. If we fix any $\mathbf{y}_1, \dots, \mathbf{y}_{i-1}$ such that E_{i-1} holds and take uniform random $\mathbf{y}_i \in [0,1)^n$ then we have by Markov's inequality and Lemma 3.3 that

$$\mathbb{P}[\boldsymbol{\rho}_i > \delta_i] \leqslant \frac{\mathbb{E}[\boldsymbol{\rho}_i]}{\delta_i} = \frac{\boldsymbol{\rho}_{i-1}^2}{\delta_i} \leqslant \frac{\delta_{i-1}^2}{Cd\delta_{i-1}^2} = \frac{1}{Cd}.$$

Integrating the complement of this inequality over all choices of $\mathbf{y}_1, \dots, \mathbf{y}_{i-1} \in [0,1)^n$ for which $E_0 \wedge \cdots \wedge E_i$ holds, we obtain by Fubini–Tonelli's theorem and induction that

$$\mathbb{P}[E_0 \wedge \cdots \wedge E_i] \geqslant \left(1 - \frac{1}{Cd}\right)\mathbb{P}[E_0 \wedge \cdots \wedge E_{i-1}] \geqslant \left(1 - \frac{1}{Cd}\right)^i,$$

which is the claimed inequality for i.

The claim now follows since

$$\mathbb{P}\left[E_{\mathrm{P}_*}\right] = \mathbb{P}[E_d] \geqslant \mathbb{P}[E_0 \wedge \cdots \wedge E_d] \geqslant \left(1 - \frac{1}{Cd}\right)^d \geqslant 1 - \frac{1}{C}. \qquad \square$$

Next, we extend the conclusion of Claim 3.5 to all elements of $\mathcal{P}$.

Claim 3.6. For every $\mathrm{P} \in \mathcal{P}$, we have

$$\mathbb{P}[E_{\mathrm{P}}] \geqslant 1 - \frac{1}{C_{3.1} + 1}.$$

In particular, with positive probability, all of the events $\{E_{\mathrm{P}} : \mathrm{P} \in \mathcal{P}\}$ occur simultaneously.

Proof of Claim 3.6. Fix $\mathrm{P} \in \mathcal{P}$. Define sets

$$S := \left\{(\mathbf{y}_1, \dots, \mathbf{y}_d) \in \mathbb{T}^n \times \cdots \times \mathbb{T}^n : E_{\mathrm{P}_*} \text{ holds}\right\} \quad \text{and}$$
$$T := \{(\mathbf{y}_1, \dots, \mathbf{y}_d) \in \mathbb{T}^n \times \cdots \times \mathbb{T}^n : E_{\mathrm{P}} \text{ holds}\}$$

Note that $\mathbb{P}\left[E_{\mathrm{P}_*}\right]$ and $\mathbb{P}[E_{\mathrm{P}}]$ are equal to $\mu(S)$ and $\mu(T)$, respectively, where μ denotes the Lebesgue measure. Recall from Claim 3.5 that $\mathbb{P}\left[E_{\mathrm{P}_*}\right] \geqslant 1 - (C_{3.1} + 1)^{-1}$. So, it suffices to show that $\mu(T) \geqslant \mu(S)$ (In fact, a straightforward modification of the argument below shows that S and T have the same Lebesgue measure).

Fix linearly independent vectors $\mathbf{w}_1, \dots, \mathbf{w}_d \in \Lambda^d$ such that

$$V(\mathrm{P}) = \left\{ \sum_{i=1}^{d} \lambda_i \mathbf{w}_i : \lambda_i \in \{0,1\} \text{ for } i \in [d] \right\}.$$

