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The Game of JumbleG

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For Béla Bollobás on his 60th birthday

JumbleG is a Maker–Breaker game. Maker and Breaker take turns in choosing edges from the complete graph K_n. Maker's aim is to choose what we call an ϵ*-regular* graph (that is, the minimum degree is at least $(\frac{1}{2}-\epsilon)n$ and, for every pair of disjoint subsets $S, T \subset V$ of cardinalities at least ϵn, the number of edges $e(S,T)$ between S and T satisfies $\left|\frac{e(S,T)}{|S|\,|T|}-\frac{1}{2}\right| \leqslant \epsilon$.) In this paper we show that Maker can create an ϵ-regular graph, for $\epsilon \geqslant 2(\log n/n)^{1/3}$. We also consider a similar game, JumbleG2, where Maker's aim is to create a graph with minimum degree at least $\left(\frac{1}{2}-\epsilon\right)n$ and maximum co-degree at most $\left(\frac{1}{4}+\epsilon\right)n$, and show that Maker has a winning strategy for $\epsilon > 3(\log n/n)^{1/2}$. Thus, in both games Maker can create a pseudo-random graph of density $\frac{1}{2}$. This guarantees Maker's win in several other positional games, also discussed here.

1. Introduction

JumbleG is a Maker–Breaker game. Maker and Breaker take turns in choosing edges from the complete graph K_n on n vertices. Maker's aim is to choose a graph which is ϵ-regular (the definition follows).

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Let $G = (V, E)$ be a graph of order n. We usually assume that the vertex set is $[n] = \{1, \ldots, n\}$. We call a pair S, T of non-empty disjoint subsets of $[n]$ *ϵ-unbiased* if

$$\left| \frac{e_G(S,T)}{|S|\,|T|} - \frac{1}{2} \right| \leqslant \epsilon, \tag{1.1}$$

where $e_G(S, T)$ is the number of $S - T$ edges in G. The graph G is *ϵ-regular* if

P1: $\delta(G) \geqslant (\frac{1}{2} - \epsilon)n$,

P2: any pair S, T of disjoint subsets of $[n]$ with $|S|, |T| \geqslant \epsilon n$ is ϵ-unbiased.

Theorem 1.1. *Maker has a winning strategy in JumbleG provided $\epsilon \geqslant 2(\log n/n)^{1/3}$ and n is sufficiently large.*

We consider also a similar game, which we denote by JumbleG2. In this game Maker's aim is to create a graph with properties P1 and P3, where

P3: maximum co-degree is at most $(\frac{1}{4} + \epsilon)n$.

(The co-degree of vertices $u, v \in V(G)$ is the number of common neighbours of u and v in G.)

Here, too, Maker can win provided ϵ is not too small.

Theorem 1.2. *Maker has a winning strategy in JumbleG2 for all $\epsilon \geqslant 3(\log n/n)^{1/2}$ if n is sufficiently large.*

Theorems 1.1 and 1.2 are proved in Section 2. As shown in Section 3, our restrictions on ϵ are best possible, up to a logarithmic factor.

Although the goals of the above two games appear to be quite different, they are in fact very similar to each other: in both, Maker tries to create a *pseudo-random graph* of density around $\frac{1}{2}$. Informally speaking, a pseudo-random graph $G = (V, E)$ is a graph on n vertices whose edge distribution resembles that of a truly random graph $G(n, p)$ of the same edge density $p = e(G)\binom{n}{2}^{-1}$. The reader can consult [12] for a recent survey on pseudo-random graphs. The fact that an ϵ-regular graph is pseudo-random with density $\frac{1}{2}$ is apparent from the definition. To see that degrees and co-degrees can guarantee pseudo-randomness, we need to recall some notions and results due to Thomason. He introduced the notion of *jumbled* graphs [17]. A graph G with vertex set $[n]$ is (α, β)-jumbled if, for every $S \subseteq [n]$, we have

$$\left| e_G(S) - \alpha\binom{|S|}{2} \right| \leqslant \beta|S|$$

where $e_G(S)$ is the number of edges of G contained in S.

Thomason showed that one can check for pseudo-randomness via jumbledness by checking degrees and co-degrees. Suppose that $G = (V, E)$ has minimum degree at least αn and no two vertices have more than $\alpha^2 n + \mu$ common neighbours. Then (see Theorem 1.1 of [17] and its proof), for every $s \leqslant n$, every set $S \subseteq V$ of size $|S| = s$ satisfies

$$\left| e(S) - \alpha\binom{s}{2} \right| \leqslant \frac{((s-1)\mu + \alpha n)^{1/2} + \alpha}{2} s, \tag{1.2}$$

and therefore G is (α, β)-jumbled with $\beta = ((\alpha n + (n-1)\mu)^{1/2} + \alpha)/2$.

