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PHASE TRANSITION OF DEGENERATE TURÁN PROBLEMS IN p-NORMS*

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Abstract. For a positive real number p, the p-norm $\|G\|_p$ of a graph G is the sum of the pth powers of all vertex degrees. We study the maximum p-norm $\mathrm{ex}_p(n,F)$ of F-free graphs on n vertices. Füredi and Kündgen [*J. Graph Theory*, 51 (2006), pp. 37–48] showed that for every bipartite graph F, there exists a threshold p_F such that for $p < p_F$, the order of $\mathrm{ex}_p(n,F)$ is governed by pseudorandom constructions, while for $p > p_F$, it is governed by star-like constructions, assuming a mild assumption on the growth rate of $\mathrm{ex}(n,F)$. The main contribution of our paper is extending this result to hypergraphs. Moreover, in the case of graphs, our proof differs from that in [Z. Füredi and A. Kündgen, *J. Graph Theory*, 51 (2006), pp. 37–48], offering the advantage of producing the correct constant factor when $p > p_F$. When $p = p_F$, Füredi and Kündgen proved a general upper bound on $\mathrm{ex}_p(n,F)$ that is tight up to a $\log n$ factor and conjectured that this factor is unnecessary. We confirm this conjecture for several well-studied bipartite graphs, including one-sided degree-bounded graphs that meet Füredi's bound and families of short even cycles.

Key words. degenerate Turán problem, degree powers, counting stars, phase transition

MSC codes. 05C07, 05C35

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1. Introduction. Given an integer $r \geq 2$, an *r-uniform hypergraph* (henceforth an *r-graph*) on a set V is a subset $\mathcal{H}$ of $\binom{V}{r} := \{X \subseteq V : |X| = r\}$. We identify a hypergraph $\mathcal{H}$ with its edge set and use $V(\mathcal{H})$ to denote its vertex set. The size of $V(\mathcal{H})$ is denoted by $v(\mathcal{H})$. The *degree* $d_{\mathcal{H}}(v)$ of v in $\mathcal{H}$ is the number of edges in $\mathcal{H}$ containing v.

Given an r-graph $\mathcal{H}$ and a real number $p \geq 0$, let the *p-norm* of $\mathcal{H}$ be defined as

$$\|\mathcal{H}\|_p := \sum_{v \in V(\mathcal{H})} d^p_{\mathcal{H}}(v),$$

where, for convenience, we write $d^p_{\mathcal{H}}(v) := (d_{\mathcal{H}}(v))^p$.

Given a family $\mathcal{F}$ of r-graphs, we say that an r-graph $\mathcal{H}$ is *$\mathcal{F}$-free* if it does not contain any member of $\mathcal{F}$ as a subgraph. The *p-norm Turán number* of $\mathcal{F}$ is defined as

$$\mathrm{ex}_p(n,\mathcal{F}) := \max\left\{ \|\mathcal{H}\|_p : v(\mathcal{H}) = n \text{ and } \mathcal{H} \text{ is } \mathcal{F}\text{-free} \right\}.$$

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The case $p=1$ corresponds to the *Turán number* $\mathrm{ex}(n,\mathcal{F})$ of $\mathcal{F}$ (differing only by a multiplicative factor of r), which represents the maximum number of edges in an n-vertex $\mathcal{F}$-free r-graph.

Extending the seminal work of Turán [Tur41], Caro–Yuster [CY00, CY04] initiated[1] the study of the p-norm Turán problem for graphs by determining the value of $\mathrm{ex}_p(n,K_{\ell+1})$ for $p\geq 1$. This line of research has since been extended to various other graphs and hypergraphs, as explored in works such as [Nik09, BN12, LLQS19, BCL22b, BCL22a, Zha22, Ger24, CIL$^+$24]. In this work, we focus on the case where $\mathcal{F}$ is degenerate.

The *Turán density* of $\mathcal{F}$ is defined as $\pi(\mathcal{F}):=\lim_{n\to\infty}\mathrm{ex}(n,\mathcal{F})/\binom{n}{r}$. A family $\mathcal{F}$ of r-graphs is called *degenerate* if $\pi(\mathcal{F})=0$. According to a classical theorem of Erdős [Erd64b], this is equivalent to stating that $\mathcal{F}$ contains at least one r-partite r-graph. Determining the growth rate of $\mathrm{ex}(n,\mathcal{F})$ for degenerate families is a central and notoriously difficult topic in extremal combinatorics, and it remains unresolved for most families. For example, the Even Cycle Problem proposed by Erdős [Erd64a, BS74], which asks for the exponent of $\mathrm{ex}(n,C_{2k})$, is still open for every k not in $\{2,3,5\}$ (see, e.g., [ERS66, Ben66, Wen91, LU93, LUW99]). For more results on degenerate Turán problems, we refer the reader to the survey [FS13].

For an r-partite r-graph F, the *partition number* $\tau_{\mathrm{part}}(F)$ of F is defined as the minimum size of a set $S_1\subseteq V(F)$ such that $V(F)\setminus S_1$ can be partitioned into $r-1$ sets $S_2,\ldots,S_r$, with each edge of F containing exactly one vertex from each S_i. The *independent covering number* $\tau_{\mathrm{ind}}(F)$ of F is defined as the minimum size of a set S such that every edge of F contains exactly one vertex from S. It is clear from the definition that $\tau_{\mathrm{ind}}(F)\leq\tau_{\mathrm{part}}(F)$ for every r-partite r-graph F, and $\tau_{\mathrm{ind}}(F)=\tau_{\mathrm{part}}(F)$ for every bipartite graph F.

Given the definitions that we have introduced, we can immediately derive the following two general lower bounds for $\mathrm{ex}_p(n,F)$.

FACT 1.1. *Let $r\geq 2$ be an integer and F be an r-partite r-graph. For every real number $p\geq 1$, we have*

$$\mathrm{ex}_p(n,F)\geq\max\left\{n\left(\frac{r\cdot\mathrm{ex}(n,F)}{n}\right)^p,(\tau_{\mathrm{ind}}(F)-1)\binom{n-\tau_{\mathrm{ind}}(F)+1}{r-1}^p\right\}.$$

The first lower bound arises from an optimal construction for $\mathrm{ex}(n,\mathcal{F})$ as well as convexity (see Corollary 2.5). The second lower bound is based on the star-like r-graph $S^r(n,t)$ for $t=\tau_{\mathrm{ind}}(F)-1$, where

$$S^r(n,t):=\left\{e\in\binom{[n]}{r}:|e\cap[t]|=1\right\},\quad\text{and}\quad[n]:=\{1,\ldots,n\}.$$

Our work is motivated by the combination of the following facts in graphs. For $p=1$, the lower bound constructions for $\mathrm{ex}_1(n,\mathcal{F})$ often exhibit certain pseudorandom properties (see, e.g., [KRS96, ARS99, MYZ18, PZ21]) and, in particular, are almost regular, meaning that the maximum and minimum degrees differ by only a constant factor. In contrast, works of Caro and Yuster [CY00], Nikiforov [Nik09], and Gerbner [Ger24] on even cycles and complete bipartite graphs showed that for large p, the lower bound constructions for $\mathrm{ex}_p(n,\mathcal{F})$ are highly structured and resemble $S^2(n,t)$ for some appropriate choice of t.

[1]According to the introduction in [FK06], it seems that $\mathrm{ex}_p(n,K_t)$ was already considered by Erdős in the 1970s (see [Erd70]).

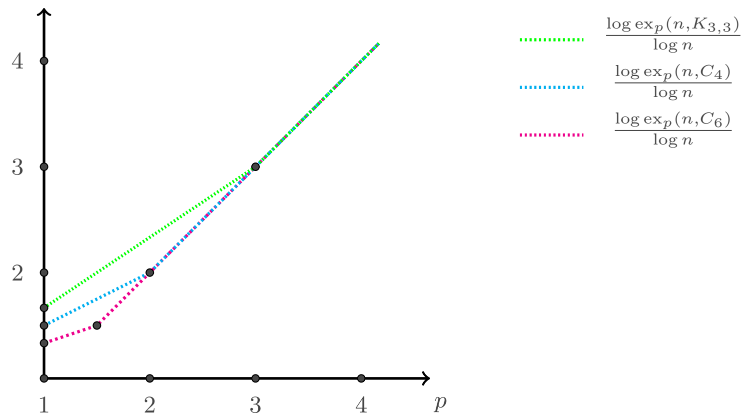

FIG. 1. *Exponents of* $\mathrm{ex}_p(n,K_{3,3})$, $\mathrm{ex}_p(n,C_4)$, *and* $\mathrm{ex}_p(n,C_6)$.

This contrast suggests that a general phenomenon (see Figure 1) may hold: for every degenerate family $\mathcal{F}$ of r-graphs with $\mathrm{ex}(n,\mathcal{F})=\Omega(n^{1+\alpha})$ for some $\alpha>r-2$, there exists a threshold $p_{\mathcal{F}}>1$ such that, for $p\in(1,p_{\mathcal{F}})$, $\mathrm{ex}_p(n,\mathcal{F})=O(n(\frac{\mathrm{ex}(n,\mathcal{F})}{n})^p)$, while for $p>p_{\mathcal{F}}$, $\mathrm{ex}_p(n,\mathcal{F})=O\left(n^{p(r-1)}\right)$. Füredi and Kündgen [FK06] showed that this holds for $r=2$. In the following theorem, we show that this holds for all $r\geq 2$.

For a family $\mathcal{F}$ of r-graphs, we define

$$\tau_{\mathrm{part}}(\mathcal{F}):=\min\{\tau_{\mathrm{part}}(F)\colon F\in\mathcal{F} \text{ is } r\text{-partite}\} \quad \text{and}$$
$$\tau_{\mathrm{ind}}(\mathcal{F}):=\min\{\tau_{\mathrm{ind}}(F)\colon F\in\mathcal{F} \text{ is } r\text{-partite}\}.$$

THEOREM 1.2. *Let $r\geq 2$ be an integer and $p>1$ be a real number. Suppose that $\mathcal{F}$ is a degenerate family of r-graphs satisfying $\mathrm{ex}(n,\mathcal{F})=O(n^{1+\alpha})$ for some constant $\alpha\in[r-2,r-1)$. Then there exists a constant $C_{\mathcal{F}}>0$ such that*

$$\mathrm{ex}_p(n,\mathcal{F})\leq\begin{cases} C_{\mathcal{F}}\cdot n^{1+p\alpha}, & \text{if} \quad 1<p<\frac{1}{r-1-\alpha},\\ (\tau_{\mathrm{part}}(\mathcal{F})-1+o(1))\binom{n}{r-1}^p, & \text{if} \quad p>\frac{1}{r-1-\alpha}.\end{cases}$$

In particular, for $r=2$, we have, for every $p>\frac{1}{1-\alpha}$,

$$\mathrm{ex}_p(n,\mathcal{F})=(\tau_{\mathrm{ind}}(\mathcal{F})-1+o(1))\,n^p.$$

Remarks.

- The Rational Exponent Conjecture of Erdős and Simonovits (see [FS13, Conjecture 1.6]) states that for every degenerate finite family $\mathcal{F}$ of graphs, there exist a rational number α and a constant $c>0$ such that
$$\lim_{n\to\infty}\frac{\mathrm{ex}(n,\mathcal{F})}{n^{1+\alpha}}=c.$$
Note that by Corollary 2.5, if this conjecture holds, then Theorem 1.2 is tight in the exponent for every $p\in(1,\frac{1}{r-1-\alpha})$ when $r=2$.
- If $\mathrm{ex}(n,\mathcal{F})=O(n^{1+\beta})$ for some $\beta\leq r-2$, then by taking $\alpha=r-2$ in Theorem 1.2, we obtain $\frac{1}{r-1-\alpha}=1$, and hence,
$$\mathrm{ex}_p(n,\mathcal{F})\leq(\tau_{\mathrm{part}}(\mathcal{F})-1+o(1))\binom{n}{r-1}^p \quad \text{for every} \quad p>1.$$

This bound is tight in the exponent unless $\mathcal{F}$ contains an r-graph F with $\tau_{\text{part}}(F) = 1$. In that case, it is straightforward to show that, for $r = 2$, either $\mathrm{ex}_p(n,\mathcal{F}) = \Theta(n)$ (if $\mathrm{ex}(n,\mathcal{F}) = \Theta(n)$) or $\mathrm{ex}_p(n,\mathcal{F}) = \Theta(1)$ (if $\mathrm{ex}(n,\mathcal{F}) = \Theta(1)$) for every $p \ge 1$. The case $r \ge 3$ seems to be more complex, even in the special case of intersection problems (when each forbidden r-graph has only 2 edges); see [FT16] for a survey.

