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On Triple Systems with Independent Neighbourhoods

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For Béla Bollobás on his 60th birthday

Let $\mathscr{H}$ be a 3-uniform hypergraph on an n-element vertex set V. The neighbourhood of $a, b \in V$ is $N(ab) := \{x : abx \in E(\mathscr{H})\}$. Such a 3-graph has independent neighbourhoods if no $N(ab)$ contains an edge of $\mathscr{H}$. This is equivalent to $\mathscr{H}$ not containing a copy of $\mathbb{F}_{3,2} := \{abx, aby, abz, xyz\}$.

In this paper we prove an analogue of the Andrásfai–Erdős–Sós theorem for triangle-free graphs with minimum degree exceeding $2n/5$. It is shown that any $\mathbb{F}_{3,2}$-free 3-graph with minimum degree exceeding $(\frac{4}{9} - \frac{1}{125})\binom{n}{2}$ is bipartite, (for $n > n_0$), *i.e.*, the vertices of $\mathscr{H}$ can be split into two parts so that every triple meets both parts.

This is, in fact, a Turán-type result. It solves a problem of Erdős and T. Sós, and answers a question of Mubayi and Rödl that

$$\text{ex}(n, \mathbb{F}_{3,2}) = \max_{\alpha} (n - \alpha)\binom{\alpha}{2}.$$

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Here the right-hand side is $\frac{4}{9}\binom{n}{3} + O(n^2)$. Moreover $e(\mathscr{H}) = \mathrm{ex}(n, \mathbb{F}_{3,2})$ is possible only if $V(\mathscr{H})$ can be partitioned into two sets A and B so that each triple of $\mathscr{H}$ intersects A in exactly two vertices and B in one.

1. Independent neighbourhoods

Consider a 3-uniform hypergraph $\mathscr{H}$, and let a and b be two distinct vertices. The *neighbourhood*, $N(ab)$, of the pair ab consists of the vertices z for which $\{a, b, z\} \in E(\mathscr{H})$. Its size, $\mu(ab) := |N(ab)|$, is called the *codegree* of ab. We say that the neighbourhoods of $\mathscr{H}$ are *independent* if $N(ab)$ contains no triple from $E(\mathscr{H})$ for any pair ab. The *link graph* G_a of a in $\mathscr{H}$ is defined as an ordinary graph on $V(\mathscr{H})$ consisting of the pairs bc for which $\{a, b, c\} \in E(\mathscr{H})$. The *degree* of a, $\deg_{\mathscr{H}}(a) = |E(G_a)|$. We use $d_{\min}(\mathscr{H})$ for the minimum degree and $\mathsf{maxcodeg}(\mathscr{H})$ or $\mu^\star$ for the max $\mu(ab)$.

When it is possible, we shall use simplified notation, discarding parentheses and commas, *e.g.*, we shall often abbreviate a triple $\{a, b, c\}$ to abc. As usual, $G[A]$ denotes the subgraph (subhypergraph) of G induced by the vertices of A. For a graph G, $G[A, B]$ denotes the bipartite subgraph defined by the edges joining A to B.

Construction 1.1. ((2, 1)-colourings and $\mathscr{H}_{2,1}(A, B)$) *A hypergraph $\mathscr{H}$ has a* (2, 1)*-colouring if there exists a partition $V = A \cup B$ such that each triple in $E(\mathscr{H})$ meets A in exactly two vertices, and meets B in one vertex. Then the neighbourhoods of $\mathscr{H}$ are independent. Denote by $\mathscr{H}_{2,1}(A, B)$ the hypergraph consisting of all* (2, 1)*-coloured triples.*

Let $\mathbb{H}_n$ denote the class of n-vertex (2, 1)-colourable hypergraphs with maximum number of edges. Then $||A| - \frac{2}{3}n| < 1$, $e(\mathscr{H}) = \frac{4}{9}\binom{n}{3} + O(n^2)$ and every degree is $\frac{4}{9}\binom{n}{2} + O(n)$. For $n = 3k + 2$ the choices $|A| = 2k + 1$ and $2k + 2$ give the same edge-number, $\mathbb{H}_n$ has 2 members. $\mathbb{H}_n$ consists of a single hypergraph for $n = 3k$, $3k + 1$.

Construction 1.2. (A hypergraph without (2, 1)-colourings) *Let $V = A \cup B \cup D \cup \{x\}$, $|V| = n$, $|A| + |B| + |D| = n - 1$, $|A| = \lfloor \frac{2}{3}n \rfloor$, $|B| > 0$ and $|D| \geqslant 2$. Define $E(\mathscr{H})$ by taking the edges of $\mathscr{H}_{2,1}(A, B \cup D)$ and having the link graph of x as follows: $E(G_x) := \{ab : a \in$*

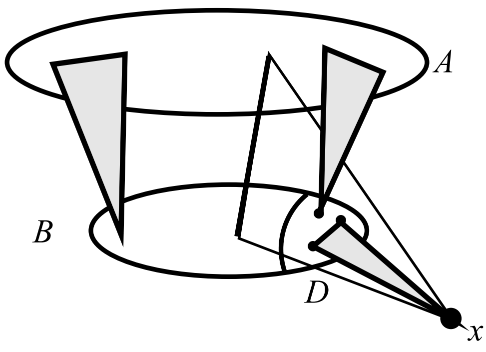

Figure 1. A hypergraph without a (2, 1)-colouring

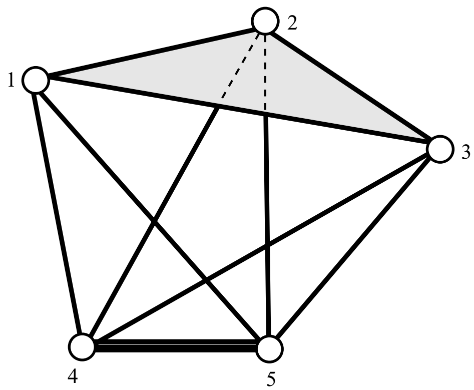

Figure 2. The excluded hypergraph $\mathbb{F}_{3,2}$

A, $b \in B\} \cup \{b_1b_2 : b_1, b_2 \in D\}$. Then the neighbourhoods of $\mathscr{H}$ are independent but $\mathscr{H}$ has no $(2,1)$-colouring, although (for $|D| = O(1)$) each vertex has degree at least $\frac{4}{9}\binom{n}{2} - O(n)$.

The vertex x can be replaced by a set X to get a hypergraph containing all triples of types AAB, AAD, ABX and DDX.

The notion of 'independent neighbourhoods' can be considered as one of the hypergraph extensions of triangle-free graphs. Andrásfai, Erdős and T. Sós [1] showed that if an n-vertex graph is triangle-free and its minimum degree exceeds $\frac{2}{5}n$, then it is bipartite. The main result of this paper is the following analogue of this important theorem. A hypergraph is *bipartite* if there exists a partition $A \cup B$ of the vertices for which every hyperedge meets both parts.

Theorem 1.3. *Let $\gamma \leqslant 1/125$ be fixed and $n > n_0$. Let $\mathscr{H}$ be an n-vertex 3-uniform hypergraph. Suppose that the neighbourhoods of $\mathscr{H}$ are independent and*

$$d_{\min}(\mathscr{H}) > \left(\frac{4}{9} - \gamma\right)\binom{n}{2}. \tag{1.1}$$

Then $\mathscr{H}$ is bipartite.

We use $n_0 = 25000$ throughout this paper, but the statements (probably) hold for much smaller values of n, too. Construction 1.2 shows that the above min-degree condition does not imply that $\mathscr{H}$ must have a (2,1)-colouring. The next example shows that $\gamma \geqslant 5/72 = 0.069\ldots$ cannot be chosen in Theorem 1.3.