Now suppose that for some $\epsilon = \Omega(1/n)$ a graph G on n vertices has minimum degree at least $\alpha n = \left(\frac{1}{2} - \epsilon\right)n$ and maximum co-degree at most $\left(\frac{1}{4} + \epsilon\right)n = \alpha^2 n + (2\epsilon - \epsilon^2)n$. Then a routine calculation, based on (1.2), shows that G is ϵ'-regular for $\epsilon' = \Omega(\epsilon^{1/4})$. Thus Theorem 1.2 can be used to show that Maker can create an ϵ-regular graph with $\epsilon = n^{-1/8+o(1)}$, a weaker result than that provided by the direct application of Theorem 1.1. Indeed, let $|S| = s$, $|T| = t \geqslant \epsilon' n$, $\mu = (2\epsilon - \epsilon^2)n$, and $\epsilon' \geqslant \Omega(\epsilon^{1/4}) \geqslant \Omega(n^{-1/4})$. Then

$$\begin{aligned}
\left|\frac{e_G(S,T)}{st} - \frac{1}{2}\right| &= \frac{1}{st}|e_G(S \cup T) - e_G(S) - e_G(T) - (\alpha + \epsilon)st| \\
&\leqslant \frac{1}{st}\left(\left|e_G(S \cup T) - \alpha\binom{s+t}{2}\right| + \left|e_G(S) - \alpha\binom{s}{2}\right| + \left|e_G(T) - \alpha\binom{t}{2}\right|\right) + \epsilon \\
&\leqslant \frac{((s+t)\mu + \alpha n)^{1/2} + \alpha}{2st}(s+t) + \frac{(s\mu + \alpha n)^{1/2} + \alpha}{2t} + \frac{(t\mu + \alpha n)^{1/2} + \alpha}{2s} + \epsilon \\
&\leqslant \frac{(s+t)^{3/2}\mu^{1/2}}{2st} + \frac{(s\mu)^{1/2}}{2t} + \frac{(t\mu)^{1/2}}{2s} + \frac{4\big((\alpha n)^{1/2} + \alpha\big)}{2\min\{s,t\}} + \epsilon \\
&\leqslant \frac{((1+\epsilon')n)^{1/2}(2\epsilon n)^{1/2}}{\epsilon' n} + \frac{(2n\epsilon n)^{1/2}}{2\epsilon' n} + \frac{(2n\epsilon n)^{1/2}}{2\epsilon' n} + \frac{2n^{1/2}}{\epsilon' n} + \epsilon \\
&\leqslant c\frac{\epsilon^{1/2}}{\epsilon'} \\
&\leqslant \epsilon'.
\end{aligned}$$

Pseudo-random graphs are known to have many nice properties. Hence, Maker's ability to create a pseudo-random graph guarantees his win in several other positional games. For example, using a result of [11], one can guarantee Maker's success in creating $\frac{n}{4} - O(n^{5/6}\log^{1/6} n)$ pairwise edge-disjoint Hamiltonian cycles. This is trivially best possible up to the error-term and confirms a conjecture of Lu [13] in a strong form. We will discuss this and other games in Section 4.

2. Playing JumbleG

In this section we prove Theorems 1.1 and 1.2. The proofs are quite similar and are based on the approach of Erdős and Selfridge [9] via potential functions.

Lemma 2.1. *If the edges of a hypergraph $\mathscr{F}$ satisfy $\sum_{X\in\mathscr{F}} 2^{-|X|} < 1/4$ then Maker can force a 2-colouring of $\mathscr{F}$.*

Proof. Let a round consist of a move of Maker followed by a move of Breaker. At the start of a round, let C_M, C_B denote the set of edges chosen so far by Maker and Breaker, let R denote the unchosen edges and for $X \in \mathscr{F}$ let $\delta_{X,M}, \delta_{X,B}$ be the indicators of $X \cap C_M \neq \emptyset$, $X \cap C_B \neq \emptyset$ respectively. Let $\delta_X = \delta_{X,M} + \delta_{X,B}$. We use the potential function

$$\Phi = \sum_{\substack{X\in\mathscr{F} \\ \delta_X \leqslant 1}} 2^{-|X\cap R|+1-\delta_X}.$$

This represents the expected number of monochromatic sets if the unchosen edges are coloured at random. Our assumption is that $\Phi < \frac{1}{2}$ at the start and we will see that it can be kept this way until the end of the last complete round. If n is odd, Maker with his last choice can at most double the value of Φ. In any case, at the end of the play $\Phi < 1$. Also, at the end $R = \emptyset$; thus $\delta_X \geqslant 2$ for all $X \in \mathscr{F}$, showing that Maker has achieved his objective.