For p at the threshold, i.e., for $p = \frac{1}{r-1-\alpha}$, Füredi and Kündgen [FK06] prove a general upper bound that is tight up to a $\log n$ factor for $\mathrm{ex}_p(n,F)$ when $r = 2$. In the following theorem, we generalize this result to $r \ge 3$.

THEOREM 1.3. *Let $r \ge 2$ be an integer. Suppose that $\mathcal{F}$ is a degenerate family of r-graphs satisfying* $\mathrm{ex}(n,\mathcal{F}) = O(n^{1+\alpha})$ *for some constant $\alpha \ge r-2$. Then*

$$\mathrm{ex}_{p_*}(n,\mathcal{F}) = O\left(n^{p_*(r-1)} \log n\right) \quad \text{where} \quad p_* := \frac{1}{r-1-\alpha}.$$

We conjecture that the $\log n$ factor in Theorem 1.3 can be removed, thus extending the conjecture of Füredi and Kündgen [FK06], who made it for $r = 2$. In support of this conjecture, we prove it for several well-studied families of bipartite graphs in the following theorem.

Given a bipartite graph F with two parts V_1 and V_2, we say F is *s-bounded* if every vertex in V_2 has degree at most s. A celebrated theorem of Füredi [Für91], later refined by Alon, Krivelevich, and Sudakov [AKS03], establishes that $\mathrm{ex}(n,F) = O(n^{2-\frac{1}{s}})$ for every s-bounded bipartite graph F. This bound is tight for graphs such as complete bipartite graphs $K_{s,t}$ when t is sufficiently large [KRS96, ARS99, Buk24].

THEOREM 1.4. *The following statements hold for sufficiently large n.*

(i) $\mathrm{ex}_{\ell/(\ell-1)}(n,\{C_4,C_6,\ldots,C_{2\ell}\}) \le 765 n^{\frac{\ell}{\ell-1}}$ *for every $\ell \ge 3$.*

(ii) $\mathrm{ex}_{3/2}(n,C_6) \le 2164 n^{3/2}$.

(iii) *Suppose that F is an s-bounded bipartite graph. Then*

$$\mathrm{ex}_s(n,F) \le 2\left(\frac{|V(F)|^s}{s!} + |V(F)|\right) n^s.$$

This paper is organized as follows. In section 2, we present some preliminary results. In section 3, we introduce a p-norm extension of the classical Δ-Almost-Regularization Theorem by Erdős and Simonovits. The proofs of Theorems 1.2, 1.3, and 1.4 are provided in sections 4, 5, and 6, respectively. Section 7 includes some open problems and concluding remarks.

Remark. After the preprint was posted on arXiv, Dániel Gerbner informed us that results similar to Theorems 1.2 and 1.4 for the case $r = 2$ were already proved by Füredi and Kündgen in [FK06, Theorem 3.3] using an elegant and concise argument. Our proofs of both theorems appear to be quite different from the approach taken by Füredi and Kündgen. In the case $p < 1/(1-\alpha)$, our proof relies on a p-norm adaptation of the classical Δ-Almost-Regularization Theorem by Erdős and Simonovits, which is of independent interest. In the case $p > 1/(1-\alpha)$, our proof has the additional advantage of providing the tight main term.

2. Preliminaries.

We present some notation and preliminary results that will be used in the subsequent proofs.

Given an r-graph $\mathcal{H}$, we use $\delta(\mathcal{H})$, $\Delta(\mathcal{H})$, and $d(\mathcal{H})$ to denote the *minimum*, *maximum*, and *average degree* of $\mathcal{H}$, respectively. For a vertex $v \in V(\mathcal{H})$, the *link* $L_{\mathcal{H}}(v)$ of v is defined as the $(r-1)$-graph consisting of all $(r-1)$-sets S such that $S \cup \{v\} \in \mathcal{H}$. We will omit the subscript $\mathcal{H}$ if it is clear from the context.

Unless otherwise stated, all asymptotic notations in this paper are considered with respect to n. Floors and ceilings will be omitted when they are not critical to the proofs. The base of log is assumed to be 2.

FACT 2.1. *Let $p \geq 1$ and $x \geq y \geq 0$ be real numbers. Then*

$$(x^p + y^p)^{1/p} \geq x \geq \frac{x+y}{2}.$$

FACT 2.2 (power mean inequality). *Let $p > q \geq 1$ be two real numbers and $x_1, \ldots, x_n$ be nonnegative real numbers. Then*

$$\left(\frac{\sum_{i\in[n]} x_i^p}{n}\right)^{1/p} \geq \left(\frac{\sum_{i\in[n]} x_i^q}{n}\right)^{1/q}.$$

FACT 2.3 (Minkowski's inequality). *Let $p \geq 1$ and $x_1, \ldots, x_n, y_1, \ldots, y_n$ be real numbers. Then*

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p}.$$

In particular, for every $p \geq 1$ and $x, y \geq 0$,

$$(x^p + y^p)^{1/p} \leq x + y.$$

FACT 2.4. *Let $p > q \geq 1$ be two real numbers and $\mathcal{H}$ be an r-graph on n vertices. Then*

$$\begin{aligned}\|\mathcal{H}\|_p = \sum_{v\in V(\mathcal{H})} d_{\mathcal{H}}^p(v) &= \sum_{v\in V(\mathcal{H})} d_{\mathcal{H}}^{q+(p-q)}(v) \\ &\leq \sum_{v\in V(\mathcal{H})} d_{\mathcal{H}}^q(v) \cdot (\Delta(\mathcal{H}))^{p-q} = \|\mathcal{H}\|_q \cdot (\Delta(\mathcal{H}))^{p-q}.\end{aligned}$$

In particular, $\|\mathcal{H}\|_p \leq \|\mathcal{H}\|_q \cdot n^{(r-1)(p-q)}$.

The following result is an immediate corollary of Fact 2.2.

COROLLARY 2.5. *Let $p > q \geq 1$ be two real numbers and $\mathcal{H}$ be an n-vertex r-graph. Then*

$$\left(\frac{\|\mathcal{H}\|_p}{n}\right)^{1/p} \geq \left(\frac{\|\mathcal{H}\|_q}{n}\right)^{1/q}.$$

Consequently, $\|\mathcal{H}\|_p \geq n(\|\mathcal{H}\|_q / n)^{p/q}$, and hence,

$$\text{ex}_p(n, \mathcal{F}) \geq n\left(\frac{\text{ex}_1(n, \mathcal{F})}{n}\right)^p = n\left(\frac{r \cdot \text{ex}(n, \mathcal{F})}{n}\right)^p. \tag{2.1}$$

Given an r-graph $\mathcal{H}$ and a vertex set $U \subseteq V(\mathcal{H})$, we use $\mathcal{H}[U]$ to denote the *induced subgraph* of $\mathcal{H}$ on U. Similarly, for r pairwise disjoint vertex sets $V_1, \ldots, V_r \subseteq V(\mathcal{H})$, we use $\mathcal{H}[V_1, \ldots, V_r]$ to denote the collection of edges in $\mathcal{H}$ that contain exactly one vertex from each V_i.

PROPOSITION 2.6. *Let $r \geq 2$ be an integer and $p \geq 1$ be a real number. Let $\mathcal{G}$ be an r-graph on an n-set V and let $U \subseteq V$ be a vertex set. For every $m \leq n$, there exists a set $W \subseteq V$ of size m such that the induced subgraph $\mathcal{H} := \mathcal{G}[U \cup W]$ satisfies*

$$\sum_{v \in U} d^p_{\mathcal{H}}(v) \geq (1 + o_m(1)) \left(\frac{m}{n}\right)^{p(r-1)} \sum_{v \in U} d^p_{\mathcal{G}}(v).$$

Proof of Proposition 2.6. Choose uniformly at random an m-set $\mathbf{W}$ from V. For each $v \in U$, an edge $e \in L_{\mathcal{G}}(v)$ is contained in $\mathbf{W}$ with probability

$$\mathbb{P}[e \subseteq \mathbf{W}] = \frac{\binom{n-(r-1)}{m-(r-1)}}{\binom{n}{m}} = (1 + o_m(1)) \left(\frac{m}{n}\right)^{r-1}.$$

For every $v \in U$, let $d_{\mathcal{G}}(v, \mathbf{W}) := |L_{\mathcal{G}}(v) \cap \binom{\mathbf{W}}{r-1}|$, noting from the equation above that $\mathbb{E}[d_{\mathcal{G}}(v, \mathbf{W})] = (1 + o_m(1)) \left(\frac{m}{n}\right)^{r-1} d_{\mathcal{G}}(v)$. Combining this with Fact 2.2 and the linearity of expectation, we obtain

$$\begin{aligned}
\mathbb{E}\left[\sum_{v \in U} d^p_{\mathcal{G}}(v, \mathbf{W})\right] &= \sum_{v \in U} \mathbb{E}\left[d^p_{\mathcal{G}}(v, \mathbf{W})\right] \\
&\geq \sum_{v \in U} \mathbb{E}\left[d_{\mathcal{G}}(v, \mathbf{W})\right]^p \\
&= \sum_{v \in U} \left((1 + o_m(1)) \left(\frac{m}{n}\right)^{r-1} d_{\mathcal{G}}(v)\right)^p \\
&= (1 + o_m(1)) \left(\frac{m}{n}\right)^{p(r-1)} \sum_{v \in U} d^p_{\mathcal{G}}(v).
\end{aligned}$$

Therefore, there exists a set $W \subseteq V$ of size m such that the induced subgraph $\mathcal{H} := \mathcal{G}[U \cup W]$ satisfies $\sum_{v \in U} d^p_{\mathcal{H}}(v) \geq \sum_{v \in U} d^p_{\mathcal{G}}(v, W) \geq (1 + o_m(1)) \left(\frac{m}{n}\right)^{p(r-1)} \sum_{v \in U} d^p_{\mathcal{G}}(v)$. ∎

THEOREM 2.7 (Erdős [Erd64b]). *For every degenerate family $\mathcal{F}$ of r-graphs on n vertices, there exists a constant $\delta > 0$ such that*

$$\mathrm{ex}(n, \mathcal{F}) = O\left(n^{r-\delta}\right).$$

We say an r-graph $\mathcal{H}$ is *semibipartite* if there exists a bipartition $V_1 \cup V_2 = V(\mathcal{H})$ such that every edge in $\mathcal{H}$ contains exactly one vertex from V_1, in which case we also call it $|V_1|$ by $|V_2|$ semibipartite. For convenience, we write $\mathcal{H} = \mathcal{H}[V_1, V_2]$ to emphasize that $\mathcal{H}$ is semibipartite with respect to the bipartition $V_1 \cup V_2 = V(\mathcal{H})$. Given a family $\mathcal{F}$ of r-graphs, we use $\mathrm{ex}(m, n, \mathcal{F})$ to denote the maximum number of edges in an m by n semibipartite $\mathcal{F}$-free r-graph. The function $\mathrm{ex}_p(m, n, \mathcal{F})$ is defined analogously: for every $p \geq 1$, $\mathrm{ex}_p(m, n, \mathcal{F})$ is the maximum p-norm of an m by n semibipartite $\mathcal{F}$-free r-graph.