Construction 1.4. (A non-bipartite triple system) *Let V be an n-element set, $V = A \cup B$, with $|A| = (3/4)n + O(1)$, $|B| = n/4 + O(1)$, and let $E_0 = \{z_1, z_2, z_3\} \subset B$. Split the pairs of A into 3 almost equal parts: $\cup_{1 \leqslant i \leqslant 3} \mathscr{E}_i = \{ab : a, b \in A\}$, $|\mathscr{E}_i| \sim \frac{1}{3}\binom{|A|}{2}$. Let $E(\mathscr{H})$ consist of E_0, the hyperedges of $\mathscr{H}_{2,1}(A, B \setminus E_0)$, and all triples of the form $\{abz_i : a, b \in A,\ z_i \in E_0,$ and $ab \in \mathscr{E}_j, 1 \leqslant j \neq i \leqslant 3\}$. Then $\mathscr{H}$ is not bipartite, its neighbourhoods are independent, and $d_{\min}(\mathscr{H}) = \frac{3}{8}\binom{n}{2} + O(n)$.*

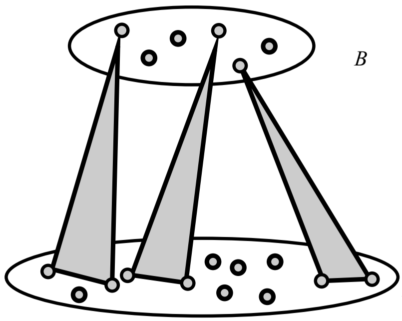

Figure 3. The extremal hypergraph

Conjecture 1.5. *Theorem 1.3 holds for every $\gamma < 5/72$.*

2. Turán's problem

Let $\mathbb{F}_{3,2}$ be the hypergraph on the vertices $1, 2, 3, 4, 5$ having 4 triples $\{1, 2, 3\}$, $\{1, 4, 5\}$, $\{2, 4, 5\}$ and $\{3, 4, 5\}$. In an ordinary graph the neighbourhoods are independent if and only if it is triangle-free. Similarly, in a 3-uniform hypergraph the neighbourhoods of pairs are independent if and only if it is $\mathbb{F}_{3,2}$-free.

More generally, given a 3-uniform hypergraph $\mathscr{F}$, let $\mathrm{ex}(n, \mathscr{F})$ denote the maximum possible size of a 3-uniform hypergraph of order n that does not contain any subhypergraph isomorphic to $\mathscr{F}$. An averaging argument shows that the ratio $\mathrm{ex}(n, \mathscr{F})/\binom{n}{3}$ is a non-increasing sequence [15], therefore $\pi(\mathscr{F}) := \lim_{n\to\infty} \mathrm{ex}(n, \mathscr{F})/\binom{n}{3}$ always exists.

Erdős and T. Sós, in connection with Ramsey–Turán problems for hypergraphs, investigated $\mathrm{ex}(n, \mathbb{F}_{3,2})$. (In [7] $\mathbb{F}_{3,2}$ is denoted by $G^{(3)}(5, 4)$, and in [8, Theorem 2] a more general class of hypergraphs is considered.) Mubayi and Rödl [20] showed that $4/9 \leqslant \pi(\mathbb{F}_{3,2}) \leqslant 1/2$ and conjectured that

$$\mathrm{ex}(n, \mathbb{F}_{3,2}) = \frac{4}{9}\binom{n}{3} + o(n^3). \tag{2.1}$$

They also conjectured that $\mathscr{H}_{2,1}(A, B)$ is the extremal hypergraph. Equation (2.1) was verified by the present authors.

Theorem 2.1. (Turán density [14]) $\pi(\{abc, ade, bde, cde\}) = 4/9$.

Here we will give a new proof. The new method also leads to structure theorems for $\mathbb{F}_{3,2}$-free hypergraphs: to the min-degree stability theorem (Theorem 1.3) stated in the previous section, and to the following further refinements and exact solutions.

Theorem 2.2. (The finer structure) *Let $\mathscr{H}$ be an n-vertex 3-uniform hypergraph not containing $\mathbb{F}_{3,2}$ and suppose that it satisfies the min-degree condition (1.1) with $\gamma \leqslant 1/125$. Suppose that n is sufficiently large, e.g., $n > \max\{n_0, 1/\gamma\}$ (where $n_0 := 25000$). Let A be a maximum independent set of vertices; denote its complement by B. Then we have the following.*

(i) *B is also independent; (A,B) is a 2-colouring.*
(ii) $||A| - \frac{2}{3}n| < \sqrt{\gamma}n$; $||B| - \frac{1}{3}n| < \sqrt{\gamma}n$.
(iii) *The structure of $\mathscr{H}$ is very close to the extremal one; all but at most $\sqrt{\gamma}n^3$ hyperedges have type AAB, the rest being of type ABB.*
(iv) *Consider any two-colouring (A_1, B_1) of $\mathscr{H}$ with $|A_1| \geqslant |B_1|$, the existence of which was stated in Theorem 1.3. If $|A| \geqslant 0.65n$ then $A_1 \subset A$ and $|A \setminus A_1| \leqslant \sqrt{\gamma}n$.*

Theorem 2.3. (Extremal) *If $\mathscr{H}$ is an n-vertex 3-uniform hypergraph not containing $\mathbb{F}_{3,2}$ and n is sufficiently large, $n > n_0$, then $e(\mathscr{H}) \leqslant \max_\alpha (n-\alpha)\binom{\alpha}{2}$. In case of equality $\mathscr{H} \in \mathbb{H}_n$.*

We show that if $e(\mathscr{H})$ is sufficiently large, then the structure of $\mathscr{H}$ is close to that of $\mathscr{H}_{2,1}(A,B)$.

Theorem 2.4. (Global stability) *Suppose that $\mathscr{H}$ is an n-vertex 3-uniform hypergraph not containing $\mathbb{F}_{3,2}$ and $e(\mathscr{H}) > (\frac{4}{9} - c)\binom{n}{3}$, where $c < 10^{-4}$. Then one can delete $O(c^{1/3})n^3$ triples from $\mathscr{H}$ so that the remaining hypergraph has a $(2,1)$-colouring (A,B) with $||A| - \frac{2}{3}n| < O(c^{1/3})n$ and $||B| - \frac{1}{3}n| < O(c^{1/3})n$.*

Taking $|A| = (\frac{2}{3} - \frac{1}{2}c)n$, $|B| = (\frac{1}{3} - \frac{1}{2}c)n$, $|D| = |X| = \frac{1}{2}cn$ in Construction 1.2, one can obtain a hypergraph satisfying the constraint of Theorem 2.4; however, one needs to remove at least $\Omega(c^3)n^3$ triples from it to make it $(2,1)$-colourable.

Theorem 2.5. (Codegree stability) *For every $\varepsilon > 0$ there exists a $c > 0$ such that the following holds. If $\mathscr{H}$ is an n-vertex 3-uniform hypergraph not containing an $\mathbb{F}_{3,2}$ and satisfying*

$$\left| \mathsf{maxcodeg}(\mathscr{H}) - \frac{2}{3}n \right| > \varepsilon n,$$

then

$$e(\mathscr{H}) \leqslant \left(\frac{4}{9} - c\right)\binom{n}{3}.$$

Theorem 1.3 is not the first Turán-type stability result concerning hypergraphs. Based on earlier works of Füredi and Kündgen [12], de Caen and Füredi proved [5] that $\pi(\mathscr{L}_7) = \frac{3}{4}$, where $\mathscr{L}_7$ is the hypergraph formed by the 7 lines of a Fano plane. This result was sharpened in [13] to a min-degree result of the same type as Theorem 1.3. A slightly weaker form of this was also proved independently by Keevash and Sudakov [17]. Further, Keevash and Mubayi obtained in [16] a min-degree version of a Turán-type result of Bollobás [2]. Keevash and Sudakov [18] also obtained a stability result, improving an extremal result of Frankl [9] on the hypergraph-triangle problem. In general, stability may not hold (see the constructions of W. G. Brown [4], and Kostochka [19] for $K_4^{(3)}$). As far as we know, at present these are the only hypergraph results of *this* type.