It remains to show that Maker can ensure that the value of Φ never increases after one complete round is played. Suppose that in some round Maker chooses an edge a and Breaker chooses an edge b. Let Φ' be the new value of Φ. Then

$$\begin{aligned}\Phi' - \Phi = &- \sum_{\substack{a,b\in X\\ \delta_X=0}} 2^{1-|X\cap R|} - \sum_{\substack{a\in X\\ \delta_{X,B}=1}} 2^{-|X\cap R|} - \sum_{\substack{b\in X\\ \delta_{X,M}=1}} 2^{-|X\cap R|}\\ &+ \sum_{\substack{a\in X, b\notin X\\ \delta_{X,M}=1}} 2^{-|X\cap R|} + \sum_{\substack{a\notin X, b\in X\\ \delta_{X,B}=1}} 2^{-|X\cap R|}\\ &\leqslant - \left(\sum_{\substack{a\in X\\ \delta_{X,B}=1}} 2^{-|X\cap R|} - \sum_{\substack{a\in X\\ \delta_{X,M}=1}} 2^{-|X\cap R|} \right) + \left(\sum_{\substack{b\in X\\ \delta_{X,B}=1}} 2^{-|X\cap R|} - \sum_{\substack{b\in X\\ \delta_{X,M}=1}} 2^{-|X\cap R|} \right),\end{aligned}$$

which is nonpositive if Maker chooses a to maximize

$$\sum_{\substack{a\in X\\ \delta_{X,B}=1}} 2^{-|X\cap R|} - \sum_{\substack{a\in X\\ \delta_{X,M}=1}} 2^{-|X\cap R|}. \qquad \square$$

Lemma 2.2. *Let $\epsilon = \epsilon(n)$ tend to zero with n. Let $\delta > 1$ be fixed. Let $t = \lceil \delta\epsilon^{-2} \log n \rceil$. Then, for all sufficiently large n, Maker can ensure that any pair of disjoint subsets of V, both of size at least t, is ϵ-unbiased.*

Proof. Assume that $t \leqslant n/2$, for otherwise there is nothing to prove. This means that $\epsilon > \left(\frac{2\log n}{n}\right)^{1/2}$.

Let $k = \lceil (\frac{1}{2} + \epsilon)\, t^2 \rceil$. Let $\mathscr{T}$ consist of pairs (S, T) of disjoint subsets of V, both of size *exactly* t. Recall that $e_M(S, T)$ counts the number of Maker's edges connecting S to T. A simple averaging argument shows that it is enough to show that Maker can guarantee that

$$t^2 - k < e_M(S, T) < k, \quad \text{for all } (S, T) \in \mathscr{T}. \tag{2.1}$$

(Indeed, let S', T' have size at least t each. The expectation of $\frac{e_M(S,T)}{t^2}$, where S, T are random t-subsets of S', T', is $\frac{e_M(S',T')}{|S'||T'|}$. By (2.1) this cannot differ from $\frac{1}{2}$ by more than ϵ, as required.)

If Maker is able to ensure that all k-element subsets of the edge-set $S : T = \{\{x, y\} \mid x \in S,\ y \in T\}$ are properly 2-coloured (*i.e.*, not monochromatic) for every $(S, T) \in \mathscr{T}$, then he has achieved his goal. A direct application of Lemma 2.1 is not possible, however: there are simply too many of these k-sets and the criterion does not hold. We need to cut down on the number of sets.

Define $\ell = \lceil 2t^2\epsilon \rceil$ and $\lambda = \lceil 2^\ell n^{-2t} \rceil$. For $(S, T) \in \mathscr{T}$ we prove the existence of a collection $\mathscr{X}_{S,T}$, of ℓ-subsets of the edge-set $S : T = \{\{x, y\} \mid x \in S,\ y \in T\}$ such that (i) $|\mathscr{X}_{S,T}| = \lambda$ and (ii) each k-set $B \subseteq S : T$ contains at least one member of $\mathscr{X}_{S,T}$. Let us show that if the elements of $\mathscr{X}_{S,T}$ are chosen at random, independently with replacement, then this property is almost surely satisfied. In estimating this probability we will use the following auxiliary inequalities: $\ell = o(t^2)$ and

$$\frac{\binom{k}{\ell}}{\binom{t^2}{\ell}} = \prod_{i=0}^{\ell-1} \frac{k-i}{t^2-i} = \left(\frac{k}{t^2}\right)^\ell \prod_{i=0}^{\ell-1}\left(1 - \frac{i(t^2-k)}{t^2k - ki}\right)$$
$$\geqslant \left(\frac{1}{2} + \epsilon\right)^\ell \exp\left\{-\frac{\ell^2}{2t^2} + O(\epsilon^2\ell)\right\}.$$