PROPOSITION 2.8. *Every r-graph $\mathcal{G}$ on n vertices contains a balanced r-partite subgraph $\mathcal{H}$ such that*

$$\|\mathcal{H}\|_p \geq \left(\frac{r!}{r^r} + o(1)\right)^p \|\mathcal{G}\|_p.$$

In particular, for $r = 2$ we have

$$\mathrm{ex}_p(n, n, \mathcal{F}) \geq \left(\frac{1}{2} + o(1)\right)^p \mathrm{ex}_p(2n, \mathcal{F}). \tag{2.2}$$

Proof of Proposition 2.8. Choose a balanced r-partition $V_1 \cup \cdots \cup V_r = V(\mathcal{G})$ uniformly at random. More specifically, we first fix integers $m_1, \ldots, m_r$ satisfying $m_r + 1 \geq m_1 \geq \cdots \geq m_r$ and $m_1 + \cdots + m_r = n$. Then we inductively select uniformly at random an m_i-set V_i from $V(\mathcal{G}) \setminus (V_0 \cup V_1 \cup \cdots \cup V_{i-1})$, where $V_0 := \emptyset$. Let $\mathbf{G} := \mathcal{G}[V_1, \ldots, V_r]$ and $V := V(\mathcal{G})$. Similarly to the proof of Proposition 2.6, it follows from Fact 2.2 and the linearity of expectation that

$$\begin{aligned}\mathbb{E}\left[\|\mathbf{G}\|_p\right] = \sum_{v \in V} \mathbb{E}\left[d_{\mathbf{G}}^p(v)\right] &\geq \sum_{v \in V} \left(\mathbb{E}\left[d_{\mathbf{G}}(v)\right]\right)^p \\ &= \sum_{v \in V} \left(\sum_{e \in \mathcal{G}\colon v \in e} \mathbb{P}\left[e \in \mathbf{G}\right] \right)^p \\ &= \sum_{v \in V} \left(\left(\frac{r!}{r^r} + o(1) \right) \cdot d_{\mathcal{G}}(v) \right)^p = \left(\frac{r!}{r^r} + o(1) \right)^p \|\mathcal{G}\|_p .\end{aligned}$$

Therefore, there exists a balanced r-partition $V_1 \cup \cdots \cup V_r = V(\mathcal{G})$ such that the r-partite subgraph $\mathcal{H} := \mathcal{G}[V_1, \ldots, V_r]$ satisfies $\|\mathcal{G}\|_p \geq \left(\frac{r!}{r^r} + o(1)\right)^p \|\mathcal{G}\|_p$. ∎

Let $K^r_{s_1,\ldots,s_r}$ be the complete r-partite r-graph with parts of sizes $s_1, \ldots, s_r$, respectively. Extending classical theorems of Kővári, Sós, and Turán [KST54] and Erdős [Erd64b], the following upper bound for $\mathrm{ex}(m, n, K^r_{s_1,\ldots,s_r})$ was proved in [HHL+23].

PROPOSITION 2.9 (see [HHL+23, Proposition 2.1]). *Suppose that $r \geq 3$, $s_r \geq \cdots \geq s_1 \geq 1$, and $m, n \geq 1$ are integers. Then*

$$\mathrm{ex}(m, n, K^r_{s_1,\ldots,s_r}) \leq \frac{(s_2 + \cdots + s_r - r + 1)^{\frac{1}{s_1}}}{r-1} m n^{r-1-\frac{1}{s_1 \cdots s_{r-1}}} + (s_1 - 1)\binom{n}{r-1}.$$

PROPOSITION 2.10. *Let $r \geq 2$ be an integer and $\mathcal{F}$ be a degenerate family of r-graphs. Suppose that $\mathrm{ex}(n, \mathcal{F}) = O(n^{1+\alpha})$ for some constant α. Then there exist constants $C_{\mathcal{F}}, N_0$ such that*

$$\mathrm{ex}(m, n, \mathcal{F}) \leq C_{\mathcal{F}} m^{1+\alpha-(r-1)} n^{r-1} \quad \textit{for all} \quad n \geq m \geq N_0.$$

Proof of Proposition 2.10. Let C, N_0 be constants such that $\mathrm{ex}(n, \mathcal{F}) \leq C n^{1+\alpha}$ for every $n \geq N_0$. Let $C_{\mathcal{F}} := 2^{2+\alpha} C$. Suppose to the contrary that there exists an $\mathcal{F}$-free m by n semibipartite r-graph $\mathcal{G} = \mathcal{G}[V_1, V_2]$ with $|\mathcal{G}| > C_{\mathcal{F}} m^{1+\alpha-(r-1)} n^{r-1}$, where $n \geq m \geq N_0$. Similar to the proof of Proposition 2.6, there exists a set $U \subseteq V_2$ of size m such that the induced subgraph $\mathcal{H} := \mathcal{G}[V_1 \cup U]$ satisfies

$$\begin{aligned}|\mathcal{H}| = \sum_{v \in V_1} d_{\mathcal{H}}(v) &\geq (1 + o(1)) \left(\frac{m}{n}\right)^{r-1} \sum_{v \in V_1} d_{\mathcal{G}}(v) \\ &\geq \frac{1}{2} \left(\frac{m}{n}\right)^{r-1} |\mathcal{G}| \\ &> \frac{1}{2} \left(\frac{m}{n}\right)^{r-1} C_{\mathcal{F}} m^{1+\alpha-(r-1)} n^{r-1} = C(2m)^{1+\alpha} \geq \mathrm{ex}\left(|V_1 \cup U|, \mathcal{F}\right),\end{aligned}$$

a contradiction. ∎

3. Regularization under the p-norm.

In this section, we prove the following extension of the classical Δ-Almost-Regularization Theorem by Erdős and Simonovits (see, e.g., [FS13, Theorem 2.19]).

LEMMA 3.1. *Let $r \geq 2$ be an integer. Let $\alpha \in (r-2, r-1)$, $p \in [1, \frac{1}{r-1-\alpha})$, and $C > 0$ be real numbers. Then for every $\varepsilon \in (0,1)$, there exist constants K and N_0 such that the following holds for every $n \geq N_0$. Suppose $\mathcal{G}$ is an r-graph on n vertices with $\|\mathcal{G}\|_p \geq Cn^{1+p\alpha}$. Then $\mathcal{G}$ contains a subgraph $\mathcal{H}$ on m vertices satisfying*

(i) $\|\mathcal{G}\|_p \geq (1-\varepsilon)Cm^{1+p\alpha}$,

(ii) $m \geq \frac{(C^{1/\delta}n)^{\frac{\delta}{1-3\delta}}}{2}$, *where* $\delta := \frac{1-p(r-1-\alpha)}{4}$,

(iii) $\Delta(\mathcal{H}) \leq (\frac{K}{1-\varepsilon} \cdot \frac{\|\mathcal{H}\|_p}{m})^{1/p}$, *and*

(iv) $|\mathcal{H}| > \hat{C}m^{1+\alpha}$, *where* $\hat{C} := \frac{(1-\varepsilon)C^{1/p}}{rK^{\frac{p-1}{p}}}$.

Proof of Lemma 3.1. Let $r \geq 2$, $\alpha \in (r-2, r-1)$, $p \in [1, \frac{1}{r-1-\alpha})$, and $C > 0$ be as assumed in Lemma 3.1. Since $p \in [1, \frac{1}{r-1-\alpha})$, the constant $\delta = \frac{1-p(r-1-\alpha)}{4}$ satisfies $0 < \delta < 1/4$. Fix $\varepsilon \in (0,1)$. Let ε_1 be the real number in $(0, \varepsilon)$ such that

$$1 - \varepsilon_1 - ((r-1)\varepsilon_1)^{1/p} = (1-\varepsilon)^{1/p}.$$

Let K be a constant satisfying

$$K^{\delta} \geq 2^{1+p(r-1)} \quad \text{and} \quad \frac{K^{1+p\alpha}}{K^{p(r-1)}} \cdot \frac{\varepsilon_1}{2^{2+p\alpha}} = K^{4\delta} \cdot \frac{\varepsilon_1}{2^{2+p\alpha}} > K^{2\delta}.$$

Let N_1 be the constant such that Proposition 2.6 holds with $o_m(1) \geq -1/2$ for all $m \geq N_1$. Let $N_0 \gg N_1$ be a sufficiently large integer and $\mathcal{G}$ be an r-graph on $n \geq N_0$ vertices with $\|\mathcal{G}\|_p \geq Cn^{1+p\alpha}$.

For convenience, for every r-graph $\mathcal{K}$, we define

$$\Phi(\mathcal{K}) := \frac{\|\mathcal{K}\|_p}{|V(\mathcal{K})|^{1+p\alpha}}.$$

Note that $\Phi(\mathcal{G}) \geq C$.

We will define a sequence of subgraphs $\mathcal{G}_0 = \mathcal{G} \supseteq \mathcal{G}_1 \supseteq \cdots \supseteq \mathcal{G}_k$ for some $k \geq 0$ such that

$$\Phi(\mathcal{G}_{i+1}) \geq K^{2\delta} \cdot \Phi(\mathcal{G}_i) \geq \Phi(\mathcal{G}_i) \quad \text{and} \quad \left(\frac{1}{K}\right)^{i+1} n \leq |V(\mathcal{G}_{i+1})| \leq \left(\frac{2}{K}\right)^{i+1} n$$

for every $i \in [0, k-1]$.

Suppose we have defined $\mathcal{G}_i$ for some $i \geq 0$. Let

$$U_i := \left\{ v \in V(\mathcal{G}_i) \colon d^p_{\mathcal{G}_i}(v) \geq \frac{K \cdot \|\mathcal{G}_i\|_p}{|V(\mathcal{G}_i)|} \right\}.$$

It follows from

$$\|\mathcal{G}_i\|_p = \sum_{v \in V(\mathcal{G}_i)} d^p_{\mathcal{G}_i}(v) \geq \sum_{v \in U_i} d^p_{\mathcal{G}_i}(v) \geq |U_i| \cdot \frac{K \cdot \|\mathcal{G}_i\|_p}{|V(\mathcal{G}_i)|}$$

that

$$|U_i| \leq \frac{|V(\mathcal{G}_i)|}{K}.$$

If $\sum_{v \in U_i} d^p_{\mathcal{G}_i}(v) < \varepsilon_1 \|\mathcal{G}_i\|_p$ or $|V(\mathcal{G}_i)| \leq N_1$, then we stop the process and set $k := i$. Otherwise, we apply Proposition 2.6 to $\mathcal{G}_i$ with U and m in the proposition corresponding to U_i and $|V(\mathcal{G}_i)|/K$ here. Let $V_{i+1} \subseteq V(\mathcal{G}_i)$ be the $\frac{|V(\mathcal{G}_i)|}{K}$-set returned by Proposition 2.6, and let $\mathcal{G}_{i+1} := \mathcal{G}_i[U_i \cup V_{i+1}]$. By Proposition 2.6, we have

$$\begin{aligned}\|\mathcal{G}_{i+1}\|_p &\geq \sum_{v\in U_i} d^p_{\mathcal{G}_{i+1}}(v)\\ &\geq (1+o(1))\left(\frac{|V(\mathcal{G}_i)|/K}{|V(\mathcal{G}_i)|}\right)^{p(r-1)} \sum_{v\in U_i} d^p_{\mathcal{G}_i}(v) \geq \frac{\sum_{v\in U_i} d^p_{\mathcal{G}_i}(v)}{2K^{p(r-1)}} \geq \frac{\varepsilon_1 \|\mathcal{G}_i\|_p}{2K^{p(r-1)}}.\end{aligned}$$

It follows that

$$\begin{aligned}\Phi(\mathcal{G}_{i+1}) = \frac{\|\mathcal{G}_{i+1}\|_p}{|V(\mathcal{G}_{i+1})|^{1+p\alpha}} &\geq \frac{\varepsilon_1 \|\mathcal{G}_i\|_p}{2K^{p(r-1)}} / \left(\frac{2|V(\mathcal{G}_i)|}{K}\right)^{1+p\alpha}\\ &= \frac{\varepsilon_1}{2K^{p(r-1)}} \cdot \frac{K^{1+p\alpha}}{2^{1+p\alpha}} \cdot \frac{\|\mathcal{G}_i\|_p}{|V(\mathcal{G}_i)|^{1+p\alpha}} > K^{2\delta} \cdot \Phi(\mathcal{G}_i). \qquad (3.1)\end{aligned}$$

Additionally, it follows from the inductive hypothesis that

$$|V(\mathcal{G}_{i+1})| = |U_i \cup V_{i+1}| \geq |V_{i+1}| \geq \frac{|V(\mathcal{G}_i)|}{K} \geq \left(\frac{1}{K}\right)^{i+1} n, \quad \text{and} \qquad (3.2)$$

$$|V(\mathcal{G}_{i+1})| = |U_i \cup V_{i+1}| \leq |U_i| + |V_{i+1}| \leq \frac{|V(\mathcal{G}_i)|}{K} + \frac{|V(\mathcal{G}_i)|}{K} \leq \left(\frac{2}{K}\right)^{i+1} n, \qquad (3.3)$$

as desired.

