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ORIGINAL PAPER

On Minimum Saturated Matrices

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Abstract Motivated both by the work of Anstee, Griggs, and Sali on forbidden submatrices and also by the extremal sat-function for graphs, we introduce sat-type problems for matrices. Let $\mathcal{F}$ be a family of k-row matrices. A matrix M is called $\mathcal{F}$*-admissible* if M contains no submatrix $F \in \mathcal{F}$ (as a row and column permutation of F). A matrix M without repeated columns is $\mathcal{F}$*-saturated* if M is $\mathcal{F}$-admissible but the addition of any column not present in M violates this property. In this paper we consider the function $\text{sat}(n, \mathcal{F})$ which is the *minimal* number of columns of an $\mathcal{F}$-saturated matrix with n rows. We establish the estimate $\text{sat}(n, \mathcal{F}) = O(n^{k-1})$ for any family $\mathcal{F}$ of k-row matrices and also compute the sat-function for a few small forbidden matrices.

Keywords Saturated matrices · Forbidden submatrices · Forbidden configurations

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1 Introduction

First, we must introduce some simple notation. Let the shortcut 'an $n \times m$-matrix' M mean a matrix with n rows (which we view as horizontal arrays) and m 'vertical' columns such that each entry is 0 or 1. For an $n \times m$-matrix M, its *order* $v(M) = n$ is the number of rows and its *size* $e(M) = m$ is the number of columns. We use expressions like 'an n-row matrix' and 'an n-row' to mean a matrix with n rows and a row containing n elements, respectively.

For an $n \times m$-matrix M and sets $A \subseteq [n]$ and $B \subseteq [m]$, $M(A, B)$ is the $|A| \times |B|$-submatrix of M formed by the rows indexed by A and the columns indexed by B. We use the following obvious shorthand: $M(A,) = M(A, [m])$, $M(A, i) = M(A, \{i\})$, etc. For example, the rows and the columns of M are denoted by $M(1,), \dots, M(n,)$ and $M(, 1), \dots, M(, m)$ respectively while individual entries – by $M(i, j)$, $i \in [n]$, $j \in [m]$.

We say that a matrix M is a *permutation* of another matrix N if M can be obtained from N by permuting its rows and then permuting its columns. We write $M \cong N$ in this case. A matrix F is a *submatrix* of a matrix M (denoted $F \subseteq M$) if we can obtain a matrix which is a permutation of F by deleting some set of rows and columns of M. In other words, $F \cong M(A, B)$ for some index sets A and B. The *transpose* of M is denoted by M^T (we use this notation mostly to denote vertical columns, for typographical reasons); $(a)^i$ is the (horizontal) sequence containing the element a i times. The $n \times (m_1 + m_2)$-matrix $[M_1, M_2]$ is obtained by concatenating an $n \times m_1$-matrix M_1 and an $n \times m_2$-matrix M_2. The *complement* $1 - M$ of a matrix M is obtained by interchanging ones and zeros in M. The *characteristic function* χ_Y of $Y \subseteq [n]$ is the n-column with ith entry being 1 if $i \in Y$ and 0 otherwise.

Many interesting and important properties of classes of matrices can be defined by listing forbidden submatrices (some authors use the term 'forbidden configurations'). More precisely, given a family $\mathcal{F}$ of matrices (referred to as *forbidden*), we say that a matrix M is $\mathcal{F}$*-admissible* (or $\mathcal{F}$*-free*) if M contains no $F \in \mathcal{F}$ as a submatrix. A *simple* matrix M (that is, a matrix without repeated columns) is called $\mathcal{F}$*-saturated* (or $\mathcal{F}$*-critical*) if M is $\mathcal{F}$-free but the addition of any column not present in M violates this property; this is denoted by $M \in \mathrm{SAT}(n, \mathcal{F})$, $n = v(M)$. Note that, although the definition requires that M is simple, we allow multiple columns in matrices belonging to $\mathcal{F}$.

One well-known extremal problem is to consider $\mathrm{forb}(n, \mathcal{F})$, the maximal size of a simple $\mathcal{F}$-free matrix with n rows or, equivalently, the maximal size of $M \in \mathrm{SAT}(n, \mathcal{F})$. Many different results on the topic have been obtained; we refer the reader to a recent survey by Anstee [2]. We just want to mention a remarkable fact that one of the first forb-type results, namely Formula (1) here, proved independently by Vapnik and Chervonenkis [22], Perles and Shelah [20], and Sauer [19], was motivated by such different topics as probability, logic, and a problem of Erdős on infinite set systems.