The probability that there is a k-subset of $S : T$ which does not contain a member of $\mathscr{X}_{S,T}$ is at most

$$\binom{t^2}{k}\left(1 - \frac{\binom{k}{\ell}}{\binom{t^2}{\ell}}\right)^\lambda \leqslant 2^{t^2} \exp\left\{-\lambda\left(\frac{1}{2} + \epsilon\right)^\ell e^{-\frac{\ell^2}{2t^2} + O(\epsilon^2\ell)}\right\}$$
$$= 2^{t^2} \exp\{-n^{-2t} e^{2\epsilon\ell - \frac{\ell^2}{2t^2} + O(\epsilon^2\ell)}\}$$
$$= 2^{t^2} \exp\{-e^{-2t\log n + (2+o(1))\epsilon^2 t^2}\} = o(1),$$

so a family $\mathscr{X}_{S,T}$ with the required property does exist.

Let $\mathscr{F} = (\binom{[n]}{2}, \mathscr{E})$ be the hypergraph with hyper-edges $\mathscr{E} = \bigcup_{(S,T)\in\mathscr{T}} \mathscr{X}_{S,T}$. (We will use the term *hyper-edges* to distinguish them from the edges of K_n.) To complete the proof it is enough to show that Maker can ensure that the choices $E_M, E_B \subset \binom{[n]}{2}$ of Maker, Breaker respectively are a 2-colouring of $\mathscr{F}$. This follows from Lemma 2.1 in view of the inequality

$$|\mathscr{E}|\, 2^{-\ell} \leqslant \binom{n}{t}^2 \lambda 2^{-\ell} = o(1). \tag{2.2}$$

□

Proof of Theorem 1.1. To ensure that all degrees of Maker's graph are appropriate we use a trick similar to the one in the proof of the previous lemma. Let $k = \lceil (1/2 + \epsilon)n \rceil$. Maker again would like to use Lemma 2.1 and ensure that all k-subsets of the edges incident with vertex i are properly 2-coloured. These are again too many; we define $\ell = \lceil 10\epsilon^{-1} \log n \rceil$, $M = \lceil 2^\ell / n^2 \rceil$, and $\mu = nM$. We want to find a collection $A_1, A_2, \ldots, A_\mu$ of ℓ-sets such that, for $1 \leqslant i \leqslant n$, every k-subset of the edges incident with i contains at least one of $A_{(i-1)M+j}$, $1 \leqslant j \leqslant M$. As before, we construct the sets A_i randomly. The probability that there is a bad k-subset (containing no chosen ℓ-set) is at most

$$\binom{n-1}{k}\left(1 - \frac{\binom{k}{\ell}}{\binom{n-1}{\ell}}\right)^M \leqslant 2^n \exp\left\{-\frac{2^\ell}{n^2}\frac{k^\ell}{n^\ell} e^{-\ell^2/n}\right\} \leqslant 2^n \exp\{-\left((1+\epsilon)e^{-\ell/n}\right)^\ell\} < n^{-2}$$

for large n, and so the desired sets exist.

For property P2 let $t = \lceil 6\epsilon^{-2} \log n \rceil$. By our assumption on ϵ, we have $t < \epsilon n$. Define $\mathscr{X}_{S,T}$ as in the proof of Lemma 2.2. Namely, let $\ell' = \lceil 2t^2\epsilon \rceil$ and $\lambda = \lceil 2^{\ell'} n^{-2t} \rceil$. For

$(S,T) \in \mathscr{T}$ (that is, S,T are disjoint t-sets) let $\mathscr{X}_{S,T}$ be a collection ℓ'-subsets of $S:T$ such that (i) $|\mathscr{X}_{S,T}| = \lambda$ and (ii) every $\lceil(\frac{1}{2}+\epsilon)t^2\rceil$-set contains at least one member of $\mathscr{X}_{S,T}$.

Let $\mathscr{F}$ be the hypergraph with the edge set $\mathscr{E}_1 \cup \mathscr{E}_2 = \{A_1, A_2, \dots, A_\mu\} \cup \bigcup_{(S,T)\in\mathscr{T}} \mathscr{X}_{S,T}$.