We claim that the process defined above stops after at most $k_* := \log_K(n/N_1)$ steps. Indeed, suppose this is not true. Then at the k_*-step, by (3.1), we would have

$$\frac{\|\mathcal{G}_{k_*}\|}{|V(\mathcal{G}_{k_*})|^{1+p\alpha}} = \Phi(\mathcal{G}_{k_*}) \geq \left(K^{2\delta}\right)^{k_*} \cdot \Phi(\mathcal{G}_0) \geq CK^{2\delta k_*}. \qquad (3.4)$$

It is trivially true that

$$\begin{aligned}\frac{\|\mathcal{G}_{k_*}\|}{|V(\mathcal{G}_{k_*})|^{1+p\alpha}} = \frac{\sum_{v\in V(\mathcal{G}_{k_*})} d^p_{\mathcal{G}_{k_*}}(v)}{|V(\mathcal{G}_{k_*})|^{1+p\alpha}} &\leq \frac{|V(\mathcal{G}_{k*})| \cdot |V(\mathcal{G}_{k*})|^{p(r-1)}}{|V(\mathcal{G}_{k_*})|^{1+p\alpha}}\\ &= |V(\mathcal{G}_{k_*})|^{p(r-1-\alpha)} \leq |V(\mathcal{G}_{k_*})|.\end{aligned}$$

Combining this with (3.3) and (3.4), we obtain

$$CK^{2\delta k_*} \leq |V(\mathcal{G}_{k_*})| \leq \left(\frac{2}{K}\right)^{k_*} n.$$

It follows that

$$n \geq CK^{2\delta k_*}\left(\frac{K}{2}\right)^{k_*} = C\left(\frac{K^{2\delta}}{2}\right)^{k_*} K^{k_*} \geq CK^{\delta k_*}K^{k_*} = C\left(\frac{n}{N_1}\right)^{1+\delta},$$

which is a contradiction since $C, \delta, N_1 > 0$ are fixed and n is sufficiently large. Therefore, the process defined above stops after at most $k_* := \log_K(n/N_1)$ steps.

Recalling that k is the final step of the process, and using (3.2), we have

$$|V(\mathcal{G}_k)| \geq \left(\frac{1}{K}\right)^k n \geq \left(\frac{1}{K}\right)^{k_*} n \geq N_1.$$

This means that the process stopped due to

$$\sum_{v\in U_k} d^p_{\mathcal{G}_k}(v) < \varepsilon_1 \|\mathcal{G}_k\|_p. \qquad (3.5)$$

Let $\mathcal{H}$ denote the induced subgraph of $\mathcal{G}_k$ on $W := V(\mathcal{G}_k) \setminus U_k$ and let $m := |W|$. Recall that

$$U_k := \left\{ v \in V(\mathcal{G}_k) \colon d^p_{\mathcal{G}_k}(v) \geq \frac{K \cdot \|\mathcal{G}_k\|_p}{|V(\mathcal{G}_k)|} \right\}.$$

We will show that $\mathcal{H}$ satisfies the assertions in Lemma 3.1.

Let $\mathcal{R} := \mathcal{G}_k \setminus \mathcal{H}$. Note that every edge in $\mathcal{R}$ contains at least one vertex from U_k. Therefore,

$$\sum_{v \in W} d_{\mathcal{R}}(v) \leq (r-1) \cdot |\mathcal{R}| \leq (r-1) \cdot \sum_{v \in U_k} d_{\mathcal{R}}(v).$$

Since $\mathcal{R} \subseteq \mathcal{G}_k$, it follows from the definition of U_k and (3.5) that

$$\begin{aligned}
\sum_{v \in W} d^p_{\mathcal{R}}(v) \leq \sum_{v \in W} d_{\mathcal{R}}(v) \cdot d^{p-1}_{\mathcal{G}_k}(v) &\leq \sum_{v \in W} d_{\mathcal{R}}(v) \cdot \left(\frac{K \cdot \|\mathcal{G}_k\|_p}{|V(\mathcal{G}_k)|} \right)^{\frac{p-1}{p}} \\
&\leq (r-1) \cdot \sum_{v \in U_k} d_{\mathcal{R}}(v) \cdot \left(\frac{K \cdot \|\mathcal{G}_k\|_p}{|V(\mathcal{G}_k)|} \right)^{\frac{p-1}{p}} \\
&\leq (r-1) \cdot \sum_{v \in U_k} d_{\mathcal{G}_k}(v) \cdot d^{p-1}_{\mathcal{G}_k}(v) \\
&= (r-1) \cdot \sum_{v \in U_k} d^p_{\mathcal{G}_k}(v) < (r-1)\varepsilon_1 \|\mathcal{G}_k\|_p .
\end{aligned}$$

If, for the sake of contradiction, it holds that $\sum_{v \in W} d^p_{\mathcal{H}}(v) = \|\mathcal{G}\|_p < (1-\varepsilon)\|\mathcal{G}_k\|_p$, then it follows from the inequality above that

$$\begin{aligned}
\sum_{v \in W} d^p_{\mathcal{G}_k}(v) &= \sum_{v \in W} (d_{\mathcal{H}}(v) + d_{\mathcal{R}}(v))^p \\
&\leq \left(\left(\sum_{v \in W} d^p_{\mathcal{H}}(v) \right)^{1/p} + \left(\sum_{v \in W} d^p_{\mathcal{R}}(v) \right)^{1/p} \right)^p \\
&\leq \left(\|\mathcal{G}\|_p^{1/p} + \left((r-1)\varepsilon_1 \|\mathcal{G}_k\|_p \right)^{1/p} \right)^p \\
&< \left(\left((1-\varepsilon) \|\mathcal{G}_k\|_p \right)^{1/p} + \left((r-1)\varepsilon_1 \|\mathcal{G}_k\|_p \right)^{1/p} \right)^p \\
&= \left((1-\varepsilon)^{1/p} + (r-1)^{1/p} \varepsilon_1^{1/p} \right)^p \|\mathcal{G}_k\|_p = (1-\varepsilon_1)^p \|\mathcal{G}_k\|_p ,
\end{aligned}$$

where the first inequality follows from Fact 2.3 and the last equality follows from the definition of ε_1. Combining this with (3.5), we obtain

$$\|\mathcal{G}_k\|_p = \sum_{v \in U_k} d^p_{\mathcal{G}_k}(v) + \sum_{v \in W} d^p_{\mathcal{G}_k}(v) < \varepsilon_1 \|\mathcal{G}_k\|_p + (1-\varepsilon_1)^p \|\mathcal{G}_k\|_p \leq \|\mathcal{G}_k\|_p ,$$

a contradiction. Therefore, we have

$$\|\mathcal{G}\|_p \geq (1-\varepsilon) \|\mathcal{G}_k\|_p , \tag{3.6}$$

which implies that

$$\Phi(\mathcal{H}) = \frac{\|\mathcal{G}\|_p}{m^{1+p\alpha}} \geq \frac{(1-\varepsilon)\|\mathcal{G}_k\|_p}{|V(\mathcal{G}_k)|^{1+p\alpha}} \geq (1-\varepsilon) \frac{\|\mathcal{G}_0\|_p}{|V(\mathcal{G}_0)|^{1+p\alpha}} \geq (1-\varepsilon)C.$$

Here, we used the fact that

$$\Phi(\mathcal{G}_k) \geq K^{2\delta} \cdot \Phi(\mathcal{G}_{k-1}) \geq \cdots \geq K^{2k\delta} \cdot \Phi(\mathcal{G}_0) \geq \Phi(\mathcal{G}_0) \geq C.$$

This completes the proof of Lemma 3.1(i).

Next, we prove Lemma 3.1(ii). Note that by $W = V(\mathcal{G}_k) \setminus |U_k|$ and $|U_k| \leq \frac{|V(\mathcal{G}_k)|}{K}$, we have $|W| \geq |V(\mathcal{G}_k)| - \frac{|V(\mathcal{G}_k)|}{K}$. Recall the following results that we have established.

Claim 3.2. *We have the following:*

(i) $|V(\mathcal{G}_k)| - \frac{|V(\mathcal{G}_k)|}{K} \leq |W| = m \leq |V(\mathcal{G}_k)|$.

(ii) $\left(\frac{1}{K}\right)^k n \leq |V(\mathcal{G}_k)| \leq \left(\frac{2}{K}\right)^k n$.

(iii) $K^{2k\delta}C \leq \frac{\|\mathcal{G}_k\|_p}{|V(\mathcal{G}_k)|^{1+p\alpha}} \leq \frac{|V(\mathcal{G}_k)|^{1+p(r-1)}}{|V(\mathcal{G}_k)|^{1+p\alpha}} = |V(\mathcal{G}_k)|^{p(r-1-\alpha)} = |V(\mathcal{G}_k)|^{1-4\delta}$.

It follows from Claim 3.2(ii) and (iii) that

$$K^{2k\delta}C \leq |V(\mathcal{G}_k)|^{1-4\delta} \leq \left(\left(\frac{2}{K}\right)^k n\right)^{1-4\delta}.$$

Since $K^\delta \geq 2^{1+p(r-1)} \geq 2^{p(r-1-\alpha)} = 2^{1-4\delta}$, the inequality above implies that

$$n^{1-4\delta} \geq \frac{K^{2k\delta} C K^{k(1-4\delta)}}{2^{k(1-4\delta)}} \geq \frac{K^{2k\delta} C K^{k(1-4\delta)}}{K^{k\delta}} = K^{k(1-3\delta)}C.$$

It follows that $K^k \leq n^{\frac{1-4\delta}{1-3\delta}} C^{-\frac{1}{1-3\delta}}$. Therefore,

$$m \geq |V(\mathcal{G}_k)| - \frac{|V(\mathcal{G}_k)|}{K} \geq \frac{|V(\mathcal{G}_k)|}{2} \geq \frac{1}{2}\left(\frac{1}{K}\right)^k n \geq \frac{C^{\frac{1}{1-3\delta}} n^{\frac{\delta}{1-3\delta}}}{2},$$

proving Lemma 3.1(ii).

It follows from the definition of U_k and $\|\mathcal{H}\|_p \geq (1-\varepsilon)\|\mathcal{G}_k\|_p$ (by (3.6)) that

$$\Delta(\mathcal{H}) < \left(\frac{K \cdot \|\mathcal{G}_k\|_p}{|V(\mathcal{G}_k)|}\right)^{1/p} \leq \left(\frac{K \cdot \|\mathcal{H}\|_p}{(1-\varepsilon)|V(\mathcal{G}_k)|}\right)^{1/p} \leq \left(\frac{K \cdot \|\mathcal{H}\|_p}{(1-\varepsilon)m}\right)^{1/p}. \tag{3.7}$$

This proves Lemma 3.1(iii).

Finally, it follows from

$$\|\mathcal{H}\|_p = \sum_{v \in W} d^p_{\mathcal{H}}(v) \leq \sum_{v \in W} d_{\mathcal{H}}(v) \cdot (\Delta(\mathcal{H}))^{p-1} = r \cdot |\mathcal{H}| \cdot (\Delta(\mathcal{H}))^{p-1}$$

and (3.7) that

$$|\mathcal{H}| \geq \frac{\|\mathcal{H}\|_p}{r\,(\Delta(\mathcal{H}))^{p-1}} > \frac{\|\mathcal{H}\|_p}{r\left(\frac{K \cdot \|\mathcal{H}\|_p}{(1-\varepsilon)m}\right)^{\frac{p-1}{p}}} = \frac{1}{r}\left(\frac{(1-\varepsilon)m}{K}\right)^{\frac{p-1}{p}} \|\mathcal{H}\|_p^{1/p}.$$

Combining this with $\|\mathcal{H}\|_p \geq (1-\varepsilon)\|\mathcal{G}_k\|_p \geq (1-\varepsilon)Cm^{1+p\alpha}$ (by (3.6) and Claim 3.2(iii)), we obtain

$$|\mathcal{H}| > \frac{1}{r}\left(\frac{(1-\varepsilon)m}{K}\right)^{\frac{p-1}{p}} \left((1-\varepsilon)Cm^{1+p\alpha}\right)^{1/p} = \frac{(1-\varepsilon)C^{1/p}m^{1+\alpha}}{rK^{\frac{p-1}{p}}},$$

which proves Lemma 3.1(iv). ∎

4. Proof of Theorem 1.2. In this section, we prove Theorem 1.2. This will be achieved through the following two propositions, first addressing the case $p < \frac{1}{r-1-\alpha}$.