For each collection $\mathbf{z}_1, \dots, \mathbf{z}_d \in \mathbb{T}^n$, let $K_{0,\mathrm{P}}(\mathbf{z}_1, \dots, \mathbf{z}_d) := K$, and for each $i \in [d]$, let

$$K_{i,\mathrm{P}}(\mathbf{z}_1, \dots, \mathbf{z}_d) := K_{i-1,\mathrm{P}}(\mathbf{z}_1, \dots, \mathbf{z}_d) \cup \left(K_{i-1,\mathrm{P}}(\mathbf{z}_1, \dots, \mathbf{z}_d) + \phi_{\mathbf{z}_1,\dots,\mathbf{z}_d}(\mathbf{w}_i)\right).$$

Similar to the proof of Claim 3.5, we have $K_{\mathrm{P}}(\mathbf{z}_1, \dots, \mathbf{z}_d) = K_{d,\mathrm{P}}(\mathbf{z}_1, \dots, \mathbf{z}_d)$.

Recall that $\{\mathbf{b}_1, \dots, \mathbf{b}_d\}$ is a basis of Λ_d and $B = [\mathbf{b}_1, \dots, \mathbf{b}_d] \in \mathbb{R}^{d\times d}$. Let $W := [\mathbf{w}_1, \dots, \mathbf{w}_d] \in \mathbb{R}^{d\times d}$, and let $M \in \mathbb{R}^{d\times d}$ be the matrix such that $WM = B$. It follows that $MB^{-1}W = I$, and thus,

$$MB^{-1}\mathbf{w}_i = \mathbf{e}_i. \tag{13}$$

Given an element $(\mathbf{y}_1, \dots, \mathbf{y}_d) \in S$, define the map $\varphi : \mathbb{T}^{d\times n} \to \mathbb{T}^{d\times n}$ by

$$\varphi([\mathbf{y}_1, \dots, \mathbf{y}_d]^t) := [\psi(\mathbf{y}_1), \dots, \psi(\mathbf{y}_d)]^t,$$

where

$$[\psi(\mathbf{y}_1), \dots, \psi(\mathbf{y}_d)] := [\mathbf{y}_1, \dots, \mathbf{y}_d]M \bmod \mathbb{Z}^{n\times d}.$$

Combining (13) with definition (11), we obtain

$$\begin{aligned}\phi_{\psi(\mathbf{y}_1),\dots,\psi(\mathbf{y}_d)}(\mathbf{w}_i) &= [\psi(\mathbf{y}_1), \dots, \psi(\mathbf{y}_d)]B^{-1}\mathbf{w}_i \\ &= [\mathbf{y}_1, \dots, \mathbf{y}_d]MB^{-1}\mathbf{w}_i = [\mathbf{y}_1, \dots, \mathbf{y}_d]\mathbf{e}_i = \mathbf{y}_i.\end{aligned}$$

It follows that

$$K_{\mathrm{P}}(\psi(\mathbf{y}_1), \dots, \psi(\mathbf{y}_d)) = K_{\mathrm{P}_*}(\mathbf{y}_1, \dots, \mathbf{y}_d).$$

Since $(\mathbf{y}_1, \dots, \mathbf{y}_d) \in S$, it follows from the definition of S that

$$\bar{\rho}(\Lambda + K_{\mathrm{P}}(\psi(\mathbf{y}_1), \dots, \psi(\mathbf{y}_d))) = \bar{\rho}\left(\Lambda + K_{\mathrm{P}_*}(\mathbf{y}_1, \dots, \mathbf{y}_d)\right) \leqslant C_{2.3}\delta^{2^d},$$

which implies that $(\psi(\mathbf{y}_1), \dots, \psi(\mathbf{y}_d)) \in T$.

Since P is a fundamental parallelepiped of Λ_d, the matrix M must be an integer matrix with determinant ± 1.