Lemma 2.2 (or rather its proof) implies that it suffices for Maker to force a 2-colouring of $\mathscr{F}$. Indeed, the definition of the sets A_i will imply property P1. To see that P2 will also hold, observe that for any $S,T \in \mathscr{T}$, we will have

$$\left|\frac{e_M(S,T)}{t^2} - \frac{1}{2}\right| \leqslant \epsilon,$$

while the claim for general $|S|,|T| \geqslant t$ follows by averaging.

It remains to check that $\mathscr{F}$ satisfies the conditions of Lemma 2.1 for large n. The initial value Φ of the potential function satisfies

$$\Phi \leqslant Mn2^{-\ell} + \Phi(\mathscr{E}_2) = o(1). \tag{2.3}$$

(Here we have used (2.2).) This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. This time for property P1 we define $\ell = \lfloor \epsilon n \rfloor$; as before, $M = \lceil 2^\ell/n^2 \rceil$, $\mu = nM$, $k = \lceil (1/2+\epsilon)n \rceil$. The family $A_1, A_2, \dots, A_\mu$ should satisfy: For $1 \leqslant i \leqslant n$, every k-subset of the edges incident with i contains at least one of $A_{(i-1)M+j}$, $1 \leqslant j \leqslant M$. We construct the A_i randomly. Suppose that we randomly choose M ℓ-subsets of $[n-1]$ independently with replacement. The probability that there is a k-subset of $[n-1]$ which contains no chosen ℓ-set is at most

$$\begin{aligned}
&\binom{n-1}{k}\left(1 - \frac{\binom{k}{\ell}}{\binom{n-1}{\ell}}\right)^M \\
&\leqslant 2^n \exp\left\{-\frac{\binom{k}{\ell}M}{\binom{n}{\ell}}\right\} \\
&= 2^n \exp\left\{-M\,\frac{k\cdots(k-\lfloor \ell/2\rfloor+1)}{n\cdots(n-\lfloor \ell/2\rfloor+1)} \cdot \frac{(k-\lfloor \ell/2\rfloor)\cdots(k-\ell+1)}{(n-\lfloor \ell/2\rfloor)\cdots(n-\ell+1)}\right\} \\
&\leqslant 2^n \exp\left\{-M\left(\frac{k-\ell/2}{n}\right)^{\lfloor \ell/2\rfloor} \cdot \left(\frac{k-\ell}{n}\right)^{\lceil \ell/2\rceil}\right\} \\
&\leqslant \exp\left\{n\log 2 - \frac{2^\ell}{n^2}\left(\frac{1}{2}+\frac{\epsilon}{2}\right)^{\lfloor \ell/2\rfloor}\left(\frac{1}{2}\right)^{\lceil \ell/2\rceil}\right\} \\
&= \exp\left\{n\log 2 - \frac{(1+\epsilon)^{\lfloor \ell/2\rfloor}}{n^2}\right\} \\
&< n^{-2}
\end{aligned}$$

for $\epsilon \geqslant 3(\log n/n)^{1/2}$, so the required family exists.

For property P3 we take a collection $B_1, B_2, \dots, B_\rho$ of ℓ-sets where $\rho = \binom{n}{2}N$ and $N = \lceil 4^\ell/n^3 \rceil$. For each pair $i,j \in [n]$ select N random ℓ-subsets of $[n]\setminus\{i,j\}$ so that each $\lceil(1/4+\epsilon)n\rceil$-set contains at least one of them. The hyper-edges are $\{(i,x) : x \in A\}$ $\cup\{(j,x) : x \in A\}$ for each random $A \subseteq [n]\setminus\{i,j\}$. $B_1, B_2, \dots, B_\rho$ are chosen randomly and now with $k = \lceil(1/4+\epsilon)n\rceil$ the probability that there is a k-subset of $[n-2]$ which contains no chosen ℓ-set is at most

$$\binom{n-2}{k}\left(1-\frac{\binom{k}{\ell}}{\binom{n-2}{\ell}}\right)^N \leqslant \exp\left\{n\log 2-\frac{(1+2\epsilon)^{\lfloor \ell/2\rfloor}}{n^3}\right\} < n^{-3}$$

for large n, and so the sets exist.

We will use Lemma 2.1 and so we need to check that the initial potential is less than 1/4. Now the initial value of the potential function is at most

$$Mn2^{1-\ell}+Nn^2 2^{1-2\ell}=o(1)$$

and this completes the proof of Theorem 1.2. □

3. Breaker's strategies

In this section we show that up to a small power of $\log n$, our restrictions on ϵ are sharp in both Theorems 1.1 and 1.2 or, even more strongly, with respect to each of properties P1–P3.