PROPOSITION 4.1. *Let $r \geq 2$ be an integer. Suppose that $\mathcal{F}$ is a degenerate family of r-graphs satisfying $\mathrm{ex}(n,\mathcal{F}) = O(n^{1+\alpha})$ for some constant α. Then for every $p \in (1, \frac{1}{r-1-\alpha})$, there exists a constant $C_{\mathcal{F}}$ such that for all sufficiently large n,*

$$\mathrm{ex}_p(n,\mathcal{F}) \leq C_{\mathcal{F}} \cdot n^{1+p\alpha}.$$

In particular, if $\mathrm{ex}(n,\mathcal{F}) = \Theta(n^{1+\alpha})$, then together with (2.1),

$$\mathrm{ex}_p(n,\mathcal{F}) = \Theta(n^{1+p\alpha}) \quad \textit{for every} \quad p \in \left(1, \frac{1}{r-1-\alpha}\right).$$

Proof of Proposition 4.1. Let $C, N_0 > 0$ be constants such that $\mathrm{ex}(n,\mathcal{F}) \leq Cn^{1+\alpha}$ for every $n \geq N_0$. Let $\delta := \frac{1-(r-1-\alpha)p}{4} \in \left(0, \frac{1}{4}\right)$. Fix $\varepsilon := \frac{1}{2}$ and let $K = K(r,\alpha,p,\varepsilon)$ and $N_2 = N_0(r,\alpha,p,\varepsilon)$ be the constants returned by Lemma 3.1. Let $C_{\mathcal{F}} := 2^p r^p K^{p-1} C^p$ and $N_1 := \max\{N_2, (2N_0)^{\frac{1-3\delta}{\delta}}/C_{\mathcal{F}}^{1/\delta}\}$.

Suppose to the contrary that there exists an $\mathcal{F}$-free r-graph $\mathcal{G}$ on $n \geq N_1$ vertices with $\|\mathcal{G}\|_p > C_{\mathcal{F}} \cdot n^{1+p\alpha}$. Then, by Lemma 3.1, there exists a subgraph $\mathcal{H} \subseteq \mathcal{G}$ on $m \geq (C_{\mathcal{F}}^{1/\delta} n)^{\frac{\delta}{1-3\delta}}/2 \geq N_0$ vertices with $|\mathcal{H}| > (1-\varepsilon) C_{\mathcal{F}}^{1/p} m^{1+\alpha}/(rK^{\frac{p-1}{p}}) = Cm^{1+\alpha}$, contradicting the $\mathcal{F}$-freeness of $\mathcal{H} \subseteq \mathcal{G}$. ∎

Next, we consider the case $p > \frac{1}{r-1-\alpha}$.

PROPOSITION 4.2. *Let $r \geq 2$ be an integer. Suppose that $\mathcal{F}$ is a degenerate family of r-graphs satisfying $\mathrm{ex}(n,\mathcal{F}) = O(n^{1+\alpha})$ for some constant $\alpha \in (r-2, r-1)$. Then for every $p > \frac{1}{r-1-\alpha}$,*

$$\mathrm{ex}_p(n,\mathcal{F}) \leq (\tau_{\mathrm{part}}(\mathcal{F}) - 1 + o(1)) \binom{n}{r-1}^p.$$

Proof of Proposition 4.2. Let $F \in \mathcal{F}$ be an r-partite r-graph satisfying $\tau_{\mathrm{part}}(F) = \tau_{\mathrm{part}}(\mathcal{F})$. Let $A_1 \cup \cdots \cup A_r = V(F)$ be an r-partition of F with $|A_1| \leq \cdots \leq |A_r|$ and $|A_1| = \tau_{\mathrm{part}}(F)$. Let $s_i := |A_i|$ for $i \in [r]$. Note that $s_1 = |A_1| = \tau_{\mathrm{part}}(F) = \tau_{\mathrm{part}}(\mathcal{F})$. Let the $(r-1)$-partite $(r-1)$-graph F_1 on $A_2 \cup \cdots \cup A_r$ be defined as

$$F_1 := \bigcup_{v \in A_1} L_F(v).$$

By Theorem 2.7, there exist constants $\delta > 0$ and $C > 0$ such that $\mathrm{ex}(n, F_1) \leq Cn^{r-1-\delta}$ for every $n \in \mathbb{N}$. By reducing δ, we may assume that $\delta \leq \min\{\frac{1}{s_1 \cdots s_{r-1}}, \frac{1}{2}\}$. Let $p_* := \frac{1}{r-1-\alpha}$. Let $\delta_1 > 0$ be a sufficiently small constant such that, in particular,

$$\delta_1 < \min\left\{p - p_*, \frac{2+\alpha-r}{p-p_*}\right\} \quad \text{and} \quad \delta_2 := \delta_1 + \frac{\delta_1}{r-1-\alpha} < \min\left\{\frac{\delta}{s_1}, \frac{p-1}{p}\right\}.$$

Fix an arbitrary small constant $\varepsilon > 0$. Let n be sufficiently large. Suppose to the contrary that there exists an n-vertex $\mathcal{F}$-free r-graph $\mathcal{H}$ with

$$\|\mathcal{H}\|_p \geq (\tau_{\mathrm{part}}(\mathcal{F}) - 1 + \varepsilon) \binom{n}{r-1}^p = (s_1 - 1 + \varepsilon) \binom{n}{r-1}^p.$$

Let

$$V := V(\mathcal{H}), \quad U := \left\{v \in V : d_{\mathcal{H}}(v) \geq n^{r-1-\delta_1}\right\}, \quad V_1 := V \setminus U, \quad \text{and} \quad \mathcal{H}_1 := \mathcal{H}[V_1].$$

CLAIM 4.3. *We have* $|U| \le n^{\delta_2}$.

Proof of Claim 4.3. Suppose to the contrary that this is not true. Let $U' \subseteq U$ be a set of size n^{δ_2}. By Proposition 2.6, there exists a set $V'' \subseteq V$ of size n^{δ_2} such that the induced subgraph $\mathcal{H}[U'' \cup V'']$ satisfies

$$|\mathcal{H}[U'' \cup V'']| \ge \frac{1}{r} \sum_{v \in U''} d_{\mathcal{H}[U'' \cup V'']}(v) \ge \frac{1}{r}(1+o(1)) \left(\frac{n^{\delta_2}}{n}\right)^{r-1} \sum_{v \in U''} d_{\mathcal{H}}(v)$$
$$\ge \frac{1}{2r} n^{-(r-1)(1-\delta_2)} \cdot n^{\delta_2} \cdot n^{r-1-\delta_1} = \frac{n^{r\delta_2-\delta_1}}{2r}.$$

Since $r\delta_2 - \delta_1 - \delta_2(1+\alpha) = \delta_2(r-1-\alpha) - \delta_1 = (r-1-\alpha)\delta_1$, we have

$$|\mathcal{H}[U'' \cup V'']| \ge \frac{n^{r\delta_2-\delta_1}}{2r} = \frac{n^{(r-1-\alpha)\delta_1}}{2^{2+\alpha} r} \cdot \left(2n^{\delta_2}\right)^{1+\alpha} > \mathrm{ex}(2n^{\delta_2}, \mathcal{F}) \ge \mathrm{ex}\left(|U'' \cup V''|, \mathcal{F}\right),$$

a contradiction. ∎

CLAIM 4.4. *We have*

$$\sum_{v \in U} d_{\mathcal{H}}(v) \le \left(s_1 - 1 + \frac{2\varepsilon}{3}\right) \binom{n}{r-1}, \quad \textit{and} \tag{4.1}$$

$$\|\mathcal{H}_1\|_p \ge \left(\left(\frac{\varepsilon}{3}\right)^{1/p} - \left(\frac{\varepsilon}{4}\right)^{1/p}\right)^p \binom{n}{r-1}^p. \tag{4.2}$$

Proof of Claim 4.4. Let $\mathcal{S}$ be the collection of edges in $\mathcal{H}$ that contain exactly one vertex from U. Note that $\mathcal{S} = \mathcal{S}[U, V_1]$ is a semibipartite r-graph. Since $F \subseteq K^r_{s_1,\ldots,s_r}$ and $\mathcal{S}$ is F-free, it follows from Proposition 2.9 and Claim 4.3 that

$$|\mathcal{S}| \le \frac{(s_2 + \cdots + s_r - r + 1)^{\frac{1}{s_1}}}{r-1} |U| n^{r-1-\frac{1}{s_1 \cdots s_{r-1}}} + (s_1 - 1)\binom{n}{r-1}$$
$$\le \frac{(s_2 + \cdots + s_r - r + 1)^{\frac{1}{s_1}}}{r-1} n^{r-1-\frac{1}{s_1 \cdots s_{r-1}}+\delta_2} + (s_1 - 1)\binom{n}{r-1}$$
$$\le \frac{\varepsilon}{2}\binom{n}{r-1} + (s_1 - 1)\binom{n}{r-1}.$$

Let $\mathcal{S}_2$ denote the set of edges in $\mathcal{H}$ that contain at least two vertices from U. It is clear that

$$|\mathcal{S}_2| \le |U|^2 \binom{n}{r-2} \le n^{2\delta_2}\binom{n}{r-2} \le \frac{\varepsilon}{6r}\binom{n}{r-1}.$$

Therefore,

$$\sum_{v \in U} d_{\mathcal{H}}(v) \le |\mathcal{S}| + r \cdot |\mathcal{S}_2| \le \frac{\varepsilon}{2}\binom{n}{r-1} + (s_1 - 1)\binom{n}{r-1} + r \cdot \frac{\varepsilon}{6r}\binom{n}{r-1}$$
$$= \left(s_1 - 1 + \frac{2\varepsilon}{3}\right)\binom{n}{r-1}. \tag{4.3}$$

This proves (4.1).

Next, we prove (4.2). First, note that for every $v\in V_1$, we have

$$d_{\mathcal{H}}(v)-d_{\mathcal{H}_1}(v)\le |U|\binom{n}{r-2}\le n^{r-2+\delta_2}.$$

Therefore, by the assumption that $\delta_2<\frac{p-1}{p}$, we have

$$\sum_{v\in V_1}(d_{\mathcal{H}}(v)-d_{\mathcal{H}_1}(v))^p\le |V_1|\cdot n^{p(r-2+\delta_2)}\le n^{p(r-2+\delta_2)+1}\le \frac{\varepsilon}{4}\binom{n}{r-1}^p.$$

Consequently, it follows from Fact 2.3 that

$$\begin{aligned}\left(\sum_{v\in V_1}d^p_{\mathcal{H}}(v)\right)^{1/p}&=\left(\sum_{v\in V_1}(d_{\mathcal{H}_1}(v)+d_{\mathcal{H}}(v)-d_{\mathcal{H}_1}(v))^p\right)^{1/p}\\&\le\left(\sum_{v\in V_1}d^p_{\mathcal{H}_1}(v)\right)^{1/p}+\left(\sum_{v\in V_1}(d_{\mathcal{H}}(v)-d_{\mathcal{H}_1}(v))^p\right)^{1/p}\\&\le\|\mathcal{H}_1\|_p^{1/p}+\left(\frac{\varepsilon}{4}\right)^{1/p}\binom{n}{r-1}.\end{aligned}$$

Suppose to the contrary that $\|\mathcal{H}_1\|_p<((\frac{\varepsilon}{3})^{1/p}-(\frac{\varepsilon}{4})^{1/p})^p\binom{n}{r-1}^p$. Then it follows from (4.3) and the inequality above that

$$\begin{aligned}\|\mathcal{H}\|_p&=\sum_{v\in U}d^p_{\mathcal{H}}(v)+\sum_{v\in V_1}d^p_{\mathcal{H}}(v)\\&<\sum_{v\in U}d_{\mathcal{H}}(v)\cdot\binom{n}{r-1}^{p-1}+\left(\left(\left(\frac{\varepsilon}{3}\right)^{1/p}-\left(\frac{\varepsilon}{4}\right)^{1/p}\right)\binom{n}{r-1}+\left(\frac{\varepsilon}{4}\right)^{1/p}\binom{n}{r-1}\right)^p\\&\le\left(s_1-1+\frac{2\varepsilon}{3}\right)\binom{n}{r-1}^p+\frac{\varepsilon}{3}\binom{n}{r-1}^p=(s_1-1+\varepsilon)\binom{n}{r-1}^p,\end{aligned}$$

a contradiction. This proves (4.2). ∎

Let $\hat{p}:=\frac{1-(p-p_*)\delta_1}{r-1-\alpha}<p_*<p$. Since $\alpha>r-2$ and $\delta_1\le\frac{2+\alpha-r}{p-p_*}$, we have $\hat{p}\ge 1$. It follows from Fact 2.4 and (4.2) that there exists a constant $\varepsilon_1>0$ satisfying

$$\begin{aligned}\|\mathcal{H}_1\|_{\hat{p}}\ge\frac{\|\mathcal{H}_1\|_p}{(\Delta(\mathcal{H}_1))^{p-\hat{p}}}\ge\frac{\|\mathcal{H}_1\|_p}{(n^{r-1-\delta_1})^{p-\hat{p}}}\ge\frac{\varepsilon_1 n^{p(r-1)}}{(n^{r-1-\delta_1})^{p-\hat{p}}}&=\varepsilon_1 n^{\hat{p}(r-1-\alpha)+(p-\hat{p})\delta_1+\hat{p}\alpha}\\&=\varepsilon_1 n^{1-(p-p_*)\delta_1+(p-\hat{p})\delta_1+\hat{p}\alpha}\\&=\varepsilon_1 n^{1+\hat{p}\alpha+(p_*-\hat{p})\delta_1}.\end{aligned}$$

Since $(p_*-\hat{p})\delta_1>0$ and n is sufficiently large, we have $\|\mathcal{H}_1\|_{\hat{p}}\gg n^{1+\hat{p}\alpha}$, which, by Proposition 4.1, implies that $\|\mathcal{H}_1\|_{\hat{p}}>\mathrm{ex}_{\hat{p}}(n,\mathcal{F})$, a contradiction. This completes the proof of Proposition 4.2. ∎

5. Proof of Theorem 1.3. We present the proof of Theorem 1.3 in this section. The following result will be useful for the proof.