The forb-problem is reminiscent of the Turán function $\mathrm{ex}(n, \mathcal{F})$: given a family $\mathcal{F}$ of forbidden graphs, $\mathrm{ex}(n, \mathcal{F})$ is the maximal size of an $\mathcal{F}$-free graph on n vertices not containing any member of $\mathcal{F}$ as a subgraph (see e.g., surveys [15,17,21]). Erdős, Hajnal, and Moon [11] considered the 'dual' function $\mathrm{sat}(n, \mathcal{F})$, the *minimal* size of a maximal $\mathcal{F}$-free graph on n vertices. This is an active area of extremal graph theory; see the dynamic survey by Faudree, Faudree, and Schmitt [12].

Here we consider the 'dual' of the forb-problem for matrices. Namely, we are interested in the value of $\mathrm{sat}(n, \mathcal{F})$, the *minimal* size of an $\mathcal{F}$-saturated matrix with n rows:

$$\mathrm{sat}(n, \mathcal{F}) = \min\{e(M) : M \in \mathrm{SAT}(n, \mathcal{F})\}.$$

We decided to use the same notation as for its graph counterpart. This should not cause any confusion as this paper will deal with matrices. Obviously, $\mathrm{sat}(n, \mathcal{F}) \leq \mathrm{forb}(n, \mathcal{F})$. If $\mathcal{F} = \{F\}$ consists of a single forbidden matrix F then we write $\mathrm{SAT}(n, F) = \mathrm{SAT}(n, \{F\})$, and so on.

We denote by T_k^l the simple $k \times \binom{k}{l}$-matrix consisting of all k-columns with exactly l ones and by K_k – the $k \times 2^k$ matrix of all possible columns of order k. Naturally, $T_k^{\leq l}$ denotes the $k \times f(k,l)$-matrix consisting of all distinct columns with at most l ones, and so on, where we use the shortcut

$$f(k,l) = \binom{k}{0} + \binom{k}{1} + \cdots + \binom{k}{l}.$$

Vapnik and Chervonenkis [22], Perles and Shelah [20], and Sauer [19] showed independently that

$$\mathrm{forb}(n, K_k) = f(n, k-1). \tag{1}$$

Formula (1) turns out to play a significant role in our study.

This paper is organized as follows. In Sect. 2 we give some general results about the sat-function, the principal one being Theorem 2 which states that $\mathrm{sat}(n, \mathcal{F}) = O(n^{k-1})$ holds for any family $\mathcal{F}$ of k-row matrices. Turning to specific matrices, in Sect. 3 we compute $\mathrm{sat}(n, K_k)$ for $k = 2$ and $k = 3$. By Theorem 2, $\mathrm{sat}(n, K_2)$ can grow at most linearly, and indeed it is linear in n. Surprisingly, though, $\mathrm{sat}(n, K_3)$ is constant for $n \geq 4$. Finally, in Sect. 4, we examine a selection of small matrices F to see how $\mathrm{sat}(n, F)$ behaves. In particular, we find some F for which the function grows and other F for which it is constant (or bounded): it would be interesting to determine a criterion for when $\mathrm{sat}(n, F)$ is bounded, but we cannot guess one from the present data.

2 General Results

Here we present some results dealing with $\mathrm{sat}(n, \mathcal{F})$ for a general family $\mathcal{F}$.

The following simple observation can be useful in tackling these problems. Let M' be obtained from $M \in \mathrm{SAT}(n, \mathcal{F})$ by *duplicating* the nth row of M, that is, we let $M'([n],) = M$ and $M'(n+1,) = M(n,)$. Suppose that M' is $\mathcal{F}$-admissible. Complete M', by adding columns in an arbitrary way, to an $\mathcal{F}$-saturated matrix. Let C be any added $(n+1)$-column. As both $M'([n],)$ and $M'([n-1] \cup \{n+1\},)$ are equal to

$M \in \mathrm{SAT}(n, \mathcal{F})$, we conclude that both $C([n])$ and $C([n-1] \cup \{n+1\})$ must be columns of M. As C is not an M'-column, $C = (C', b, 1-b)$ where $b \in \{0, 1\}$ and C' is some $(n-1)$-column such that both $(C', 0)$ and $(C', 1)$ are columns of M. This implies that $\mathrm{sat}(n+1, \mathcal{F}) \le e(M) + 2d$, where d is the number of pairs of equal columns in M after we delete the nth row. In particular, the following theorem follows.