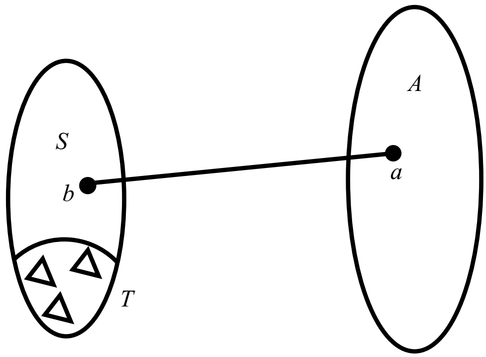

Figure 4. The partition. A and S are independent. The line (a,b) indicates that we shall sum multiplicities $\mu(a,b)$

3. Proofs

3.1. The partition

Let $\mathscr{H}$ be an $\mathbb{F}_{3,2}$-free triple system on the vertex set V, $|V| = n$, satisfying (1.1), the assumption of Theorem 1.3. Let α denote the maximum size of an independent (*i.e.*, hyperedge-free) subset of V. Let A be a maximal independent set with $|A| = \alpha$ and denote its complement by B, $B = V(\mathscr{H}) \setminus A$. Take as many independent (*i.e.*, pairwise disjoint) triples in B as possible, denote their number by v. That is, $\mathscr{M} := \{E^1, E^2, \ldots, E^v\} \subseteq E(\mathscr{H})$, $E^i \cap E^j = \emptyset$, $E^i \cap A = \emptyset$. Their vertices form T, the remaining part is S. So V is partitioned into $A \cup S \cup T$, where A and S are independent, and $|T| = 3v$.

Since the neighbourhoods are independent, we have $\alpha \geqslant \mu^\star$. Clearly,

$$n \times d_{\min}(\mathscr{H}) \leqslant \sum_{x \in V} \deg_{\mathscr{H}}(x) = 3e(\mathscr{H}) = \sum_{a,b \in V} \mu(ab) \leqslant \binom{n}{2} \mu^\star.$$

Then (1.1) implies that

$$\alpha \geqslant \mu^\star \geqslant \frac{2d_{\min}}{n-1} > \left(\frac{4}{9} - \gamma\right) n > \frac{2}{5} n.$$

Define the rectangular domain

$$D_1 := \{(x,y) : 2/5 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1/5\}.$$

Then $\alpha \geqslant \frac{2}{5} n$ gives $v \leqslant \frac{1}{3}(n - \alpha) \leqslant \frac{1}{5} n$. Hence, for all possible values of the pairs (α, v),

$$\frac{1}{n}(\alpha, v) \in D_1. \tag{3.1}$$

3.2. Sketch of the proof of Theorem 1.3

We shall classify the triples according to their positions relative to A, S and T. The number of hyperedges of type XYZ is denoted by Δ_{XYZ}. Since A and S are independent sets,

$$e(\mathscr{H}) = \Delta_{AAS} + \Delta_{AAT} + \Delta_{ASS} + \Delta_{AST} + \Delta_{ATT} + \Delta_{SST} + \Delta_{STT} + \Delta_{TTT}.$$

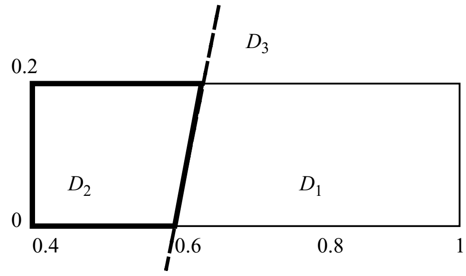

Figure 5.

In Section 3.3 we collect some inequalities for degree-3 polynomials in two variables we will use later. In Sections 3.4–3.9 we are going to give several upper bounds for different combinations of Δs which hold for every $\mathbb{F}_{3,2}$-free hypergraph. In Section 3.10 we show that the min-degree condition (1.1), used for the vertices of $A \cup B$, implies bipartiteness. This will complete the proof of Theorem 1.3. Meanwhile, in Section 3.6 we make a short digression to prove $\pi(\mathbb{F}_{3,2}) = 4/9$.

3.3. Three inequalities on polynomials

Define the polynomial

$$f(x,y) = \frac{3}{2}\left((1-x)x^2 - \frac{4}{27}\right) + y\left(\frac{2}{3} - 2x^2 + (1-x-y)(1-x-3y)\right).$$

Lemma 3.1. $f(x,y) \leqslant 0$ *for every point* (x,y) *in the rectangle* D_1.

Proof (standard optimization). The determinant of the Hessian matrix of $f(x,y)$ is negative for every $(x,y) \in D_1$ (see Section A.1). This implies that f does not have local minima or maxima inside D_1: the extrema must be on the boundary ∂D_1. Then the maximum can be found by checking the behaviour of $f(x,y)$ on the four boundary segments of ∂D_1. On each line $f(x,y)$ reduces to a third-degree polynomial of one variable, whose extrema can be identified by the roots of its derivative, a second-degree polynomial. The details of the calculations are postponed to the Appendix (Section A.1). □

The very same optimization method establishes the next two lemmas, too. Details are postponed to the Appendix. Define the domain $D_2 \subset D_1$ (a trapezium) by the lines $y = 0$, $y = 1/5$, $x = 2/5$, and $y = 5x - 3$ (the vertices are $(2/5, 0)$, $(2/5, 1/5)$, $(3/5, 0)$ and $(64/100, 2/5)$). Define the polynomial

$$F(x,y) := f(x,y) + \frac{1}{2}y(1-3y).$$

Lemma 3.2. $F(x,y) < -1/4000$ *for every* $(x,y) \in D_2$.

Let D_3 be the open half-plane $\{(x, y) : y < 5x - 3\}$. Note that $D_2 \cap D_3 = \emptyset$. Define

$$g(x, y) := -7x^2 - xy - \frac{3}{2}y^2 + 7x + y - \frac{31}{18} + 5\gamma. \tag{3.2}$$

Lemma 3.3. $g(x, y) < -1/2500$ *for every* $(x, y) \in D_3$.

3.4. Estimating $e(\mathscr{H})$

(1) Let $X \cup Y \cup Z = V(\mathscr{H})$ be a partition of the vertices. Add up the codegrees of the pairs (x, y) with $x \in X, y \in Y$. For every x, y we have $\mu(xy) \leqslant \mu^\star \leqslant \alpha$, so

$$\sum_{x \in X, y \in Y} \mu(xy) \leqslant |X||Y|\mu^\star \leqslant |X||Y|\alpha. \tag{3.3}$$

In the left-hand side of this sum we count the hyperedges of type XXY and XYY exactly twice, the types XYZ occur only once. Therefore

$$2\Delta_{XXY} + 2\Delta_{XYY} + \Delta_{XYZ} \leqslant |X||Y|\alpha.$$

Apply this to $(X, Y, Z) = (A, S, T)$:

$$2\Delta_{AAS} + 2\Delta_{ASS} + \Delta_{AST} \leqslant (n - \alpha - 3v)\alpha^2. \tag{3.4}$$

For $X = A$, $Y = B$ (and $Z = \emptyset$) we obtain

$$\Delta_{AAB} + \Delta_{ABB} \leqslant \frac{1}{2}(n - \alpha)\alpha^2. \tag{3.5}$$

The main goal in the rest of the proof (Sections 3.4–3.10) is to show that B is independent, because (3.5) and $\Delta_{BBB} = 0$ imply the upper bound $e(\mathscr{H}) \leqslant (2/27)n^3$. Next we show (in Sections 3.11–3.12) that $\mu^\star$ is strictly less than $|A|$. Then $\mu^\star < \alpha$ and (3.3) will imply the exact upper bound for $\mathrm{ex}(n, \mathbb{F}_{3,2})$.