PROPOSITION 5.1. *Let $r\ge 2$ be an integer and $p\ge 1$ be a real number. Suppose that $\mathcal{G}=\mathcal{G}[V_1,\dots,V_r]$ is an r-partite r-graph with* $\min\{|V_i|: i\in[r]\}\ge 2$. *Then there exists a nonempty set $U\subseteq V_1$ such that*

$$|\mathcal{G}[U,V_2,\ldots,V_r]| \geq \frac{|U|^{1-\frac{1}{p}}}{2}\left(\frac{\sum_{v\in V_1} d_{\mathcal{G}}^p(v)}{\lceil \log\left(|V_2|\cdots|V_r|\right)\rceil}\right)^{1/p} \geq \frac{|U|^{1-\frac{1}{p}}}{4}\left(\frac{\sum_{v\in V_1} d_{\mathcal{G}}^p(v)}{\log\left(|V_2|\cdots|V_r|\right)}\right)^{1/p}.$$

Proof of Proposition 5.1. Let $N := |V_2|\cdots|V_r|$ and $t := \lceil \log N\rceil$. For each $i \in [t]$, let

$$U_i := \left\{v \in V_1 \colon d_{\mathcal{G}}(v) \in [2^{i-1}, 2^i)\right\}.$$

Since $\sum_{v\in V_1} d_{\mathcal{G}}^p(v) = \sum_{i\in[t]}\sum_{v\in U_i} d_{\mathcal{G}}^p(v)$, by the Pigeonhole Principle, there exists $i_* \in [t]$ such that

$$\sum_{v\in U_{i_*}} d_{\mathcal{G}}^p(v) \geq \frac{\sum_{v\in V_1} d_{\mathcal{G}}^p(v)}{t}.$$

Let $U := U_{i^*}$ and $m := |U|$. It follows from the definition of U_{i_*} that

$$m\cdot 2^{pi_*} = |U|\cdot 2^{pi_*} \geq \sum_{v\in U_{i_*}} d_{\mathcal{G}}^p(v) \geq \frac{\sum_{v\in V_1} d_{\mathcal{G}}^p(v)}{t},$$

which implies that

$$2^{i_*-1} \geq \frac{1}{2}\left(\frac{\sum_{v\in V_1} d_{\mathcal{G}}^p(v)}{m\cdot t}\right)^{1/p}.$$

Therefore,

$$|\mathcal{G}[U,V_2,\ldots,V_r]| = \sum_{v\in U} d_{\mathcal{G}}(v) \geq m\cdot 2^{i_*-1} \geq \frac{1}{2}\left(\frac{\sum_{v\in V_1} d_{\mathcal{G}}^p(v)}{t}\right)^{1/p} m^{1-\frac{1}{p}}.$$

This proves Proposition 5.1. ∎

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Recall that $p_* := \frac{1}{r-1-\alpha}$. Let n be sufficiently large. Suppose that $\mathcal{G}$ is an $\mathcal{F}$-free r-graph on n vertices. By Proposition 2.8, there exists a balanced r-partition $V_1 \cup \cdots \cup V_r = V(\mathcal{G})$ such that the r-partite subgraph $\mathcal{H} := \mathcal{G}[V_1,\ldots,V_r]$ satisfies

$$\|\mathcal{H}\|_{p_*} \geq \left(\frac{r!}{r^r} + o(1)\right)^{p_*}\|\mathcal{G}\|_{p_*} \geq \frac{1}{2}\left(\frac{r!}{r^r}\right)^{p_*}\|\mathcal{G}\|_{p_*}.$$

Since $\|\mathcal{H}\|_{p_*} = \sum_{i\in[r]}\sum_{v\in V_i} d_{\mathcal{H}}^{p_*}(v)$, by the Pigeonhole Principle, there exists V_i such that

$$\sum_{v\in V_i} d_{\mathcal{H}}^{p_*}(v) \geq \frac{\|\mathcal{H}\|_{p_*}}{r} \geq \frac{1}{2r}\left(\frac{r!}{r^r}\right)^{p_*}\|\mathcal{G}\|_{p_*}.$$

By symmetry, we may assume that $i = 1$.

Applying Proposition 5.1 to $\mathcal{H}$, we obtain a nonempty set $U \subseteq V_1$ of size m for some $m \le |V_1|$ such that

$$\begin{aligned}|\mathcal{H}[U,V_2,\dots,V_r]| &\ge \frac{|U|^{1-\frac{1}{p_*}}}{4}\left(\frac{\sum_{v\in V_1} d_{\mathcal{H}}^{p_*}(v)}{\log(|V_2|\cdots|V_r|)}\right)^{1/p_*}\\ &\ge \frac{m^{1-\frac{1}{p_*}}}{4}\left(\frac{1}{2r}\left(\frac{r!}{r^r}\right)^{p_*}\frac{\|\mathcal{G}\|_{p_*}}{r\cdot\log n}\right)^{1/p_*}\\ &= \frac{m^{1+\alpha-(r-1)}}{4}\left(\frac{1}{2r}\left(\frac{r!}{r^r}\right)^{p_*}\frac{\|\mathcal{G}\|_{p_*}}{r\cdot\log n}\right)^{1/p_*}.\end{aligned}$$

Since $\mathrm{ex}(n,\mathcal{F}) = O(n^{1+\alpha})$, it follows from Proposition 2.10 that $|\mathcal{H}[U,V_2,\dots,V_r]| \le C_{\mathcal{F}} m^{1+\alpha-(r-1)} n^{r-1}$ for some constant $C_{\mathcal{F}} > 0$. Therefore,

$$\frac{m^{1+\alpha-(r-1)}}{4}\left(\frac{1}{2r}\left(\frac{r!}{r^r}\right)^{p_*}\frac{\|\mathcal{G}\|_{p_*}}{r\cdot\log n}\right)^{1/p_*} \le C_{\mathcal{F}} m^{1+\alpha-(r-1)} n^{r-1},$$

which implies that

$$\|\mathcal{G}\|_{p_*} \le C_{\mathcal{F}}^{p_*}\cdot 4^{p_*}\cdot 2r\cdot\left(\frac{r^r}{r!}\right)^{p_*} r\log n\cdot n^{p_*(r-1)} = C_{\mathcal{F}}^{p_*} 2^{2p^*+1} r\left(\frac{r^r}{r!}\right)^{p_*} n^{p_*(r-1)}\log n.$$

This proves Theorem 1.3. ∎

6. Proof of Theorem 1.4.

In this section, we prove Theorem 1.4. For convenience, for every integer $\ell \ge 3$, let $C_{\le 2\ell} := \{C_4, C_6, \dots, C_{2\ell}\}$. The following two theorems will be useful for us.

THEOREM 6.1 (Lam and Verstraëte [LV05]). *Let $\ell \ge 3$ be an integer. For every $n \in \mathbb{N}$,*

$$\mathrm{ex}(n, C_{\le 2\ell}) \le \frac{1}{2} n^{1+\frac{1}{\ell}} + 2^{\ell^2} n = \left(\frac{1}{2} + o(1)\right) n^{1+\frac{1}{\ell}}.$$

THEOREM 6.2 (Naor and Verstraëte [NV05]). *Let $\ell \ge 2$ be an integer. Then*

$$\mathrm{ex}(m,n,C_{\le 2\ell}) \le \begin{cases} 4\left((nm)^{\frac{1}{2}+\frac{1}{2\ell}} + n + m\right), & \text{if } \ell \text{ is odd,}\\ 4\left((nm)^{\frac{1}{2}} m^{\frac{1}{\ell}} + n + m\right), & \text{if } \ell \text{ is even.}\end{cases}$$

In particular, for every $\ell \ge 2$ and for every $n \ge m \ge 1$,

$$\mathrm{ex}(m,n,C_{\le 2\ell}) \le 4\left((nm)^{\frac{1}{2}+\frac{1}{2\ell}} + n + m\right),$$

and if $m \le n^{\frac{\ell-1}{\ell+1}}$, then $\mathrm{ex}(m,n,C_{\le 2\ell}) \le 4(n+n+m) \le 12n$.

Recall that an ordered sequence of vertices $v_1,\dots,v_{\ell+1} \in V(G)$ is a *walk* of length ℓ in a graph G if $v_i v_{i+1} \in G$ for all $i \in [\ell]$. We use $W_{\ell+1}(G)$ to denote the number of walks of length ℓ in G.

The following result will be useful for the proof of Theorem 1.4(i). The case where k is even appears in [ES82, Theorem 4], while the case where both k and ℓ are odd follows from the more general result of Sağlam [Sağ18, Theorem 1.3].

THEOREM 6.3 (Erdős and Simonovits [ES82]; Sağlam [Sağ18]). *Suppose that $k \ge \ell \ge 1$ are integers such that k is even or ℓ is odd. Then for every graph G on n vertices, we have*

$$\left(\frac{W_{k+1}(G)}{n}\right)^{1/k} \ge \left(\frac{W_{\ell+1}(G)}{n}\right)^{1/\ell}.$$

PROPOSITION 6.4. *For every graph G we have*

$$W_4(G) \ge \frac{\|G\|_{3/2}^2}{|V(G)|}.$$

Proof of Proposition 6.4. It follows from the Cauchy–Schwarz inequality that

$$\left(\sum_{uv\in G} d_G^{1/2}(v)\right)^2 = \left(\sum_{uv\in G} (d_G(u)d_G(v))^{1/2}\cdot\left(\frac{1}{d_G(u)}\right)^{1/2}\right)^2 \le \left(\sum_{uv\in G} d_G(u)d_G(v)\right)\cdot\left(\sum_{uv\in G} d_G^{-1}(u)\right).$$

Consequently,

$$W_4(G) = \sum_{uv\in G} d_G(u)d_G(v) \ge \frac{\left(\sum_{uv\in G} d_G^{1/2}(v)\right)^2}{\sum_{uv\in G} d_G^{-1}(u)} = \frac{\left(\sum_{v\in V(G)} d_G^{1/2}(v)\cdot d_G(v)\right)^2}{\sum_{u\in V(G)} d_G^{-1}(u)\cdot d_G(u)} = \frac{\|G\|_{3/2}^2}{|V(G)|},$$

as desired. ∎

First, we prove the upper bound for $\mathrm{ex}_{\ell/(\ell-1)}(n,\{C_4,\dots,C_{2\ell}\})$.