Theorem 1 *Suppose that F is a matrix with no two equal rows. Then either* $\mathrm{sat}(n, F)$ *is constant for large n, or* $\mathrm{sat}(n, F) \ge n+1$ *for every n.*

Proof If some $M \in \mathrm{SAT}(n, F)$ has at most n columns, then a well-known theorem of Bondy [7] (see, e.g., Theorem 2.1 in [6]) implies that there is $i \in [n]$ such that the removal of the ith row does not create two equal columns. Since F has no two equal rows, the duplication of any row cannot create a forbidden submatrix, so $\mathrm{sat}(n+1, F) \ge \mathrm{sat}(n, F)$. However, by the remark made just prior to the theorem, the duplication of the ith row gives an $(n+1)$-row F-saturated matrix, implying $\mathrm{sat}(n+1, F) \le \mathrm{sat}(n, F)$, as required. □

Suppose that $\mathcal{F}$ consists of k-row matrices. Is there any good general upper bound on $\mathrm{forb}(n, \mathcal{F})$ or $\mathrm{sat}(n, \mathcal{F})$? There were different papers dealing with general upper bounds on $\mathrm{forb}(n, \mathcal{F})$, for example, by Anstee and Füredi [3], by Frankl, Füredi and Pach [14] and by Anstee [1], until the conjecture of Anstee and Füredi [3] that $\mathrm{forb}(n, \mathcal{F}) = O(n^k)$ for any fixed $\mathcal{F}$ was elegantly proved by Füredi (see [4] for a proof).

On the other hand, we can show that $\mathrm{sat}(n, \mathcal{F}) = O(n^{k-1})$ for any family $\mathcal{F}$ of k-row matrices (including infinite families). Note that the exponent $k-1$ cannot be decreased in general since, for example, $\mathrm{sat}(n, T_k^k) = f(n, k-1)$.

Theorem 2 *For any family $\mathcal{F}$ of k-row matrices,* $\mathrm{sat}(n, \mathcal{F}) = O(n^{k-1})$.

Proof We may assume that K_k is $\mathcal{F}$-admissible (i.e., every matrix of $\mathcal{F}$ contains a pair of equal columns) for otherwise we are home by (1) as then $\mathrm{sat}(n, \mathcal{F}) \le \mathrm{forb}(n, K_k) = O(n^{k-1})$.

Let us define some parameters l, d, and m that depend on $\mathcal{F}$. Let $l = l(\mathcal{F}) \in [0, k]$ be the smallest number such that there exists s for which $[sT_k^{\le l}, T_k^{>l}]$ is not $\mathcal{F}$-admissible (clearly, such l exists: if we set $l = k$, then $sT_k^{\le l} = sK_k$ contains any given k-row submatrix for all large s). Let $d = d(\mathcal{F})$ be the maximal integer such that $[sT_k^{<l}, dT_k^l, T_k^{>l}]$ is $\mathcal{F}$-admissible for every s. Note that $d \ge 1$ as $[sT_k^{<l}, T_k^l, T_k^{>l}] = [sT_k^{<l}, T_k^{\ge l}]$ cannot contain a forbidden submatrix by the choice of l. Choose the minimal $m = m(\mathcal{F}) \ge 0$ such that $[mT_k^{<l}, (d+1)T_k^l, T_k^{>l}]$ is not $\mathcal{F}$-admissible. The subsequent argument will be valid provided n is large enough, which we shall tacitly assume.

We consider the two possibilities $l(\mathcal{F}) < k$ and $l(\mathcal{F}) = k$ separately. Suppose first that $l(\mathcal{F}) < k$. Consider the following set system:

$$H = \bigcup_{j \in [d-1]} \left\{ Y \in \binom{[n]}{l+1} : \sum_{y \in Y} y \equiv j \pmod{n} \right\}.$$

Here $\binom{X}{i} = \{Y \subseteq X : |Y| = i\}$ denotes the set of all subsets of X of size i.