(2) For each pair $a, b \in V$ and edge $E \in E(\mathscr{H})$ we have $|N(ab) \cap E| \leqslant 2$, otherwise we would have an $\mathbb{F}_{3,2} \subseteq \mathscr{H}$. This implies that $|\{z \in T : abz \in E(\mathscr{H})\}| \leqslant 2v$. Hence

$$\Delta_{AAT} = \sum_{\{a,b\} \subset A} |\{z \in T : abz \in E(\mathscr{H})\}| \leqslant 2v\binom{\alpha}{2} \leqslant v\alpha^2. \tag{3.6}$$

(3) We claim that

$$\Delta_{SST} \leqslant \frac{1}{2}v(n - \alpha - 3v)^2 + 2v. \tag{3.7}$$

Consider again the maximum independent family $\mathscr{M} = \{E^1, \ldots, E^v\}$. Define the multigraph G^i with vertex set S by $E(G^i) := \{E \setminus E^i : E \in E(\mathscr{H}), |E \cap E^i| = 1, |E \cap S| = 2\}$, where the multiplicity of ab is the number of Es with $E \setminus E^i = \{a, b\}$. Clearly, $\Delta_{SST} = \sum_{1 \leqslant i \leqslant v} e(G^i)$. We claim that

$$e(G^i) \leqslant \frac{1}{2}|S|^2 + 2,$$

which implies (3.7). Indeed, every edge of G^i has multiplicity at most 2. If all the edges are of multiplicity 0 or 1, then $e(G^i) \leqslant \binom{|S|}{2}$ and we are done. If, on the other hand, there exists a pair s_1s_2 with multiplicity 2, say $x_1s_1s_2$, $x_2s_1s_2 \in E(\mathscr{H})$, $x_1x_2x_3 = E^i$, then every other edge of G^i must meet s_1s_2. If not, say $xs_3s_4 \in E(\mathscr{H})$ with $x \in E^i$, then either $x \neq x_1$ and we can replace E^i in $\mathscr{M}$ by the triples $x_1s_1s_2$ and xs_3s_4, obtaining $v+1$ triples in $S \cup T$, which contradicts the maximality of $\mathscr{M}$ or $x = x_1$ and we can replace E^i in $\mathscr{M}$ by the triples $x_2s_1s_2$ and $x_1s_3s_4$. Hence $|E(G^i)| \leqslant \deg(s_1) + \deg(s_2) - 2 \leqslant 2 \times 2(|S|-1) - 2 \leqslant |S|^2/2 + 2$. □

(4) We show that

$$\Delta_{STT} \leqslant 2v^2(n-\alpha-3v) + v(2n-4). \tag{3.8}$$

Indeed, pick two hyperedges of the maximum matching $E^i, E^j \in \mathscr{M}$. Consider the 6-vertex, bipartite link graph $F_z := G_z[E^i, E^j]$ for every $z \in S$. Let $\mathscr{F}^{i,j}$ be the multigraph formed by their union, the multiplicity of the edge ab (where $a \in E^i$, $b \in E^j$) is $|\{z : z \in S, abz \in E(\mathscr{H})\}| \leqslant |S|$. Let $\mathscr{F}_3^{i,j}$ be the multigraph defined by the edges of multiplicities at least 3. Counting the edges with multiplicities, we have

$$e(\mathscr{F}^{i,j}) = e(\mathscr{F}_3^{i,j}) + (\text{the number of edges of multiplicities} \leqslant 2) \leqslant e(\mathscr{F}_3^{i,j}) + 18.$$

The graph $\mathscr{F}_3^{i,j}$ contains no 3 disjoint edges, otherwise they could be extended by vertices from S to 3 disjoint triples. Then we could replace E^i, E^j in $\mathscr{M}$ by these 3 triples, obtaining a larger matching: this would contradict the maximality of v. The König–Hall theorem (applied to this 6-vertex graph) implies that the edges of the bipartite graph $\mathscr{F}_3^{i,j}$ can be covered by two vertices, say, a and b; every edge of $\mathscr{F}_3^{i,j}$ is adjacent to either a or b. In every F_z every degree is at most 2 (because $\mathbb{F}_{3,2} \not\subset \mathscr{H}$). Hence F_z has at most 2×2 edges in $\mathscr{F}_3$ (namely, those adjacent to either a or b). Thus $e(\mathscr{F}_3) = \sum_{z \in S} e(\mathscr{F}_3 \cap F_z) \leqslant 4|S|$.

$$\begin{aligned}\Delta_{STT} &= \sum_{i \neq j} |\{abz \in E(\mathscr{H}) : a \in E^i, b \in E^j, z \in S\}| \\ &\quad + \sum_i |\{abz \in E(\mathscr{H}) : a, b \in E^i, z \in S\}| \\ &\leqslant \binom{v}{2}(4|S|+18) + 3v|S| \leqslant 2v^2(n-\alpha-3v) + v(2n-4).\end{aligned}$$

In the last step we used $\alpha \geqslant n/3$ (see (3.1)) and wrote the upper bound in a form convenient to use later.

3.5. Estimating the degrees in $A \cup S$

Add up the degrees in A and S:

$$\sum_{x \in A} \deg_{\mathscr{H}}(x) = 2\Delta_{AAB} + \Delta_{ABB},$$

$$\sum_{a \in S} \deg_{\mathscr{H}}(a) = 2\Delta_{ASS} + 2\Delta_{SST} + \Delta_{AAS} + \Delta_{AST} + \Delta_{STT}.$$

Adding them up and using $\Delta_{AAB} = \Delta_{AAS} + \Delta_{AAT}$, we obtain

$$\begin{aligned}|A \cup S| \times d_{\min}(\mathscr{H}) &\leqslant (\Delta_{AAB} + \Delta_{ABB}) + (2\Delta_{AAS} + 2\Delta_{ASS} + \Delta_{AST}) \\ &\quad + \Delta_{AAT} + \Delta_{STT} + 2\Delta_{SST}.\end{aligned}$$

Here the right-hand side can be estimated by (3.5), (3.4), (3.6), (3.8), and (3.7):

$$(n-3v)\times d_{\min}(\mathscr{H}) \leqslant \frac{1}{2}(n-\alpha)\alpha^2 + (n-\alpha-3v)\alpha^2 \\ + v\alpha^2 + 2v^2(n-\alpha-3v) + 2vn + v(n-\alpha-3v)^2.$$

Rearranging, we get

$$(n-3v)\times\left(d_{\min}(\mathscr{H}) - \frac{2}{9}n^2\right) \leqslant \frac{3}{2}\left((n-\alpha)\alpha^2 - \frac{4}{27}n^3\right) \\ + v\left(\frac{2}{3}n^2 - 2\alpha^2 + (n-\alpha-v)(n-\alpha-3v)\right) + 2vn \\ = n^3 f\left(\frac{\alpha}{n},\frac{v}{n}\right) + 2vn. \tag{3.9}$$

Using that $12v \leqslant 7n$ (see (3.1)) and rearranging again,

$$(n-3v)\times\left(d_{\min}(\mathscr{H}) - \left(\frac{4}{9}-\gamma\right)\binom{n}{2}\right) \leqslant \frac{3}{2}\left((n-\alpha)\alpha^2 - \frac{4}{27}n^3\right) \\ + v\left(\frac{2}{3}n^2 - 2\alpha^2 + (n-\alpha-v)(n-\alpha-3v)\right) + \frac{1}{2}\gamma n^2(n-3v) + n^2 \\ = n^3 F\left(\frac{\alpha}{n},\frac{v}{n}\right) + n^2. \tag{3.10}$$

3.6. Detour: The asymptotic density

Here we prove Theorem 2.1. We have to prove only an upper bound.