Proof of Theorem 1.4(i). Fix an integer $\ell\ge 3$. Let $p := \frac{\ell}{\ell-1}$. Let $C := 52\cdot 2^p < 765/3^p$ and let $\varepsilon > 0$ be sufficiently small. Notice from Proposition 2.8 that for large n,

$$\mathrm{ex}_p(n, C_{\le 2\ell}) \le \mathrm{ex}_p(2n, C_{\le 2\ell}) \le (2+o(1))^p\cdot \mathrm{ex}_p(n,n,C_{\le 2\ell}) \le 3^p\cdot \mathrm{ex}_p(n,n,C_{\le 2\ell}).$$

So it suffices to prove that $\mathrm{ex}_p(n,n,C_{\le 2\ell}) < Cn^p$ for all large n. Suppose to the contrary that this fails. Then there exists a $C_{\le 2\ell}$-free bipartite graph $G = G[V_1,V_2]$ with $|V_1| = |V_2| = n$ such that $\|G\|_p = Cn^p$. By symmetry, we may assume that

$$\sum_{v\in V_1} d_G^p(v) \ge \frac{1}{2}\left(\sum_{v\in V_1} d_G^p(v) + \sum_{v\in V_2} d_G^p(v)\right) = \frac{\|G\|_p}{2} \ge \frac{C}{2}n^p. \tag{6.1}$$

Let

$$U_1 := \left\{v\in V_1 : d_G(v) \ge n^{1-\varepsilon}\right\} \quad\text{and}\quad U_2 := \left\{v\in V_1 : d_G(v) \in [n^{1/\ell+\varepsilon}, n^{1-\varepsilon})\right\}.$$

CLAIM 6.5. *We have $\sum_{v\in U_1} d_G^p(v) \le 12n^p$.*

Proof of Claim 6.5. Since G is $C_{\le 2\ell}$-free, it follows from Theorem 6.1 (see also [AHL02]) that $|G| \le \mathrm{ex}(2n, C_{\le 2\ell}) \le (2^{1/\ell} + o(1))n^{1+1/\ell} \le 2n^{1+1/\ell}$. Therefore,

$$|U_1| \le \frac{|G|}{n^{1-\varepsilon}} \le \frac{2n^{1+1/\ell}}{n^{1-\varepsilon}} = 2n^{1/\ell+\varepsilon}.$$

Since $\frac{1}{\ell} + \varepsilon < \frac{\ell-1}{\ell+1}$ for $\ell \ge 3$, it follows from Theorem 6.2 that

$$|G[U_1, V_2]| \le 12n.$$

Combining this with Fact 2.4, we obtain

$$\sum_{v\in U_1} d_G^p(v) \le \sum_{v\in U_1} d_G(v)\cdot n^{p-1} = |G[U_1,V_2]|\cdot n^{p-1} \le 12n^p,$$

which proves Claim 6.5. ◻

CLAIM 6.6. *We have* $\sum_{v\in U_2} d_G^p(v) \le n^p$.

Proof of Claim 6.6. Let $t := \lceil \log n \rceil$. For every $i \in [t]$, let

$$W_i := \left\{ v \in U_2 \colon d_G(v) \in [2^{i-1}\cdot n^{1/\ell+\varepsilon}, 2^i \cdot n^{1/\ell+\varepsilon}) \right\}.$$

Suppose to the contrary that $\sum_{v\in U_2} d_G^p(v) > n^p$. Then, it follows from the Pigeonhole Principle that there exists $i \in [t]$ with

$$\sum_{v\in W_i} d_G^p(v) \ge \frac{\sum_{v\in U_2} d_G^p(v)}{t} \ge \frac{n^p}{t}.$$

Let $\beta \in [1/\ell + \varepsilon, 1-\varepsilon]$ be the real number such that $n^\beta = 2^{i-1}n^{1/\ell+\varepsilon}$. It follows from the definition of W_i that

$$\sum_{v\in W_i} d_G(v) \ge \sum_{v\in W_i} \frac{d_G^p(v)}{(2n^\beta)^{p-1}} = \frac{\sum_{v\in W_i} d_G^p(v)}{2^{p-1}n^{(p-1)\beta}} \ge \frac{n^{p-(p-1)\beta}}{2^{p-1}t}.$$

Consequently,

$$\begin{aligned} |G[W_i,V_2]| = \sum_{v\in W_i} d_G(v) &= \left(\sum_{v\in W_i} d_G(v)\right)^{\frac{1}{2}+\frac{1}{2\ell}} \left(\sum_{v\in W_i} d_G(v)\right)^{\frac{1}{2}-\frac{1}{2\ell}} \\ &\ge \left(|W_i|\cdot n^\beta\right)^{\frac{1}{2}+\frac{1}{2\ell}} \left(\frac{n^{p-(p-1)\beta}}{2^{p-1}t}\right)^{\frac{1}{2}-\frac{1}{2\ell}}. \end{aligned}$$

Since $p = \frac{\ell}{\ell-1}$ and $\beta \ge \frac{1}{\ell} + \varepsilon$, we have

$$\beta\cdot\left(\frac{1}{2}+\frac{1}{2\ell}\right) + (p-(p-1)\beta)\cdot\left(\frac{1}{2}-\frac{1}{2\ell}\right) = \frac{1+\beta}{2} \ge \frac{1}{2} + \frac{1}{2\ell} + \frac{\varepsilon}{2}.$$

Therefore,

$$|G[W_i,V_2]| \ge (|W_i|\cdot n)^{\frac{1}{2}+\frac{1}{2\ell}} \frac{n^{\varepsilon/2}}{(2^{p-1}t)^{\frac{1}{2}-\frac{1}{2\ell}}}.$$

Since $\varepsilon > 0$ and n is sufficiently large, it follows from Theorem 6.2 that $|G[W_i, V_2]| > \mathrm{ex}(|W_i|, |V_2|, C_{\le 2\ell})$, a contradiction. ∎

Let $V_1'' := V_1 \setminus (U_1 \cup U_2)$. It follows from (6.1) and Claims 6.5 and 6.6 that

(6.2)
$$\sum_{v\in V_1''} d_G^p(v) = \sum_{v\in V_1} d_G^p(v) - \left(\sum_{v\in U_1} d_G^p(v) + \sum_{v\in U_2} d_G^p(v)\right) \ge \frac{C}{2}n^p - 12n^p - n^p \ge \frac{C}{4}n^p.$$

Let

$$G_1 := G[V_1'', V_2], \quad \tilde{U} := \left\{v \in V_2 \colon d_{G_1}(v) \ge n^{1/\ell+\varepsilon}\right\}, \quad \text{and} \quad G_2 := G[V_1'', \tilde{U}].$$

Similar to Claims 6.5 and 6.6, we have

$$\sum_{v\in\tilde{U}} d_{G_1}^p(v) \le 12n^p + n^p = 13n^p.$$

Combining this with Fact 2.4, we obtain

$$\begin{aligned}\sum_{v\in V_1''} d_{G_2}^p(v) \le \sum_{v\in V_1''} d_{G_2}(v)\cdot\left(n^{1/\ell+\varepsilon}\right)^{p-1} &= \sum_{u\in\tilde{U}} d_{G_2}(u)\cdot\left(n^{1/\ell+\varepsilon}\right)^{p-1}\\ &\le \sum_{u\in\tilde{U}} d_{G_2}^p(u) \le 13n^p.\end{aligned} \tag{6.3}$$

Let

$$V_2'' := V_2 \setminus \tilde{U} \quad \text{and} \quad H := G[V_1'', V_2''].$$

It is clear from the definitions of V_1'' and V_2'' that $\Delta(H) \le n^{1/\ell+\varepsilon}$.

Claim 6.7. *We have* $\|H\|_p \ge \sum_{v\in V_1''} d_H^p(v) \ge 4n^p$.

Proof of Claim 6.7. Suppose to the contrary that $\sum_{v\in V_1''} d_H^p(v) < 4n^p$. Then it follows from Fact 2.3 and (6.3) that

$$\begin{aligned}\sum_{v\in V_1''} d_G^p(v) = \sum_{v\in V_1''} (d_{G_2}(v) + d_H(v))^p &\le \left(\left(\sum_{v\in V_1''} d_{G_2}^p(v)\right)^{1/p} + \left(\sum_{v\in V_1''} d_H^p(v)\right)^{1/p}\right)^p\\ &\le \left((13n^p)^{1/p} + (4n^p)^{1/p}\right)^p < \frac{C}{4}n^p,\end{aligned}$$

contradicting (6.2). Therefore, $\sum_{v\in V_1''} d_H^p(v) \ge 4n^p$. ∎

Claim 6.8. *We have* $W_{\ell+1}(H) \ge 4^{\ell-1}n^2$.

Proof of Claim 6.8. It follows from Theorem 6.3, Proposition 6.4, and Corollary 2.5 that

$$\begin{aligned}\left(\frac{W_{\ell+1}(H)}{v(H)}\right)^{1/\ell} \ge \left(\frac{W_4(H)}{v(H)}\right)^{1/3} &\ge \left(\frac{\|H\|_{3/2}^2}{(v(H))^2}\right)^{1/3}\\ &= \left(\frac{\|H\|_{3/2}}{v(H)}\right)^{2/3} \ge \left(\frac{\|H\|_{\frac{\ell}{\ell-1}}}{v(H)}\right)^{\frac{\ell-1}{\ell}}.\end{aligned}$$

Combining this with Claim 6.7, we obtain

$$W_{\ell+1}(H) \geq v(H) \cdot \left(\frac{\|H\|_{\frac{\ell}{\ell-1}}}{v(H)} \right)^{\ell-1} = \frac{\|H\|_{\frac{\ell}{\ell-1}}^{\ell-1}}{(v(H))^{\ell-2}} \geq \frac{(4n^p)^{\ell-1}}{n^{\ell-2}} = 4^{\ell-1} n^2.$$

This proves Claim 6.8. ∎

It follows from Claim 6.8 that the number of paths of length ℓ in H, denoted by $P_{\ell+1}(H)$, satisfies

$$\begin{aligned} P_{\ell+1}(H) &\geq \frac{1}{2} \left(W_{\ell+1}(H) - \binom{\ell+1}{2} \cdot 2n \cdot (\Delta(H))^{\ell-1} \right) \\ &\geq \frac{1}{2} \left(4^{\ell-1} n^2 - 2 \binom{\ell+1}{2} n^{1+\left(\frac{1}{\ell}+\varepsilon\right)(\ell-1)} \right) > \binom{2n}{2}. \end{aligned}$$

Therefore, there exist two paths of length ℓ that share the same endpoints. Since H is bipartite, this implies that G contains a copy of C_{2i} for some $i \in [2, \ell]$, a contradiction. ∎

Theorem 1.4(ii) is an immediate consequence of Theorem 1.4(i) and the following theorem.

THEOREM 6.9 (Füredi, Naor, and Verstraëte [FNV06, Theorem 3.2]). *Every C_6-free bipartite graph G contains a $\{C_4, C_6\}$-free subgraph H such that for every $v \in V(G)$,*

$$d_H(v) \geq \frac{d_G(v)}{2}.$$

Next, we present the proof of Theorem 1.4(iii). The proof is a minor adaption of the Dependent Random Choice (see, e.g., [FS11]).

Proof of Theorem 1.4(iii). Let $F = F[W_1, W_2]$ be a bipartite graph such that $d_F(v) \leq s$ for every $v \in W_2$. Let $t := |W_2|$. Let $C := 2(\frac{|V(F)|^s}{s!} + |V(F)|)$. Let $G = G[V_1, V_2]$ be an n by n bipartite graph with $\|G\|_s \geq Cn^s$.

By symmetry, we may assume that $\sum_{v \in V_1} d_G^s(v) \geq \frac{\|G\|_s}{2} \geq \frac{Cn^s}{2}$. Choose uniformly at random s vertices (with repetitions allowed) $v_1, \ldots, v_s$ from V_2. Let $\mathbf{X} := N_G(v_1) \cap \cdots \cap N_G(v_s) \subseteq V_1$. It is easy to see that

$$\mathbb{E}[|\mathbf{X}|] = \sum_{v \in V_1} \left(\frac{d_G(v)}{n} \right)^s = \frac{\sum_{v \in V_1} d_G^s(v)}{n^s} \geq \frac{Cn^s/2}{n^s} = \frac{C}{2}.$$

We call an s-set in $\mathbf{X}$ *bad* if it has at most t common neighbors. Let $\mathbf{Y}$ denote the collection of bad s-sets in $\mathbf{X}$. Notice that an s-set S is contained in $\mathbf{X}$ only if $\{v_1, \ldots, v_s\} \subseteq \bigcap_{u \in S} N_G(u)$. Therefore,

$$\mathbb{E}[|\mathbf{Y}|] \leq \binom{|V_1|}{s} \left(\frac{t}{n} \right)^s \leq \frac{t^s}{s!}.$$

It follows that

$$\mathbb{E}[|\mathbf{X}| - |\mathbf{Y}|] \geq \frac{C}{2} - \frac{t^s}{s!} \geq |V(F) \geq |W_1|.$$

By deleting one vertex from each bad set, we see that there exists a selection of s vertices $\{v_1, \ldots, v_s\} \subseteq V_2$ along with a set $X'' \subseteq N_G(v_1) \cap \cdots \cap N_G(v_s) \subseteq V_1$ of size at least $|W_1|$ such that every s-subset of X'' has at least t common neighbors. It is clear that F can be greedily embedded into $G[X'', V_2]$ with $W_1 \subseteq X''$ and $W_2 \subseteq V_2$, a contradiction. ∎

7. Concluding remarks. Let F be an r-partite r-graph satisfying $\mathrm{ex}(n,F) = O(n^{1+\alpha})$. Recall from Fact 1.1 and Theorem 1.2 that for every $p > \frac{1}{r-1-\alpha}$, we have

$$(\tau_{\mathrm{ind}}(F) - 1 + o(1)) \binom{n}{r-1}^p \leq \mathrm{ex}_p(n,F) \leq (\tau_{\mathrm{part}}(F) - 1 + o(1)) \binom{n}{r-1}^p.$$

Additionally, recall that this provides an asymptotically tight bound for $\mathrm{ex}_p(n,F)$ in the case $r = 2$, as $\tau_{\mathrm{ind}}(F) = \tau_{\mathrm{part}}(F)$ for every bipartite graph F. Unfortunately, the equality $\tau_{\mathrm{ind}}(F) = \tau_{\mathrm{part}}(F)$ does not necessarily hold for $r \geq 3$, as shown by the following example.