Property P1

Theorem 1.2 gives immediately that Maker can guarantee a graph with minimum degree at least $n/2-3\sqrt{n\log n}$. A similar result has been previously obtained by Székely [16], by applying a lemma of Beck [2, Lemma 3], which in turn is based on the Erdős–Selfridge method. This comes quite close to a result of Beck [3] who proved that Breaker can force the minimum degree of Maker's graph to be $n/2-\Omega(\sqrt{n})$.

Property P2

Let $c>0$ be any constant which is less than $6^{-1/3}$, n be large, and $\epsilon = cn^{-1/3}\log^{1/3} n$.

Here we prove that *no* graph of order n can satisfy property P2 for this ϵ, which shows that the restriction on ϵ in Theorem 1.1 is sharp up to a multiplicative constant. The proof is based on ideas of Erdős and Spencer [10].

Let G be an arbitrary graph of order n. Let $m=\lceil \epsilon n\rceil$. Let X be a random m-subset of $V(G)$ chosen uniformly. For $y\in V(G)$, let $\mathscr{E}_y$ be the event that $y\notin X$ and $|\,|\Gamma(y)\cap X|-m/2|>\epsilon m$, where $\Gamma(y)$ denoted the set of neighbours of y in G.

Let us show that for every y,

$$\mathbf{Pr}(\mathscr{E}_y)\geqslant \frac{2m}{n}. \tag{3.1}$$

Let $d=d(y)$ be the degree of y. By symmetry, we can assume that $d\leqslant \frac{n-1}{2}$. For such d we bound from below the probability p that $y\notin X$ and $|\Gamma(y)\cap X|\leqslant m/2-\epsilon m$, which equals

$$p=\sum_{i<m/2-\epsilon m}\binom{d}{i}\binom{n-1-d}{m-i}\binom{n}{m}^{-1}.$$

The combinatorial meaning of p implies that it decreases with d, so it is enough to bound p for $d=\lfloor \frac{n-1}{2}\rfloor$ only. Let us consider the summands s_h corresponding to $i=m/2-h$ with, say, $\epsilon m < h \leqslant \epsilon m + n^{1/3}$. Let

$$f(x) = (1+x)^{\frac{1+x}{2}}(1-x)^{\frac{1-x}{2}}.$$

Its Taylor series at 0 is $1 + \frac{x^2}{2} + O(x^4)$. By Stirling's formula, we obtain that each summand

$$\begin{aligned} s_h &= \Omega\left(\frac{n^{-1/3}(\log n)^{1/6}}{f^m(\frac{2h}{m})\, f^{2d-m}(\frac{2h}{2d-m})}\right) \\ &= \exp\left(-\frac{1}{3}\log n - \frac{2h^2}{m} - \frac{2h^2}{2d-m} + O(\log\log n)\right) \\ &= n^{-1/3-2c^3-o(1)}. \end{aligned}$$

Thus

$$\sum_{h=\epsilon m}^{\epsilon m + n^{1/3}} s_h = n^{-2c^3-o(1)} \geqslant \frac{2m}{n}.$$

It follows that there is a choice of an m-set X such that $|Y| \geqslant 2m$, where Y consists of the vertices for which R_x holds. By definition $Y \cap X = \emptyset$.

Assume without loss of generality that we have $d_X(y) < m - \epsilon m$ for at least half of the vertices of Y. Let $Z \subset Y$ consist of any m of these vertices. This pair (X, Z), both sets having at least ϵn elements, has the required bias.

Property P3

Here we show that Breaker can force Maker to create a co-degree of at least $\frac{n}{4} + c\sqrt{n}$. Our argument is based on a theorem of Beck [5], which states that Breaker can force Maker's graph to have maximum degree at least $n/2 + \sqrt{n}/20$. Then the following lemma shows that Breaker also succeeds in forcing a high co-degree in Maker's graph.

Lemma 3.1. *Assume that $c_1 > 0$ is constant. Then, for sufficiently large n, the following holds. Let $G = (V, E)$ be a graph on n vertices with $n(n-1)/4$ edges. If G has a vertex of degree at least $n/2 + c_1\sqrt{n}$, then G has a pair of vertices w_1, w_2 whose co-degree is at least $n/4 + c_1\sqrt{n}/10$.*

Proof. Let $c_2 = c_1/10$. Let v be a vertex of maximum degree in G. Denote $N_1 = N(v)$, $N_2 = V - N_1$. Then $|N_2| \leqslant n/2 - c_1\sqrt{n}$. If there is $u \in V$ such that $d(v, N_1) \geqslant n/4 + c_2\sqrt{n}$, we are done. Otherwise, for every u, $d(u, N_1) \leqslant n/4 + c_2\sqrt{n}$, implying:

$$\begin{aligned} A &\stackrel{\text{def}}{=} \sum_{u \in V} d(u, N_2) \\ &\geqslant \sum_{u \in V} (d(u) - d(u, N_1) - 1) \\ &\geqslant 2|E| - n(n/4 + c_2\sqrt{n}) - n \\ &= n^2/4 - c_2 n^{3/2} - 3n/2. \end{aligned}$$