Lemma 3.4. *Let $\mathscr{H}$ be an arbitrary $\mathbb{F}_{3,2}$-free hypergraph. Then*

$$d_{\min}(\mathscr{H}) \leqslant \frac{2}{9}n^2 + n.$$

Proof. If the min-degree condition (1.1) does not hold then there is nothing to prove. Otherwise $\frac{1}{n}(\alpha, v) \in D_1$ by (3.1). Then Lemma 3.1 gives that $f(\alpha/n, v/n) \leqslant 0$. Since $(n-3v) \geqslant \frac{2}{5}n$, from (3.9) we get that

$$\frac{2}{5}n \times \left(d_{\min}(\mathscr{H}) - \frac{2}{9}n^2\right) \leqslant 2vn \leqslant \frac{2}{5}n^2. \qquad \square$$

Proof of Theorem 2.1. Since the above lemma gives that

$$\mathrm{ex}(n, \mathbb{F}_{3,2}) \leqslant \mathrm{ex}(n-1, \mathbb{F}_{3,2}) + \frac{2}{9}n^2 + n,$$

it follows that $\mathrm{ex}(n, \mathbb{F}_{3,2}) \leqslant \sum_{i\leqslant n}(\frac{2}{9}i^2 + i) = \frac{2}{27}n^3 + O(n^2)$. $\square$

Recall the following lemma from [10]. If $\mathscr{F}$ is a k-uniform hypergraph such that every pair of its vertices is contained in some edge, then for every n

$$\pi(\mathscr{F})\binom{n}{k} \leqslant \mathrm{ex}(n,\mathscr{F}) \leqslant \pi(\mathscr{F})\frac{n^k}{k!}.$$

This can be applied to $\mathbb{F}_{3,2}$. So $\pi(\mathbb{F}_{3,2}) = \frac{4}{9}$ gives the following.

Corollary 3.5. $\mathrm{ex}(n,\mathbb{F}_{3,2}) \leqslant \frac{2}{27}n^3$ *holds for every* n. □

3.7. Estimating the degrees in T

Corollary 3.5 gives

$$\Delta_{TTT} \leqslant \mathrm{ex}(3v,\mathbb{F}_{3,2}) \leqslant 2v^3. \tag{3.11}$$

Consider the link graphs $G_z[B]$ restricted to B. A linear combination of (3.7), (3.8) and (3.11) with a little calculation give that

$$\begin{aligned}\sum_{z\in T} e(G_z[B]) &= \Delta_{SST} + 2\Delta_{STT} + 3\Delta_{TTT}\\ &\leqslant \frac{1}{2}v(n-\alpha-3v)^2 + 4v^2(n-\alpha-3v) + 4vn + 6v^3.\end{aligned}$$

Since T is the union of v edges, for $v > 0$ we can divide the above inequality by v. There exists an edge $E^\star \in \mathscr{M}$ meeting at most that many triples of type BBB. We obtain

$$\sum_{z\in E^\star} e(G_z[B]) \leqslant \frac{1}{2}(n-\alpha-v)^2 + 2v(n-\alpha-v) + 4n. \tag{3.12}$$

3.8. Common links in A

Consider an arbitrary $z \in B$. Since A is maximal, there exists an $abz \in E(\mathscr{H})$, $a, b \in A$. Since $\mathscr{H}$ is $\mathbb{F}_{3,2}$-free, no pair uv with $u \in A$, $v \in B$ belongs to all the three bipartite graphs $G_a[A,B]$, $G_b[A,B]$ and $G_z[A,B]$. We obtain that

$$e(G_a[A,B]) + e(G_b[A,B]) + e(G_z[A,B]) \leqslant 2|A||B| = 2\alpha(n-\alpha). \tag{3.13}$$

Note that the link graph G_a has no edges in A and its maximum degree is at most $\mu^\star$. Therefore

$$e(G_a[B]) + e(G_a) = \sum_{y\in B} \deg_{G_a}(y) \leqslant |B|\mu^\star. \tag{3.14}$$

Similarly for b,

$$e(G_b[B]) + e(G_b) \leqslant |B|\mu^\star. \tag{3.15}$$

Adding up (3.13), (3.14) and (3.15) and using $|B|\mu^\star = (n-\alpha)\mu^\star \leqslant (n-\alpha)\alpha$, we get

$$\begin{aligned}e(G_a[A,B]) + e(G_b[A,B]) + e(G_z[A,B]) + e(G_a[B]) + e(G_a) + e(G_b[B]) + e(G_b)&\\ \leqslant 2\alpha(n-\alpha) + 2|B|\mu^\star \leqslant 4\alpha(n-\alpha).&\end{aligned} \tag{3.16}$$

On the left-hand side we can use

$$\deg(a) = e(G_a) = e(G_a[B]) + e(G_a[A,B]),$$

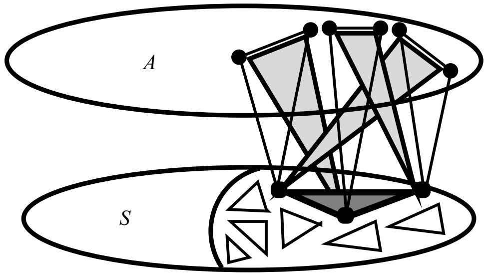

Figure 6. The special configuration H_9

since $G_a[A]$ has no edge. Similarly, $\deg(b) = e(G_b[B]) + e(G_b[A, B])$. So the left-hand side of (3.16) is $e(G_z[A, B]) + 2\deg(a) + 2\deg(b)$. This gives

$$e(G_z[A, B]) \leqslant 4\alpha(n - \alpha) - 4d_{\min}(\mathscr{H}) \quad \text{for every} \quad z \in B. \tag{3.17}$$

Lemma 3.6. *Suppose that $\mathscr{H}$ satisfies the min-degree condition* (1.1) *and $E^\star \subset B$ satisfies* (3.12). *Suppose further that $\frac{1}{n}(\alpha, v) \in D_3$, $n > n_0$. Then, for every pair $z_1, z_2 \in E^\star$, there are at least 4α common edges of $G_{z_1}[A]$ and $G_{z_2}[A]$.*

Proof. Indeed, assuming the contrary, we would also have that $e(G_{z_1}[A]) + e(G_{z_2}[A]) < \binom{\alpha}{2} + 4\alpha$. Using this, (3.17) and (3.12) we get

$$\begin{aligned} 2d_{\min} &\leqslant \deg_{\mathscr{H}}(z_1) + \deg_{\mathscr{H}}(z_2) = e(G_{z_1}[A]) + e(G_{z_2}[A]) \\ &\quad + e(G_{z_1}[A, B]) + e(G_{z_2}[A, B]) + e(G_{z_1}[B]) + e(G_{z_2}[B]) \\ &\leqslant \binom{\alpha}{2} + 4\alpha + 8\alpha(n - \alpha) - 8d_{\min} + \frac{1}{2}(n - \alpha - v)^2 + 2v(n - \alpha - v) + 4n. \end{aligned}$$

Rearranging, we get

$$0 \leqslant 5\gamma n^2 - \frac{31}{18}n^2 + 7\alpha(n - \alpha) + v(n - \alpha) - \frac{3}{2}v^2 + 10n = n^2 g\left(\frac{\alpha}{n}, \frac{v}{n}\right) + 10n.$$

By Lemma 3.3, $g \leqslant -1/2500$ on D_3. This is a contradiction for $n > 25000$. □

3.9. Small substructures

Suppose that $\mathscr{H}$ is an arbitrary $\mathbb{F}_{3,2}$-free hypergraph, A is a maximum independent set, $\alpha := |A|$, $B := V \setminus A$. Define a *well-positioned* $\mathbf{H}_9 \subseteq \mathscr{H}$ as follows. There are 6 vertices $x_1, \ldots, x_6 \in A$ and 3 vertices $\{z_1, z_2, z_3\} \subseteq B$ and the following 7 edges belong to $E(\mathscr{H})$: (z_1, z_2, z_3), $(z_i, x_{2i-1}x_{2i})$ and $(z_{i+1}, x_{2i-1}x_{2i})$ for $i = 1, 2, 3$ (where $z_4 = z_1$).

