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The codegree threshold of K_4^-

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Abstract

The codegree threshold $\mathrm{ex}_2(n, F)$ of a non-empty 3-graph F is the minimum $d = d(n)$ such that every 3-graph on n vertices in which every pair of vertices is contained in at least $d+1$ edges contains a copy of F as a subgraph. We study $\mathrm{ex}_2(n, F)$ when $F = K_4^-$, the 3-graph on 4 vertices with 3 edges. Using flag algebra techniques, we prove that

$$\mathrm{ex}_2(n, K_4^-) = \frac{n}{4} + O(1).$$

This settles in the affirmative a conjecture of Nagle [20]. In addition, we obtain a stability result: for every near-extremal configuration G, there is a quasirandom tournament T on the same vertex set such that G is close in the edit distance to the 3-graph $C(T)$ whose edges are the cyclically oriented triangles from T. For infinitely

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many values of n, we are further able to determine $\mathrm{ex}_2(n, K_4^-)$ exactly and to show that tournament-based constructions $C(T)$ are extremal for those values of n.

Keywords: extremal combinatorics, hypergraphs, codegree treshold, flag algebras.

1 Introduction

Interest in the extremal theory of hypergraphs (and of 3-graphs in particular), dates back to Turán's celebrated 1941 paper [25]. Despite significant efforts from the research community, however, the problem of determining the Turán density of a given 3-graph F is open in all but a small number of cases — see Keevash's survey of the field [14]. The difficulty of the problem has lead researchers to investigate a number of other notions of extremal density.

The *codegree* of a pair $\{x, y\} \subseteq V(G)$ is the number $d(x, y)$ of edges of a 3-graph G containing the pair $\{x, y\}$. The *minimum codegree* of G, which we denote by $\delta_2(G)$, is the minimum of $d(x, y)$ over all pairs $\{x, y\} \subseteq V(G)$. The *codegree threshold* $\mathrm{ex}_2(n, F)$ of a nonempty 3-graph F is the maximum of $\delta_2(G)$ over all F-free 3-graphs on n vertices. It can be shown [19] that the limit

$$\pi_2(F) := \lim_{n\to\infty} \frac{\mathrm{ex}_2(n, F)}{n-2}$$

exists; this quantity is called the *codegree density* of F. A simple averaging argument shows that

$$0 \le \pi_2(F) \le \pi(F) \le 1,$$

and it is known that $\pi_2(F) \neq \pi(F)$ in general.

In the late 1990s, Nagle [20] and then Czygrinow and Nagle [4] made conjectures on the values of the codegree densities $\pi_2(K_4^-)$ and $\pi_2(K_4)$, respectively, where $K_4^- = ([4], \{123, 124, 134\})$. In other words, K_4^- is the unique (up to isomorphism) 3-graph on 4 vertices with 3 edges. From the perspective of Turán-type problems, the 3-graph K_4^- is the smallest non-trivial 3-graph.

In this work, we focus on the value of $\pi_2(K_4^-)$ and settle the following conjecture in the affirmative.

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Conjecture 1.1 (Nagle) $\pi_2(K_4^-) = 1/4$.

The lower bound in Nagle's conjecture comes from an old construction originally due to Erdős and Hajnal [6]:

Construction 1.2 (Erdős-Hajnal construction) *Given a tournament T on the vertex set $[n]$, define a 3-graph $C(T)$ on the same vertex set by setting $E(C(T))$ to consist of all the triples of vertices from $[n]$ inducing a cyclically oriented triangle in T.*

It is easily checked that no tournament on 4 vertices can contain more than 2 cyclically oriented triangles, whence this construction $C(T)$ gives a K_4^--free 3-graph. Furthermore, if the tournament T is chosen uniformly at random then standard Chernoff and union bounds yield that $\delta_2(C(T)) = n/4 - o(n)$ with high probability.

Mubayi [18] determined the codegree density of the Fano plane, and Keevash and Zhao [15] later extended Mubayi's work to other projective geometries. The precise codegree threshold of the Fano plane was determined for large enough n by Keevash [13] using hypergraph regularity, and DeBiasio and Jiang [5] later found a second, regularity-free proof of the same result. Mubayi and Zhao [19] established a number of theoretical properties of the codegree density, while Falgas-Ravry [8] gave evidence that codegree density problems for complete 3-graphs are not always stable. Finally, Falgas-Ravry, Marchant, Pikhurko and Vaughan [9] determined the codegree threshold of the 3-graph $F_{3,2} = \{abc, abd, abe, cde\}$ for all n sufficiently large.

Our main result adds a new example to this scant list of known non-trivial codegree densities by showing $\pi_2(K_4^-) = 1/4$, As the smallest non-trivial 3-graph from the perspective of Turán-type problems, K_4^- has received extensive attention from researchers in the area. Its Turán density is not known, but is conjectured by Mubayi [17] to be $2/7 = 0.2857\ldots$, with the lower bound coming from a recursive construction of Frankl and Füredi [11]. Matthias [16] and Mubayi [17] proved upper bounds on $\pi(K_4^-)$, before the advent of Razborov's flag algebra framework [21], and in particular his semi-definite method, led to computer-aided improvements by Razborov [22] and Baber and Talbot [1], with a current best upper bound of $0.2868\ldots$ [10].