Therefore by convexity,

$$B \stackrel{\text{def}}{=} \sum_{u \in V} \binom{d(u, N_2)}{2} \geqslant n \binom{A/n}{2} \geqslant n^3/32 - c_2 n^{5/2} - O(n^2).$$

On the other hand,

$$B = \sum_{w_1 \neq w_2 \in N_2} \text{co-degree}(w_1, w_2),$$

and thus there is a pair $w_1, w_2 \in N_2$ such that:

$$\begin{aligned} \text{co-degree}(w_1, w_2) &\geqslant |B| / \binom{|N_2|}{2} \\ &\geqslant \frac{n^3/32 - c_2 n^{5/2} - O(n^2)}{\binom{n/2 - c_1 n^{1/2}}{2}} \\ &\geqslant n/4 + c_2\sqrt{n}. \end{aligned}$$

□

4. Consequences

As we have already mentioned in the Introduction, Maker's ability to create a pseudo-random graph of density about $\frac{1}{2}$ allows him to win quite a few other combinatorial games. We will describe some of them below. All these games are played on the complete graph K_n unless stated otherwise; Maker and Breaker choose one edge alternately, Maker's aim being to create a graph that possesses a desired graph property.

Edge-disjoint Hamilton cycles. In this game Maker's aim is to create as many pairwise edge-disjoint Hamilton cycles as possible. Lu proved [13] that Maker can always produce at least $\frac{1}{16}n$ Hamilton cycles and conjectured that Maker should be able to make $(\frac{1}{4} - \epsilon)n$ for any fixed $\epsilon > 0$. This conjecture follows immediately from our Theorem 1.1 and Theorem 2 of [11]. In [11], Frieze and Krivelevich show that a 2ϵ-regular graph contains at least $(\frac{1}{2} - 6.5\epsilon)n$ edge-disjoint Hamilton cycles, for all $\epsilon > 10(\log n/n)^{1/6}$. Our argument applies equally to the bipartite version of the problem where the game is played on the complete bipartite graph $K_{n,n}$. Thus Maker can always produce at least $(\frac{1}{4} - \epsilon)n$ edge-disjoint Hamilton cycles, verifying another conjecture of Lu [14, 15]. Finally, there is an analogous game that can be played on the complete digraph D_n and here Maker can always produce at least $(\frac{1}{2} - \epsilon)n$ edge-disjoint Hamilton cycles.

Vertex-connectivity. Theorem 1.2 can be used to show that Maker can always force an $(n/2 - 3\sqrt{n \log n})$-vertex-connected graph. Indeed, let Maker's graph M have minimum degree at least $n/2 - 3\sqrt{n \log n}$ and maximum co-degree at most $n/4 + 3\sqrt{n \log n}$. Suppose that the removal of some set R disconnects M, say $V(M) \setminus R = A \cup B$ with $|A| \leqslant |B|$. If $|A| = 1$, then obviously all neighbours of $a \in A$ are in R, implying $|R| \geqslant \delta(M) \geqslant n/2 - 3\sqrt{n \log n}$. If $|A| \geqslant 2$, let a_1, a_2 be two distinct vertices in A. Then all neighbours of a_1, a_2 lie in $A \cup R$, and therefore

$$|A| + |R| \geqslant \deg_M(a_1) + \deg_M(a_2) - \text{co-deg}_M(a_1, a_2) \geqslant \frac{3n}{4} - 9\sqrt{n \log n}.$$

If $|A| \geqslant n/4 - 6\sqrt{n \log n}$, then $|B| \geqslant |A| \geqslant n/4 - 6\sqrt{n \log n}$ as well, and by the $o(1)$-regularity of M there is an edge between A and B, a contradiction. We conclude that $|A| \leqslant n/4 - 6\sqrt{n \log n}$, implying $|R| \geqslant n/2 - 3\sqrt{n \log n}$, as required.

The result of Beck [3] showing that Breaker can force a vertex which has degree at most $n/2 - \Omega(\sqrt{n})$ in Maker's graph indicates that the error term in our result about the connectivity game is tight up to a logarithmic factor.

$c \log n$-universality. A graph G is called *r-universal* if it contains an induced copy of every graph H on r vertices. We can show the following result.