It follows from Lemma 3.4 that the map φ on $\mathbb{T}^{d\times n}$ is measure-preserving. Therefore,

$$\mathbb{P}[E_{\mathrm{P}}] = \mu(T) \geqslant \mu(S) = \mathbb{P}\left[E_{\mathrm{P}_*}\right] \geqslant 1 - \frac{1}{C_{3.1}+1},$$

which proves Claim 3.6. □

Fix a collection $(\mathbf{y}_1, \ldots, \mathbf{y}_d) \subseteq \mathbb{T}^{n\times d}$ such that all of the events $\{E_{\mathrm{P}} : \mathrm{P} \in \mathcal{P}\}$ occur simultaneously. For convenience, let $\tilde{\Lambda} := \tilde{\Lambda}(\mathbf{y}_1, \ldots, \mathbf{y}_d)$ and $\phi := \phi_{\mathbf{y}_1,\ldots,\mathbf{y}_d}$.

For every lattice point $\mathbf{z} \in \Lambda_d$, define

$$\Lambda_{\mathbf{z}} := \left\{\mathbf{x} : (\mathbf{x}, \mathbf{z}) \in \tilde{\Lambda}\right\}.$$

Note that, by definition, for every $\mathbf{z} \in \Lambda_d$, $\Lambda_{\mathbf{z}}$ can be written as

$$\Lambda_{\mathbf{z}} = \Lambda + \phi(\mathbf{z}). \tag{14}$$

This means that $\Lambda_{\mathbf{z}}$ is simply a translation of the lattice Λ (which we assumed to be $\mathbb{Z}^n$).

Fix a point $\mathbf{w} \in \mathbb{R}^d$. Since (Λ_d, B_r^d) is robust, there exists a fundamental parallelepiped $\mathrm{P}_{\mathbf{w}}$ that is contained in the ball $B_r^d(\mathbf{w})$. Fix a point $\mathbf{s} \in V(\mathrm{P}_{\mathbf{w}})$ and let $\mathrm{P}'_{\mathbf{w}} := \mathrm{P}_{\mathbf{w}} - \mathbf{s}$ be obtained from $\mathrm{P}_{\mathbf{w}}$ by translating by $-\mathbf{s}$. Note that $\mathrm{P}'_{\mathbf{w}}$ is a fundamental parallelepiped contained in B_{2r}^d and $\mathbf{0}$ is a vertex in $\mathrm{P}'_{\mathbf{w}}$, that is, $\mathrm{P}'_{\mathbf{w}} \in \mathcal{P}$.

Consider the following restriction of the set $\tilde{K} + \tilde{\Lambda}$:

$$(\tilde{K} + \tilde{\Lambda})\mid_{\mathbf{w}} := \left\{\mathbf{x} \in \mathbb{R}^n : (\mathbf{x}, \mathbf{w}) \in \tilde{K} + \tilde{\Lambda}\right\}.$$

Since $\tilde{K} = K \times B_r^d$, a point $\mathbf{x} \in \mathbb{R}^n$ is covered by $(\tilde{K} + \tilde{\Lambda})\mid_{\mathbf{w}}$ if there exists a point $\tilde{\mathbf{w}} \in \Lambda_d \cap B_r(\mathbf{w})$ such that $\mathbf{x} \in \Lambda_{\tilde{\mathbf{w}}} + K$. In particular, the set $\bigcup_{\tilde{\mathbf{w}} \in V(\mathrm{P}_{\mathbf{w}})} (\Lambda_{\tilde{\mathbf{w}}} + K) \subseteq \mathbb{R}^n$ is covered by $(\tilde{K} + \tilde{\Lambda})\mid_{\mathbf{w}}$. By (14) and the linearity of ϕ, we have

$$\begin{aligned}\bigcup_{\tilde{\mathbf{w}} \in V(\mathrm{P}_{\mathbf{w}})} (\Lambda_{\tilde{\mathbf{w}}} + K) &= \Lambda + \phi(V(\mathrm{P}_{\mathbf{w}})) + K = \Lambda + \phi(V(\mathrm{P}_{\mathbf{w}}) - \mathbf{s} + \mathbf{s}) + K \\ &= \phi(\mathbf{s}) + \Lambda + \phi\left(V\left(\mathrm{P}'_{\mathbf{w}}\right)\right) + K = \phi(\mathbf{s}) + \Lambda + K_{\mathrm{P}'_{\mathbf{w}}}(\mathbf{y}_1, \ldots, \mathbf{y}_d).\end{aligned}$$