Let F denote the 3-graph with vertex set $\{a, b, c, a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3\}$ and edge set

$$\left\{\{a, b_i, c_j\} \colon (i,j) \in [3]^2\right\} \cup \left\{\{a_i, b, c_j\} \colon (i,j) \in [3]^2\right\} \cup \left\{\{a_i, b_j, c\} \colon (i,j) \in [3]^2\right\}.$$

It is easy to verify that $\tau_{\mathrm{part}}(F) = 4$ while $\tau_{\mathrm{ind}}(F) = 3$ (with $\{a, b, c\}$ serving as a witness).

PROBLEM 7.1. *Let $r \geq 3$. Suppose that $\mathcal{F}$ is a degenerate family of r-graphs satisfying $\mathrm{ex}(n,\mathcal{F}) = O(n^{1+\alpha})$ for some constant $\alpha > 0$. Determine if $\lim_{n\to\infty} \mathrm{ex}_p(n,\mathcal{F})/n^{p(r-1)}$ exists for $p > \frac{1}{r-1-\alpha}$, and, if so, find its value.*

On the other hand, drawing parallels to the Exponent Conjecture of Erdős and Siminovits, we propose the following bold conjecture for hypergraphs, which, if true, would show that Theorem 1.2 is tight in the exponent for the case $p < \frac{1}{r-1-\alpha}$ as well.

CONJECTURE 7.2. *Let $r \geq 3$. Suppose that $\mathcal{F}$ is a degenerate finite family of r-graphs satisfying $\mathrm{ex}(n,\mathcal{F}) = \Omega(n^{1+\alpha})$ for some constant $\alpha > r-2$. Then there exist constants $\beta > 0$, $c > 0$, and $C > 0$ such that for all sufficiently large n,*

$$c \leq \frac{\mathrm{ex}(n,\mathcal{F})}{n^{1+\beta}} \leq C.$$

Remark. Several results such as those in [MYZ18, PZ21] provide some evidence supporting this conjecture. On the other hand, examples in [RS78, FG21] show that the requirement $\alpha > r-2$ cannot be removed in general.

Recall from Theorem 1.3 that we provided a general upper bound for $\mathrm{ex}_p(n,\mathcal{F})$ when p is the threshold. An interesting problem is to explore whether the $\log n$ factor can be removed from this upper bound.

PROBLEM 7.3. *Let $r \geq 2$ be an integer. Suppose that $\mathcal{F}$ is a degenerate family of r-graphs satisfying $\mathrm{ex}(n,\mathcal{F}) = O(n^{1+\alpha})$ for some constant $\alpha > 0$. Is it true that*

$$\mathrm{ex}_{p_*}(n,\mathcal{F}) = O\left(n^{p_*(r-1)}\right) \quad \textit{for} \quad p_* = \frac{1}{r-1-\alpha}?$$

Given integers $r > t \geq 1$ and a real number $p > 0$, let the (t,p)-*norm* of an r-graph $\mathcal{H}$ be defined as

$$\|\mathcal{H}\|_{t,p} := \sum_{T \in \binom{V(\mathcal{H})}{t}} d_{\mathcal{H}}^p(T).$$

Similarly, for a family $\mathcal{F}$ of r-graphs, define the (t,p)-*norm Turán number* of $\mathcal{F}$ as

$$\mathrm{ex}_{t,p}(n,\mathcal{F}) := \max\left\{\|\mathcal{H}\|_{t,p} : v(\mathcal{H}) = n \text{ and } \mathcal{H} \text{ is } \mathcal{F}\text{-free}\right\}.$$

The (t,p)-norm Turán number $\mathrm{ex}_{t,p}(n,\mathcal{F})$ was systematically studied in [CIL+24] for nondegenerate families $\mathcal{F}$. However, many degenerate cases remain unexplored.

PROBLEM 7.4. *Let $r > t \geq 2$ be integers and $\mathcal{F}$ be a finite family of r-partite r-graphs such that* $\mathrm{ex}(n,\mathcal{F}) = n^{\beta+o(1)}$. *Determine the exponent of* $\mathrm{ex}_{t,p}(n,\mathcal{F})$ *for all* $p > 1$.

Given two graphs Q and G, we use $N(Q,G)$ to denote the number of copies of Q in G. The *generalized Turán number* $\mathrm{ex}(n,Q,\mathcal{F})$ is the maximum number of copies of Q in an n-vertex $\mathcal{F}$-free graph. The generalized Turán problem was first considered by Erdős in [Erd62] and was systematically studied by Alon–Shikhelman in [AS16].

Given integers $p \geq r > t \geq 0$, the (r,t)-*book with p-pages*, denoted by $B_{t,r,p}$, is the graph constructed as follows:

- Take p sets $V_1,\ldots,V_p$, each of size r, such that there exists a t-set C satisfying $V_i \cap V_j = C$ for all $1 \leq i < j \leq p$.
- Place a copy of K_r on each V_i.

Observe that $B_{1,2,p}$ is simply a star graph with p edges.

In parallel, one could define the (t,r,p)-*norm* of a graph as follows: Given a graph G and a t-set $S \subseteq V(G)$ that induces a copy of K_t, let $d_{G,r}(S)$ denote the number of copies of K_r in G that contain S. Let

$$\|G\|_{t,r,p} := \sum d^p_{G,r}(S),$$

where the summation is taken over all t-subsets $S \subseteq V(G)$ that induce a copy of K_t in G. Similarly, let

$$\mathrm{ex}_{t,r,p}(n,F) := \max\left\{\|G\|_{t,r,p} : v(G) = n \text{ and } G \text{ is } F\text{-free}\right\}.$$

One could consider extending results in this paper to the function $\mathrm{ex}_{t,r,p}(n,F)$. This will provide an upper bound for the generalized Turán number $\mathrm{ex}(n,B_{t,r,p},F)$, since for every graph G,

$$N(B_{t,r,p},G) = \sum \binom{d_{G,r}(S)}{p} \leq \frac{1}{p!}\sum d^p_{G,r}(S) = \frac{\|G\|_{t,r,p}}{p!},$$

where the summation is taken over all t-subsets $S \subseteq V(G)$ that induce a copy of K_t in G.

For a bipartite graph $G[V_1,V_2]$ with parts V_1 and V_2, define

$$\|G\|_{p,\mathrm{left}} := \sum_{v\in V_1} d^p_G(v) \quad \text{and} \quad \|G\|_{p,\mathrm{right}} := \sum_{v\in V_2} d^p_G(v).$$

Note that $\|G\|_{1,\mathrm{left}} = \|G\|_{1,\mathrm{right}} = |G|$ and $\|G\|_p = \|G\|_{p,\mathrm{left}} + \|G\|_{p,\mathrm{right}}$ for every $p \geq 1$.

An important variation of the Turán problem is the Zarankiewicz problem. Given bipartite graphs F and G with fixed bipartitions $V(F) = W_1 \cup W_2$ and $V(G) = V_1 \cup V_2$, an *ordered copy* of $F[W_1,W_2]$ in $G[V_1,V_2]$ is a copy of F where W_1 is contained in V_1 and W_2 is contained in V_2. Given integers $m,n \geq 1$, the *Zarankiewicz number* $Z(m,n,F[W_1,W_2])$ is the maximum number of edges in a bipartite graph $G = G[V_1,V_2]$ with $|V_1| = m$ and $|V_2| = n$ that does not contain an ordered copy of $F[W_1,W_2]$.

Extending the Zarankiewicz number to the p-norm, for every $p \geq 1$, let $Z_{p,\mathrm{left}}(m,n, F[W_1,W_2])$ (resp., $Z_{p,\mathrm{right}}(m,n,F[W_1,W_2])$) denote the maximum value of $\|G\|_{p,\mathrm{left}}$ (resp., $\|G\|_{p,\mathrm{right}}$) over all bipartite graphs $G = G[V_1,V_2]$ with $|V_1| = m$ and $|V_2| = n$ that do not contain an ordered copy of $F[W_1,W_2]$. When the order $[W_1,W_2]$ is clear from the context, for simplicity, we will use $Z(m,n,F)$, $Z_{p,\mathrm{left}}(m,n,F)$, and $Z_{p,\mathrm{right}}(m,n,F)$ to represent $Z(m,n,F[W_1,W_2])$, $Z_{p,\mathrm{left}}(m,n,F[W_1,W_2])$, and $Z_{p,\mathrm{right}}(m,n,F[W_1,W_2])$, respectively.

The following theorem can be derived through relatively straightforward modifications to the proofs presented in this paper, so we omit the details and refer the reader to its arXiv:2411.15579 version for a sketch of the proof.

THEOREM 7.5. *Suppose that $F = F[W_1,W_2]$ is a bipartite graph such that $Z(m,n,F) = O(m^\alpha n^\beta + n + m)$ for some constants $\alpha,\beta \in (0,1)$ and every $n,m \geq 1$. Then there exists a constant $C_{\mathcal{F}} = C_{\mathcal{F}}(p) > 0$ such that*

$$Z_{p,\mathrm{left}}(m,n,F) \leq \begin{cases} C_{\mathcal{F}}\left(m^{1-p(1-\alpha)}n^{\beta p} + (m+n^p)\log^{\frac{p_*-1}{\delta}+1} n\right), & \text{if } \ p \in \left[1, \dfrac{1}{2-\alpha-\beta}\right), \\ C_{\mathcal{F}}\left(m^{1-p(1-\alpha)}n^{p\beta} + m + n^p\right)\log n, & \text{if } \ p = \dfrac{1}{2-\alpha-\beta}, \\ (\tau_{\mathrm{ind}}(\mathcal{F}) - 1)\, n^p + o_n(n^p) + o_m(m^p), & \text{if } \ p > \dfrac{1}{2-\alpha-\beta}, \end{cases}$$

and

$$Z_{p,\mathrm{right}}(m,n,F) \leq \begin{cases} C_{\mathcal{F}}\left(m^{\alpha p}n^{1-p(1-\beta)} + (m^p+n)\log^{\frac{p_*-1}{\delta}+1} m\right), & \text{if } \ p \in \left[1, \dfrac{1}{2-\alpha-\beta}\right), \\ C_{\mathcal{F}}\left(m^{p\alpha}n^{1-p(1-\beta)} + m^p + n\right)\log m, & \text{if } \ p = \dfrac{1}{2-\alpha-\beta}, \\ (\tau_{\mathrm{ind}}(\mathcal{F}) - 1)\, m^p + o_n(n^p) + o_m(m^p), & \text{if } \ p > \dfrac{1}{2-\alpha-\beta}. \end{cases}$$

Here, $p_* := \frac{1}{2-\alpha-\beta}$ and $\delta := \frac{1-p(2-\alpha-\beta)}{2}$.

Remark. By summing $Z_{p,\mathrm{left}}(m,n,F)$ and $Z_{p,\mathrm{right}}(m,n,F)$, we obtain an upper bound for $\mathrm{ex}_p(m,n,F)$ and the two-sided $Z_p(m,n,F[W_1,W_2])$. Here, $Z_p(m,n,F[W_1,W_2])$ represents the maximum p-norm of a bipartite graph $G = G[V_1,V_2]$ with $|V_1| = m$ and $|V_2| = n$ that does not contain an ordered copy of $F[W_1,W_2]$.

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