Theorem 4.1. *Let $r = r(n)$ be an integer, which satisfies*

$$\frac{n-r+1}{r}\left(\frac{1}{2} - \epsilon\right)^{r-1} \geqslant \frac{2\log n}{\epsilon^2},$$

for some $\epsilon = \epsilon(n) \to 0$. Then for all sufficiently large n Maker can ensure that his graph M is r-universal.

Proof. Let $t = \lfloor \frac{2\log n}{\epsilon^2} \rfloor$. Let n be sufficiently large so that the conclusion of Lemma 2.2 is valid. Let M be an arbitrary graph satisfying this property, that is, any pair of disjoint subsets of $V(M)$, both of size at least t, is ϵ-unbiased. Let G be any graph on $[r]$. We will show that G is an induced subgraph of M.

Partition $V(M) = \cup_{i=1}^r V_i$ into r parts, each having at least $\frac{n-r+1}{r}$ vertices. Initially, let $A_i = V_i$, $i \in [r]$. We define $f : [r] \to V(M)$ with $f(i) \in A_i$ inductively.

Suppose we have already defined f on $[i-1]$. It will be the case that $|A_j| \geqslant \frac{n-r+1}{r}\eta^{i-1}$ for any $j \geqslant i$, where for brevity $\eta = \frac{1}{2} - \epsilon$. We will choose $f(i) = v \in A_i$ so that for any $j > i$ we have

$$|A_{ji}(v)| \geqslant \eta |A_j|, \tag{4.1}$$

where we define $A_{ji}(v) = A_j \cap \Gamma_M(v)$ if $\{i,j\} \in E(G)$ and $A_{ji}(v) = A_j \setminus \Gamma_M(v)$ otherwise. (Here $\Gamma_M(v)$ is the set of neighbours of v in M.)

Let B_{ji} be the set of vertices of A_i violating (4.1), *i.e.*, $\{v \in A_i : |A_{ji}(v)| < \eta|A_j|\}$. Then $|B_{ji}| < t$ as the pair (B_{ji}, A_j) is not ϵ-unbiased. (Observe that $|A_j| \geqslant \frac{n-r+1}{r}\eta^{r-1} \geqslant t$.) Update A_i by deleting B_{ji} for all $j \in [i+1, r]$. Thus at least $\frac{n-r+1}{r}\eta^{i-1} - (r-i)t \geqslant t$ vertices still remain in A_i. This inequality is true for $i = r$ by our assumption and for any other i, because $\eta \leqslant \frac{1}{2}$. So a suitable $f(i)$ can always be found. Now, replace A_j with $A_{ji}(f(i))$ for $j > i$. This completes the induction step. At the end of the process $f([r])$ induces a copy of G in M. □

It follows from Theorem 4.1 that Maker can create an$r$-universal graph with $r = (1+o(1))\log_2 n$. On the other hand, Maker cannot achieve $r = 2\log_2 n - 2\log_2\log_2 n + C$ because, as was shown by Beck [4, Theorem 4], Breaker can prevent K_r in Maker's graph.

There is a remarkable parallel between random graphs and Maker–Breaker games: see *e.g.*, Chvátal and Erdős [8], Beck [3, 4] and Bednarska and Łuczak [6]. As shown by Bollobás and Thomason [7], the largest r such that a random graph of order n is almost surely r-universal is around $2\log_2 n$. We conjecture that games have the same universality threshold (asymptotically).

Conjecture 4.2. *Maker can claim an r-universal graph with $r = (2+o(1))\log_2 n$.*

The following related result improves the unbiased case of Theorem 4 in Beck [3]. (His assumption $n \geqslant 100r^3v3^{r+1}$ is stronger than ours.)

Theorem 4.3. *Let integers r, v and a real $\epsilon > 0$ (all may depend on n) satisfy $\epsilon \to 0$ and*

$$\frac{n-r+1}{r}\left(\frac{1}{2}-\epsilon\right)^{r-1} \geqslant v + \frac{2\log n}{\epsilon^2}.$$

Then for sufficiently large n, Maker can ensure that any graph G of order at most v and maximum degree less than r is a subgraph (not necessarily induced) of Maker's graph M.

Outline of proof. Use the method of Theorem 4.1 with the following changes. Take a proper colouring $c : V(G) \to [r]$. The desired f will map $i \in V(G)$ into $A_{c(i)}$. The proof goes the same way except that when choosing $f(i)$ we have to worry only about those $j \geqslant i$ which are neighbours of i in G and make sure that there are at least v good choices for $f(i) \in A_{c(i)}$ (so that we can ensure that f is injective). The details are left to the reader. □

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