In addition, 'smooth' variants of the Turán density problem for K_4^- have been studied. The *δ-linear density* of a 3-graph G is the minimum edge-density attained by an induced subgraph of G on at least $\delta v(G)$ vertices. Motived by the analogous positive results for graphs (see, for example, [24]), Erdős and Sós [7] asked whether having a δ-linear density bounded away from 0 for sufficiently small δ is enough to ensure the existence of a copy of K_4^- in sufficiently large 3-graphs. Füredi observed however that the tournament construction of Erdős and Hajnal described above gives a negative answer to this question: a linear-density of at least $1/4$ is required for the existence of a K_4^--subgraph. In recent work, Glebov, Král' and Volec [12] showed this $1/4$ lower bound is tight, using flag algebraic techniques amongst other ingredients in their proof. From the proof, it also follows that the Erdős-Hajnal construction is asymptotically the unique K_4^--free $1/4$-linear dense 3-graph. Even more recently, Reiher, Rödl and Schacht [23] reproved the result of [12] and established the edge-density at which weakly quasirandom 3-graphs must contain a copy of K_4^-, for various notions of 'weakly quasirandom'. The extremal problem for K_4^- under both a codegree and a smoothness assumption had been studied earlier by Kohayakawa, Rödl and Szemerédi (see [20,23]).

2 Our results

Our main result is the full solution of Conjecture 1.1.

Theorem 2.1 (Codegree density) $\pi_2(K_4^-) = 1/4$.

We obtain this result using flag algebra techniques: applying the semi-definite method of Razborov [22], we establish an asymptotic identity between 7-vertex subgraph densities of K_4^--free 3-graphs, from which Nagle's conjecture easily follows. Further, by analysing this identity, we deduce that in all near-extremal 3-graphs G, between almost any two pairs of vertices uv and xy we can find a tight-path with three edges connecting them. This allows coupling such a G with a tournament T on the same vertex-set in a way that almost all edges of G correspond to cyclically oriented triangles in T. The codegree assumption on G and standard results on quasirandom tournaments (see [2,3]) yield that T must be quasirandom.

Theorem 2.2 (Stability) *Let G be a K_4^--free 3-graph on $[n]$ with $\delta_2(G) \geq n/4 - o(n)$. Then there exists a quasirandom tournament T on $[n]$ such that the edit distance between G and the 3-graph $C(T)$ is $o(n^3)$.*

Using the stability result, we show that if n is sufficiently large, then the maximum value of $\mathrm{ex}(n, K_4^-)$ is always attained by a tournament-type construction. This allows us to fully determine the exact value of $\mathrm{ex}(n, K_4^-)$ for infinitely many values of n, and relate it to the existence of certain combinatorial designs: A *skew Hadamard matrix* is a square matrix A with ± 1 entries such that (i) the rows of A are pairwise orthogonal, and (ii) $A^t = -A$. The existence of such a matrix relates to the codegree threshold of K_4^- in the following way.

Theorem 2.3 (Codegree threshold) *For all n sufficiently large,*

$$\mathrm{ex}_2(n, K_4^-) \leq \left\lfloor \frac{n+1}{4} \right\rfloor .$$

Further, if there exists a skew Hadamard matrix of order $4k+4$, then for $n = 4k+3$ and $n = 4k+2$ sufficiently large, then we have equality in the equation above and every extremal construction for $n = 4k+3$ is an Erdős-Hajnal tournament-type construction.

Seberry's conjecture states that skew Hadamard matrices actually exist for every $n \equiv 0 \mod 4$. It is known to hold for all $n < 276$, and all n of the form $2^t \prod_{i \in I}(q_i + 1)$, where $t \in \mathbb{Z}_{\geq 0}$, I is a non-empty set of indices and for each $i \in I$, q_i is a prime power congruent to $3 \mod 4$.

Corollary 2.4 *If Seberry's conjecture is true, then for all n sufficiently large*

$$\mathrm{ex}_2(n, K_4^-) = \begin{cases} \lfloor \frac{n+1}{4} \rfloor & \text{if } n \equiv 2, 3 \mod 4, \\ \lfloor \frac{n+1}{4} \rfloor \text{ or } \lfloor \frac{n-3}{4} \rfloor & \text{if } n \equiv 0, 1 \mod 4. \end{cases}$$

Finally, we prove that Seberry's conjecture is actually equivalent to the tightness of Theorem 2.3 in the case $n \equiv 3 \mod 4$.

Proposition 2.5 *For $n \equiv 3 \mod 4$, the value of $\mathrm{ex}_2(n, K_4^-) = \lfloor \frac{n+1}{4} \rfloor$ if and only if there exists a skew Hadamard matrix of order $n+1$.*

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