Note that any $A \in \binom{[n]}{l}$ is contained in at most $d-1$ members of H, as there are at most $d-1$ possibilities to choose $i \in [n] \setminus A$ so that $A \cup \{i\} \in H$: namely, $i \equiv j - \sum_{a \in A} a \pmod n$ for $j \in [d-1]$.

On the other hand, the collection H', of all l-subsets of $[n]$ contained in fewer than $d-1$ members of H, has size at most $2(d-1)\binom{n}{l-1}$. Indeed, if $A \in H'$ then, using the previous observation, it must be that for some $j \in [d-1]$ and $x \in A$ we have $2x \equiv j - \sum_{a \in A \setminus \{x\}} a \pmod n$: hence, once $A \setminus \{x\}$ and j have been chosen, there are at most 2 choices for x.

Call $X \in \binom{[n]}{k}$ *bad* if, for some $A \in \binom{X}{l}$,

$$|\{Y \in H : Y \cap X = A\}| \le d-2.$$

To obtain a bad k-set X, we either complete some $A \in H'$ to any k-set, or we take any l-set A and let X contain some member of H that contains A. Therefore, the number of bad sets is at most

$$2(d-1)\binom{n}{l-1}\binom{n}{k-l} + \binom{n}{l}(d-1)\binom{n}{k-l-1} = O(n^{k-1}).$$

Let $M' = [N, T_n^l]$, where N is the $n \times |H|$ incidence matrix of H. Then we have that

$$M'(X,) \subseteq \left[e(M')T_k^{<l}, dT_k^l, T_k^{l+1}\right], \quad \text{for any } X \in \binom{[n]}{k}.$$

Hence, M' cannot contain a forbidden submatrix by the definition of d. Now complete it to arbitrary $M = [M', M''] \in \mathrm{SAT}(n, \mathcal{F})$ by adding new columns as long as no forbidden submatrix is created.

Suppose that $e(M'') \neq O(n^{k-1})$. Then, by (1), $K_k \cong M''(X, Y)$ for some X, Y. Now, remove the columns corresponding to Y from M'' and repeat the procedure as long as possible to obtain more than $O(n^{k-1})$ column-disjoint copies of K_k in M''. No $X \in \binom{[n]}{k}$ can appear more than d times: otherwise (because $T_n^l(X,) \supseteq mT_k^{<l}$ for all large n) we have that $M(X,) = [M', M''](X,) \supseteq [mT_k^{<l}, (d+1)K_k]$ is not $\mathcal{F}$-admissible. Since we have $O(n^{k-1})$ bad k-sets of rows and, by above, each has at most d column-disjoint copies of K_k, we have that $K_k \subseteq M''(X,)$ for at least one *good* (i.e., not bad) $X \in \binom{[n]}{k}$. But then $N(X,) \supseteq (d-1)T_k^l$. Moreover, since $T_n^l(X,) \supseteq [mT_k^{<l}, T_k^l]$ for all large n, we obtain

$$\begin{aligned} M(X,) = [N, T_n^l, M''](X,) &\supseteq [(d-1)T_k^l, mT_k^{<l}, T_k^l, K_k] \\ &= [(m+1)T_k^{<l}, (d+1)T_k^l, T_k^{>l}]. \end{aligned}$$

Thus, $M(X,)$ contains a forbidden matrix. This contradiction proves the required bound for $l < k$.

Consider now the other possibility, that $l = l(\mathcal{F})$ equals k. The above argument does not work in this case because the size of $M' \supseteq T_n^l$ is too large. Let $\mathcal{F}^*$ consist of those k-row matrices F such that $[dT_k^k, F]$ is not $\mathcal{F}$-admissible, where $d = d(\mathcal{F})$.

Note that $[sT_k^{<k}, T_k^k] \in \mathcal{F}^*$ for all large s by the definition of d. Thus $l(\mathcal{F}^*) < k$ and by the above argument we can find $L \in \mathrm{SAT}(n-d, \mathcal{F}^*)$ with $O(n^{k-1})$ columns. Define

$$M' = \begin{bmatrix} dT_{n-d}^{n-d} & L \\ T_d^1 & e(L)T_d^0 \end{bmatrix},$$

that is, M' is obtained from $[dT_{n-d}^{n-d}, L]$ by adding d extra rows that encode the sets $\{i\}$, $i \in [d]$. Note that M' does not have multiple columns even if T_{n-d}^{n-d} is a column of L because $d \geq 1$.