Lemma 3.7. *Suppose that $\alpha > \frac{3}{5}n$, $E^\star \subset B$ belongs to a well-positioned $\mathbf{H}_9$ and it satisfies* (3.12). *Then $d_{\min}(\mathscr{H}) < 0.434\binom{n}{2}$ holds for $n > n_0$.*

Proof. We claim that

$$\frac{3}{2}\sum_{k=1,2,3} e(G_{z_k}[A,B]) + \sum_{1\leqslant j\leqslant 6} e(G_{x_j}[A,B]) \leqslant 6\alpha(n-\alpha). \tag{3.18}$$

To see this, let us define the weight $w(ab)$ for $a \in A$, $b \in B$ as $w(ab) := \frac{3}{2}m_E(ab) + m_X(ab)$ where $m_E(ab)$ is the number of triples of the form abz_k and $m_X(ab)$ denotes the number of triples of the form abx_j ($1 \leqslant k \leqslant 3$, $1 \leqslant j \leqslant 6$). The left-hand side of (3.18) is equal to $\sum_{a\in A, b\in B} w(ab)$. We will show that $w(ab) \leqslant 6$. This is certainly true if $m_E(ab) = 0$ since, obviously, $m_X(ab) \leqslant 6$. For the case $m_E(ab) = 2$, *i.e.*, if $N(ab)$ meets $E^\star$ in two vertices, observe that $\mathbb{F}_{3,2} \not\subset \mathscr{H}$ implies that $N(ab)$ contains at most one from each pair x_{2i-1}, x_{2i}, hence $m_X(ab) \leqslant 3$. Finally, it is easy to check that $m_E(ab) = 1$ implies $m_X \leqslant 4$, completing the proof of (3.18).

Consider the identity

$$\begin{aligned}
&\frac{3}{2}\sum_{k=1,2,3} \deg_{\mathscr{H}}(z_k) + 2\sum_{1\leqslant j\leqslant 6} \deg_{\mathscr{H}}(x_j)\\
&\quad= \frac{3}{2}\sum_{k=1,2,3} e(G_{z_k}[A]) + \frac{3}{2}\sum_{k=1,2,3} e(G_{z_k}[A,B]) + \sum_{1\leqslant j\leqslant 6} e(G_{x_j}[A,B])\\
&\qquad+ \sum_{1\leqslant j\leqslant 6} (e(G_{x_j}) + e(G_{x_j}[B])) + \frac{3}{2}\sum_{k=1,2,3} e(G_{z_k}[B]).
\end{aligned}$$

Estimating the right-hand side, for the first sum we can use that no pair of vertices belongs to all the three $E(G_{z_i})$s. In the second and third sum we use (3.18), in the fourth sum each term is at most $\alpha(n-\alpha)$ by (3.14), and for the last sum we use (3.12). We obtain

$$\frac{33}{2}d_{\min} \leqslant \frac{3}{2}\times 2\binom{\alpha}{2} + 12\alpha(n-\alpha) + \frac{3}{2}\left(\frac{1}{2}(n-\alpha-v)^2 + 2v(n-\alpha-v) + 4n\right).$$

Using that $\frac{3}{2}\left(\frac{1}{2}(x-v)^2 + 2v(x-v)\right) \leqslant x^2$ holds for all x, v, we get

$$\frac{33}{2}d_{\min} \leqslant \frac{3}{2}\alpha^2 + 12\alpha(n-\alpha) + (n-\alpha)^2 + 6n = n^2 + 10\alpha n - \frac{19}{2}\alpha^2 + 6n.$$

Here $10\alpha n - (19/2)\alpha^2$ is monotone decreasing for $\alpha > \frac{10}{19}n$. Substituting $\alpha = \frac{3}{5}n$ we get the upper bound $d_{\min}(\mathscr{H}) \leqslant (179/825)n^2 + (12/33)n$, implying our claim. □

3.10. $\mathscr{H}$ is bipartite

Here we prove Theorem 1.3. The above lemmas hold, except Lemma 3.7, for every $\mathbb{F}_{3,2}$-free hypergraph satisfying $\alpha \geqslant \frac{2}{5}n$. From now on we will use that $\mathscr{H}$ satisfies the min-degree condition (1.1). Consider the partition $V = A \cup B = A \cup S \cup T$ defined in Section 3.1. We will prove that B is independent. (3.1) asserts $\frac{1}{n}(\alpha, v) \in D_1$. We will show that $v = 0$.

Our estimates could not handle all cases of (α, v) simultaneously, so we divide the domain D_1 into two parts by the line $y = 5x - 3$ and use different estimates for each of them. We chose this line to (somewhat) maximize the value of γ achievable with the method presented, and, simultaneously, to keep the required calculations minimal. Taking $y = 5x - 3.0053$ one can push γ up to 0.0084.

Consider first the case when $\frac{1}{n}(\alpha, v) \in D_2$. Then Lemma 3.2 gives that $F(\alpha/n, v/n) \leqslant -1/4000$. Since $(n - 3v) \geqslant \frac{2}{5}n$, we get from (3.10) that

$$\frac{2}{5}n \times \left(d_{\min}(\mathscr{H}) - \left(\frac{4}{9} - \gamma\right)\binom{n}{2}\right) \leqslant -\frac{1}{4000}n^3 + n^2.$$

Here the left-hand side is nonnegative, but the right-hand side is negative for $n > 4000$. This contradiction implies that $\frac{1}{n}(\alpha, v) \in D_3 \cap D_1$, in particular $\alpha > \frac{3}{5}n$.

Suppose, on the contrary, that $v > 0$. Then there exists an edge $E^\star = (z_1, z_2, z_3)$, $E^\star \subset B$ satisfying (3.12). Lemma 3.6 implies that $E^\star$ can be extended into a well-positioned $\mathbf{H}_9$. Therefore we can apply Lemma 3.7. This leads to the contradiction $d_{\min}(\mathscr{H}) < 0.434\binom{n}{2}$. Thus only $v = 0$ is possible and $\mathscr{H}$ is bipartite. □

3.11. The finer structure of $\mathscr{H}$

In this part we prove Theorem 2.2(i)–(iii). Let A be a maximum independent set. The proof of Theorem 1.3 implies that $\mathscr{H}$ is bipartite, and B is independent (for $n > n_0$). Then all triples have types AAB or ABB. So the min-degree condition (1.1) and (3.5) give

$$\frac{1}{3}n\left(\frac{4}{9} - \gamma\right)\binom{n}{2} \leqslant e(\mathscr{H}) \leqslant \frac{1}{2}(n - \alpha)\alpha^2.$$

Multiplying by 6 and rearranging, we obtain

$$\left(\frac{2}{3}n - \alpha\right)^2 (n + 3\alpha) \leqslant \left(\gamma + \left(\frac{4}{9} - \gamma\right)\frac{1}{n}\right)n^3,$$

implying Theorem 2.2(ii) for $n > \max\{n_0, 1/\gamma\}$:

$$\left|\alpha - \frac{2}{3}n\right| < \sqrt{\gamma} \cdot n. \tag{3.19}$$

Proof of Theorem 2.2(iii). Use that $\mathscr{H}$ is bipartite, Corollary 3.5, (1.1) and (3.19):

$$\begin{aligned}\Delta_{BBA} &= 2(\Delta_{BBA} + \Delta_{BAA}) - (\Delta_{BBA} + 2\Delta_{BAA})\\ &= 2e(\mathscr{H}) - \sum_{x \in A} \deg_{\mathscr{H}}(x) \leqslant 2 \times \frac{2}{27}n^3 - \alpha\left(\frac{4}{9} - \gamma\right)\binom{n}{2}\\ &= \frac{2}{9}\left(\frac{2}{3}n - \alpha\right)n^2 + \gamma\alpha\binom{n}{2} + \frac{2}{9}n\alpha < \sqrt{\gamma}n^3.\end{aligned}$$ □

3.12. The case $|A| \geqslant 0.65n$

In this section we suppose that $\mathscr{H}$ is an $\mathbb{F}_{3,2}$-free hypergraph satisfying the conditions of Theorem 2.2 and also suppose that $\alpha \geqslant 0.65n$.