Since $\mathrm{P}'_{\mathbf{w}} \in \mathcal{P}$, the choice of vectors $\{\mathbf{y}_1, \ldots, \mathbf{y}_d\}$ guarantees that

$$\bar{\rho}\left(\Lambda + K_{\mathrm{P}'_{\mathbf{w}}}(\mathbf{y}_1, \ldots, \mathbf{y}_d)\right) \leqslant C_{2.3}\delta^{2^d}.$$

Since translation (by $\phi(\mathbf{s})$) does not affect the density of uncovered points, we also have

$$\bar{\rho}\left(\bigcup_{\tilde{\mathbf{w}} \in V(\mathrm{P}_{\mathbf{w}})} (\Lambda_{\tilde{\mathbf{w}}} + K)\right) \leqslant C_{2.3}\delta^{2^d}.$$

Since $\mathbf{w} \in \mathbb{R}^d$ was arbitrary, it follows that $\bar{\rho}\left(\tilde{\Lambda} + \tilde{K}\right) \leqslant C_{2.3}\delta^{2^d}$. This completes the proof of Lemma 2.3. □

4 | CONCLUDING REMARKS

Rogers' original proof in [22] is essentially the same as our proof of Theorem 1.3 in the special case $d = 1$. We hope that the idea of using an iterative step where the dimension increases by more than 1 will lead to further improvements (via Theorem 1.3 or some other estimates).

Unfortunately, we have little intuition about the optimal robust lattice covering density in dimensions $d \geqslant 4$. The presented constant β in Theorem 1.1 was the best one that came from our sporadic search for $d \leqslant 3$. So, the natural question motivated by Theorem 1.3 is the following.

Problem 4.1. Determine $\tilde{\Theta}_n$. In particular, what is the infimum of

$$\frac{1}{n} \log_2 \left(\tilde{\Theta}_n / \nu_n\right)?$$

A lower bound $\tilde{\Theta}_n \geqslant \nu_n / 2^n$ can be established via the following argument. Let (Λ, B_r^n) be a robust lattice covering of $\mathbb{R}^n$ with $\det(\Lambda) = 1$. By definition, B_r^n contains a fundamental parallelepiped P. It is not hard to show that the largest volume of a parallelepiped (not necessarily a Λ-parallelepiped) contained in B_r^n is $\left(2r/\sqrt{n}\right)^n$, attained by an inscribed cube centred at the origin. Since $\mathrm{P} \subseteq B_r^n$ is a parallelepiped with volume $\det(\Lambda) = 1$, it follows that

$$\left(2r/\sqrt{n}\right)^n \geqslant \mathrm{vol}(\mathrm{P}) = |\det \Lambda| = 1,$$

which implies that $r \geqslant \sqrt{n}/2$. Therefore, we obtain the bound

$$\tilde{\Theta}_n \geqslant \frac{\mathrm{vol}(B_r^n)}{|\det(\Lambda)|} \geqslant \mathrm{vol}\left(B^n_{\sqrt{n}/2}\right) \geqslant \frac{1}{2^n}\,\mathrm{vol}\left(B^n_{\sqrt{n}}\right) = \frac{\nu_n}{2^n}.$$

In particular, this yields $\frac{1}{n} \log_2 \left(\tilde{\Theta}_n / \nu_n\right) \geqslant -1$, which means that the best possible value we can hope for γ via an application of Theorem 1.3 as stated is at least $\frac{1}{2}\log_2(2\pi \mathrm{e}) - 1 = 1.0471\ldots$.

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