Take arbitrary $X \in \binom{[n]}{k}$. If $X \subseteq [n-d]$, then $M'(X,) = [dT_k^k, L(X,)]$ is $\mathcal{F}$-admissible because L is $\mathcal{F}^*$-admissible; otherwise $M'(X,) \subseteq [e(M')T_k^{<k}, T_k^k]$ is $\mathcal{F}$-admissible because $l(\mathcal{F}) = k$. Thus M' is $\mathcal{F}$-free.

Complete M' to an arbitrary $M \in \mathrm{SAT}(n, \mathcal{F})$. Let C be any added column. Since

$$[M', C]([n-d],) = \left[dT_{n-d}^{n-d}, L, C([n-d])\right]$$

is $\mathcal{F}$-free, we have that $[L, C([n-d])]$ is $\mathcal{F}^*$-free. By the $\mathcal{F}^*$-saturation of L, we have that $C([n-d])$ is a column of L. Hence

$$\mathrm{sat}(n, \mathcal{F}) \leq e(M) \leq 2^d\, e(L) + d = O(n^{k-1}),$$

proving the theorem. □

Remark 1 Theorem 2 is the matrix analog of the main result in [18] that $\mathrm{sat}(n, \mathcal{F}) = O(n^{k-1})$ for any finite family $\mathcal{F}$ of k-graphs.

3 Forbidding Complete Matrices

Let us investigate the value of $\mathrm{sat}(n, K_k)$ (recall that K_k is the $k \times 2^k$-matrix consisting of all distinct k-columns). We are able to settle the cases $k = 2$ and $k = 3$.

We will use the following trivial lemma a couple of times.

Lemma 1 *Each row of any $M \in \mathrm{SAT}(n, K_k)$, $n \geq k$, contains at least l ones and at least l zeros, where $l = 2^{k-1} - 1$.*

Proof Suppose on the contrary that the first row $M(1,)$ has m_0 zeros followed by m_1 ones with $m_0 \geq m_1$ and $l > m_1$.

For $i \in [m_0]$, let C_i equal the ith column of M with the first entry 0 replaced by 1. Then the addition of C_i to M cannot create a new copy of K_k, because the first row of M' contains too few 1's, while $C_i([2, n])$ is already a column of $M([2, n],)$, which does not contain K_k. Therefore, C_i must be a column of M. Since $i \in [m_0]$ was arbitrary, we have $m_0 = m_1$.

But then M has at most $2^k - 2$ columns, which is a contradiction. □

Theorem 3 *For* $n \geq 1$*, we have* $\mathrm{sat}(n, K_2) = n + 1$.

Proof The upper bound is given by $T_n^{\leq 1} \in \mathrm{SAT}(n, K_2)$.

Suppose that the statement is not true, that is, there exists a K_2-saturated matrix with its size not exceeding its order. By Theorem 1, $\mathrm{sat}(n, K_2)$ is eventually constant so we can find an $n \times m$-matrix $M \in \mathrm{SAT}(n, K_2)$ having two equal rows for some $n \in \mathbb{N}$.

As we are free to complement and permute rows, we may assume that, for some $i \geq 2$, $M(1,) = \cdots = M(i,)$ while $M(j,) \neq M(1,)$ and $M(j,) \neq 1 - M(1,)$ for any $j \in [i + 1, n]$. Note that $i < n$ as we do not allow multiple columns in M (and $m \geq e(K_2) - 1 = 3$).

Let $j \in [i + 1, n]$. By Lemma 1, the jth row $M(j,)$ contains both 0's and 1's. By the definition of i, $M(j,)$ is not equal to $M(1,)$ nor to $1 - M(1,)$. It easily follows that there are $f_j, g_j \in [m]$ with $M(1, f_j) = M(1, g_j)$ and $M(j, f_j) \neq M(j, g_j)$. Again by Lemma 1, we can furthermore find $h_j \in [m]$ with $M(1, h_j) = 1 - M(1, f_j)$. Let $b_j = M(j, h_j)$. By exchanging f_j and g_j if necessary, we can assume that $M(j, g_j) = b_j$.