Claim 3.8. *If C is an independent set of vertices with $|C \cap A| \geqslant \frac{1}{4}n$ then $C \subseteq A$.*

Proof. Assuming the contrary, we may fix a $z \in C \cap B$. Consider G_z. It has no edges in B (by Theorem 1.3), or in $C \cap A$, and only very few edges joining A and B (by (3.17)).

We get

$$d_{\min} \leqslant e(G_z[A,B]) + e(G_z[A]) \leqslant 4\alpha(n-\alpha) - 4d_{\min} + \binom{\alpha}{2} - \binom{|A \cap C|}{2}.$$

This is a contradiction for $d_{\min} > \left(\frac{4}{9} - \gamma\right)\binom{n}{2}$, $\alpha \geqslant 0.65n$ and $|A \cap C| \geqslant 0.25n$. □

Claim 3.9. $\mu^\star < \alpha$.

Proof. Consider a pair $x, y \in V$ with $\mu(xy) = \mu^\star$, and let $C := N(xy)$. If $|C| < 0.65n \leqslant |A|$, then there is nothing to prove. Otherwise $|C| > |B| + \frac{1}{4}n$, so it has at least $n/4$ common vertices with A. Since C is independent, too, Claim 3.8 implies that $C \subseteq A$. Thus $\mu^\star = \alpha$ is only possible if $C = A$ and $x, y \in B$.

Consider $E(G_x[A])$ and $E(G_y[A])$. If they have a common triangle, say abc, then the hyperedges xab, xac, xay and bcy form an $\mathbb{F}_{3,2}$, a contradiction. So we can apply the Turán–Mantel theorem for $E(G_x[A]) \cap E(G_y[A])$. We get

$$e(G_x[A]) + e(G_y[A]) \leqslant \binom{|A|}{2} + |E(G_x[A]) \cap E(G_y[A])| \leqslant \binom{|A|}{2} + \left\lfloor \frac{1}{4}\alpha^2 \right\rfloor.$$

Since B is independent (by Theorem 2.2(i)) we get $e(G_x) = e(G_x[A]) + e(G_x[A,B])$. Use this for x and y and apply (3.17):

$$2d_{\min} \leqslant e(G_x[A]) + e(G_y[A]) + e(G_x[A,B]) + e(G_y[A,B]) \leqslant \frac{3}{4}\alpha^2 + 8\alpha(n-\alpha) - 8d_{\min}.$$

For $\alpha \geqslant 0.65n$ this gives $d_{\min} \leqslant 0.43\binom{n}{2}$. This contradiction shows that $\alpha = \mu^\star$ is not possible. □

Proof of Theorem 2.2(iv). Consider an arbitrary 2-colouring (A_1, B_1) of $V(\mathscr{H})$ with A_1 and B_1 being independent. Then one of the colour classes meets A in at least $\frac{1}{4}n$ vertices, say $|A_1 \cap A| \geqslant n/4$. Claim 3.8 states that $A_1 \subseteq A$. We show that

$$|A \setminus A_1| \leqslant \sqrt{\gamma}n.$$

Indeed, every triple meeting $A \setminus A_1$ must meet both A_1 and B. Thus for $x \in A \setminus A_1$ we have $d_{\min} \leqslant \deg_{\mathscr{H}}(x) \leqslant |B||A_1|$. This is equivalent to $|B|(|A| - |A_1|) \leqslant |B||A| - d_{\min} = \alpha(n-\alpha) - d_{\min}$. Rearranging, we get

$$|B|(|A| - |A_1|) \leqslant \left(\frac{2}{3}n - \alpha\right)\left(-\frac{1}{3}n + \alpha\right) + \left(\frac{4}{9}\binom{n}{2} - d_{\min}\right) + \frac{2}{9}n.$$

Then (3.19) implies $|A| - |A_1| \leqslant \sqrt{\gamma}n$. □

3.13. The extremal hypergraph

Here we prove Theorem 2.3. Suppose that $\mathscr{H}$ is an n-vertex $\mathbb{F}_{3,2}$-free triple system of maximum cardinality,

$$e(\mathscr{H}) \geqslant \max_a \frac{1}{2}(n-a)a(a-1) := e(n).$$

The degrees of any two vertices of $\mathscr{H}$ differ by at most $n-2$. Otherwise one can delete the vertex of smaller degree and duplicate the other, thus increasing the size of $\mathscr{H}$. Thus we may suppose that (for $n > n_0$) $d_{\min}(\mathscr{H}) > (\frac{4}{9} - 10^{-4})\binom{n}{2}$. Apply Theorems 1.3 and 2.2(i). We obtain that $\mathscr{H}$ has a 2-colouring (A, B) where $|A|$ is the maximal independent set, $|A| = \alpha$. Then Theorem 2.2(ii) implies that $\alpha > 0.65n$. Then Claim 3.9 gives $\mu^\star \leqslant \alpha - 1$. We obtain the desired upper bound:

$$2e(\mathscr{H}) = \sum_{\substack{a\in A\\ b\in B}} \mu(ab) \leqslant \alpha(n-\alpha)\mu^\star \leqslant 2(n-\alpha)\binom{\alpha}{2} \leqslant 2e(n).$$

Moreover, equality can hold only if $\mu(ab) = \alpha - 1$ for every crossing pair $a \in A$, $b \in B$. Then $|N(ab)|$ is large, so by Claim 3.8 it is contained in A. Thus all hyperedges must be of type AAB. □

3.14. Reduction to minimum degree

Here we prove Theorems 2.4 and 2.5. Both deal with hypergraphs satisfying

$$e(\mathscr{H}) > \left(\frac{4}{9} - c\right)\binom{n}{3}. \tag{3.20}$$

Lemma 3.10. *Let $\gamma = c^{2/3}$. Then for $n > n_0(c)$ one can find a subset $V_1 \subseteq V$, $|V_1| = n_1 > (1 - c^{1/3})n$, such that, for $\mathscr{H}_1 := \mathscr{H}[V_1]$,*

$$\deg_{\mathscr{H}_1}(x) > \left(\frac{4}{9} - \gamma\right)\binom{n_1}{2} \tag{3.21}$$

holds for every $x \in V_1$.

Proof. Delete a vertex from V if its degree is at most $(\frac{4}{9} - \gamma)\binom{|V|}{2}$. Repeat this if we can find another vertex of small degree. This way the average degree goes up slightly, but it cannot go too high. A routine counting shows that the process stops within $c^{1/3}n$ steps. We omit the details. □

To prove Theorem 2.4 consider the hypergraph $\mathscr{H}$. Using Lemma 3.10, deleting at most $c^{1/3}n$ vertices (and $O(c^{1/3}n^3)$ edges) we get the hypergraph $\mathscr{H}_1$ satisfying (3.21). Apply Theorem 1.3 to $\mathscr{H}_1$ to obtain a bipartition (A, B). Apply Theorem 2.2(iii) to $\mathscr{H}_1$ to obtain a (2, 1)-colourable hypergraph after deleting another $O(\sqrt{\gamma}n^3) = O(c^{1/3}n^3)$ edges. □

To prove Theorem 2.5 consider a triple system $\mathscr{H}$ satisfying (3.20). Apply Lemma 3.10 to get $\mathscr{H}_1$. Then Theorem 2.2(ii), more exactly (3.19), implies that $\alpha(\mathscr{H}_1)$ is about $\frac{2}{3}|V_1|$. Since $|V \setminus V_1|$ is small this gives an upper bound for the maximum codegree,

$$\mathsf{maxcodeg}(\mathscr{H}_1) \leqslant \mathsf{maxcodeg}(\mathscr{H}) \leqslant \alpha(\mathscr{H}) \leqslant \alpha(\mathscr{H}_1) + |V \setminus V_1| \leqslant \frac{2}{3}n + O(\sqrt{\gamma})n.$$

Finally, since B_1 is independent,

$$e(\mathscr{H}_1) = \frac{1}{2} \sum_{a \in A_1, b \in B_1} \mu(ab) \leqslant \frac{1}{2} \alpha(\mathscr{H}_1)(|V_1| - \alpha(\mathscr{H}_1))\mu^\star(\mathscr{H}_1).$$

Then (3.20), and $|V_1| \sim n$, $\alpha(\mathscr{H}_1) \sim \frac{2}{3}n$ give the lower bound for $\mathsf{maxcodeg}(\mathscr{H}_1)$. □

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Appendix: Proof of the lemmas on polynomials

A.1. Proof of Lemma 3.1

First, we show that the Hessian of f is indefinite on the open halfplane $D_4 := \{(x, y) : x > 1/3\}$, so the extrema of f on $D_1 \subset D_4$ must be on the four boundary line-segments of ∂D_1. Let f_{xx}, f_{xy}, f_{yy} denote the partial derivatives, and let $\mathbf{J}(x, y)$ be the determinant of the Hessian. Then $f_{xx} = -9x - 2y + 3$, $f_{yy} = 8x + 18y - 8$, $f_{xy} = -2x + 8y - 2$ and

$$\mathbf{J}(x, y) = f_{xx} f_{yy} - f_{xy}^2 = 88x - 76x^2 - 146xy - 28 + 102y - 100y^2.$$

We have that $\mathbf{J}(x, y) \neq 0$ for every $x > 1/3$, because solving $\mathbf{J}(x, y) = 0$ for y, the discriminant $-2271x^2 + 1354x - 199$ is negative for $x > 1/3$. Since there is at least one point where $\mathbf{J}$ is negative, *e.g.*, $\mathbf{J}(1, 0) = -16 < 0$, continuity implies that $\mathbf{J}$ is negative for every point of D_4.

Let us check the behaviour of $f(x, y)$ on the boundary ∂D_1. On each segment it reduces to a third-degree polynomial of one variable, whose extrema can be identified by the roots of its derivative, a second-degree polynomial.

(1) First check the lower horizontal boundary, $I_1 := \{(x, 0) : 2/5 \leqslant x \leqslant 1\}$. Consider $f(x, 0) = \frac{3}{2}\big((1 - x)x^2 - \frac{4}{27}\big) = -\frac{3}{2}(\frac{2}{3} - x)^2(\frac{1}{3} + x)$. So $f \leqslant 0$ on I_1.

(2) Next check the left vertical boundary line $x = 2/5$. Let $\varphi_2(y) := f(2/5, y) = (225y^3 - 180y^2 + 53y)/75 - 88/1125$ and $I_2 := \{(2/5, y) : 0 \leqslant y \leqslant 1/5\}$. The derivative φ_2' has no real roots. So φ_2 is strictly increasing and takes its maximum on $[0, 1/5]$ at $y = 1/5$. Then $\varphi_2(1/5) = -2/225$ implies $f < 0$ on I_2.

(3) The third part to be checked is the boundary segment on $x = 1$. Let $\varphi_3(y) := f(1, y) = (27y^3 - 12y - 2)/9$ and $I_3 := \{(1, y) : 0 \leqslant y \leqslant 1/5\}$. We have $\varphi_3'(y) = (27y^2 - 4)/3$; its roots are $\pm\frac{2}{9}\sqrt{3} \sim \pm 0.3849$. So φ_3' is negative on $[0, 1/5]$ and φ_3 is decreasing and takes its maximum at $y = 0$. Then $\varphi_3(0) = -2/9$ implies $f < 0$ on I_3.

(4) Finally, check the segment on $y = 1/5$. Let $\varphi_4(x) := f(x, 1/5) = (-75x^3 + 65x^2 - 12x)/50 + 28/1125$ and $I_4 := \{(x, 1/5) : 2/5 \leqslant x \leqslant 1\}$. The derivative of φ_4 is $(-225x^2 + 130x - 12)/50$ and it has one root, $x_2 := (13 + \sqrt{61})/45 \sim 0.462$ in I_4. Hence φ_4' is positive on $[2/5, x_2)$ and negative on $(x_2, 1]$, and φ_4 has its maximum at x_2. We have $\varphi_4(x_2) = (61\sqrt{61} - 665)/30375 \sim -0.0062$ so $f < 0$ on I_4. □

A.2. Proof of Lemma 3.2

It is enough to prove the next two lemmas for $\gamma = 1/125$.

Since $F(x,y)$ and $f(x,y)$ differ by a linear term, their Hessians coincide. So $\mathbf{J}$ is negative on $D_2 \subset D_1$, too. So the extrema of F must be on some of the four boundary segments of ∂D_2.

(5) Consider the lower horizontal boundary, $I_5 := \{(x,0) : 2/5 \leqslant x \leqslant 3/5\}$. Define $\varphi_5(x) := F(x,0) = \frac{3}{2}\left((1-x)x^2 - \frac{4}{27}\right) + \frac{1}{2}\gamma$. Then $\varphi_5' = \frac{3}{2}(2x-3x^2)$ and it is positive on I_5. So φ_5 is increasing and takes its maximum at $x = \frac{3}{5}$. Then $\varphi_5(3/5) = \frac{1}{2}\gamma - 7/1125 = -1/450$, implying $F < -1/4000$ on I_5.

(6) Check F on the left vertical boundary line $x = 2/5$. Let $\varphi_6(y) := F(2/5, y) = (6750y^3 - 5400y^2 + 1563y - 167)/2250$ and $I_6 := \{(2/5, y) : 0 \leqslant y \leqslant 1/5\} = I_2$. Then φ_6' has no real roots; it is positive on $[0, 1/5]$. So φ_6 takes its maximum at $y = 1/5$. We have $\varphi_6(1/5) = -41/5625 \sim -0.007289$, so $F < -1/4000$ on I_6.

(7) Consider the boundary segment on the tilted line $y = 5x - 3$. Let $\varphi_7(x) := F(x, 5x-3) = (210825x^3 - 405225x^2 + 258873x - 54982)/450$ and $I_7 := \{(x, 5x-3) : 3/5 \leqslant x \leqslant 0.64\}$. The derivative of φ_7 is $(210825x^2 - 270150x + 86291)/150$, and it has one root $x_1 = (9005 - \sqrt{235358})/14055 \sim 0.6061$ in I_7. Then φ_7' is positive on $[3/5, x_1)$ and negative on $(x_1, 0.64]$. So φ_7 takes its maximum at x_1. We have $\varphi_7(x_1) \sim -2.58 \times 10^{-4}$, so $F < -1/4000$ on I_7.

(8) Finally, check the upper boundary on $y = 1/5$. Let $\varphi_8(x) := F(x, 1/5) = \varphi_4(x) + 1/625$ and $I_8 := \{(x, 1/5) : 2/5 \leqslant x \leqslant 0.64\} \subset I_4$. Since $\varphi_4 < -0.0062$, on I_4 we get $\varphi_4(x) + 0.0016 = \varphi_8(x) < -0.0046$. So $F < -1/4000$ on I_8. □

A.3. Proof of Lemma 3.3

Consider the closed half-plane $\overline{D_3} := \{(x,y) : y \leqslant 5x - 3\}$. We claim that

$$g(x,y) = -7x^2 - xy - \frac{3}{2}y^2 + 7x + y - \frac{31}{18} + 5\gamma,$$

defined in (3.2), takes its maximum on $\overline{D_3}$ at the point $(x_0, y_0) := (20/33, 1/33)$ on the boundary line and $g(x_0, y_0) = -4/99 + 5\gamma = -1/2475 < -1/2500$. Indeed, $g(x,y) = 0$ is an ellipse lying outside D_3, its centre being $(20/41, 7/41)$. Further, g has an absolute maximum at this centre; it is a concave function so its maximum on $\overline{D_3}$ should be on the boundary. Finally, $g(x, 5x-3) = -99x^2/2 + 60x - 4091/225$ and its maximum can easily be calculated. □

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