Now, as $M \in \mathrm{SAT}(n, K_2)$, the addition of the column

$$C = (1, (0)^{i-1}, b_{i+1}, \ldots, b_n)^T$$

(which is not in M because $C(1) \neq C(2)$) must create a new K_2-submatrix, say in the xth and yth rows for some $1 \leq x < y \leq n$. Clearly, $\{x, y\} \nsubseteq [i]$ because each column of $M([i],)$ is either $((0)^i)^T$ or $((1)^i)^T$. Also, it is impossible that $x \in [i]$ and $y \in [i + 1, n]$ because then, for some $a_1, a_2 \in [m]$, $M(y, a_1) = M(y, a_2) = 1 - C(y) = 1 - b_y$, $M(x, a_1) = 1 - M(x, a_2)$ and we can see that K_2 is isomorphic to $M(\{x, y\}, \{a_1, a_2, g_y, h_y\})$, which contradicts $K_2 \nsubseteq M(\{x, y\},)$. So we have to assume that $i < x < y \leq n$.

As $K_2 \nsubseteq M(\{x, y\},)$, no column of $M(\{x, y\},)$ can equal $C(\{x, y\}) = (b_x, b_y)^T$. In particular, since $M(x, g_x) = M(x, h_x) = b_x$ and similarly for y, we must have $\{g_x, h_x\} \cap \{g_y, h_y\} = \emptyset$, and moreover $M(y, g_x) = M(y, h_x) = 1 - b_y$. But then

$$K_2 \cong M(\{1, y\}, \{g_x, h_x, g_y, h_y\}),$$

which is a contradiction proving our theorem. □

Note that $\mathrm{forb}(n, K_2) = n + 1$ for $n \geq 1$; the upper bound follows, for example, from Formula (1) with $k = 2$. Thus Theorem 3 yields that $\mathrm{sat}(n, K_2) = \mathrm{forb}(n, K_2)$ which, in our opinion, is rather surprising. A greater surprise is yet to come as we are going to show now that $\mathrm{sat}(n, K_3)$ is constant for $n \geq 4$.

Theorem 4 *For* K_3 *the following holds:*

$$\mathrm{sat}(n, K_3) = \begin{cases} 7, & \text{if } n = 3, \\ 10, & \text{if } n \geq 4. \end{cases}$$

Proof The claim is trivial for $n = 3$, so assume $n \geq 4$. A computer search [10] revealed that

$$\text{sat}(4, K_3) = \text{sat}(5, K_3) = \text{sat}(6, K_3) = \text{sat}(7, K_3) = 10,$$

which suggested that $\text{sat}(n, K_3)$ is constant. An example of a K_3-saturated 6×10-matrix is the following:

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

It is possible (but very boring) to check by hand that M is indeed K_3-saturated as is, in fact, any $n \times 10$-matrix M' obtained from M by duplicating any row, *cf.* Theorem 1 (the symmetries of M shorten the verification). A K_3-saturated 5×10-matrix can be obtained from M by deleting one row (any). For $n = 4$, we have to provide a special example:

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

So $\text{sat}(n, K_3) \leq 10$ for each $n \geq 4$ and, to prove the theorem, we have to show that no K_3-saturated matrix M with at most 9 columns and at least 4 rows can exist. Let us assume the contrary.

Claim 1 *Any row of* $M \in \text{SAT}(n, K_3)$ *necessarily contains at least four 0's and at least four 1's, for* $n \geq 4$.

Proof of Claim Suppose, contrary to the claim, that the first row $M(1,)$ contains only three 0's, say in the first three columns (by Lemma 1 we must have at least three 0's).

If we replace the ith of these 0's by 1, $i \in [3]$, then the obtained column C_i, if added to M, does not create any K_3-submatrix. Indeed, the first row of $[M, C_i]$ contains at most three 0's, while $C_i([2, n])$ is a column of $M([2, n],) \not\supseteq K_3$. As M is K_3-saturated, C_1, C_2 and C_3 are columns of M. These columns differ only in the first entry from $M(, 1)$, $M(, 2)$ and $M(, 3)$ respectively. Therefore, for each $A \in \binom{[2,n]}{3}$, the matrix $M(A,)$ can contain at most $e(M) - 3 \leq 6$ distinct columns. But then any column C which is not a column of M and has top entry 1 (C exists as $n \geq 4$) can be added to M without creating a K_3 submatrix, because the first row of $[M, C]$ contains at most three 0's. This contradiction proves Claim 1. □

Therefore, $e(M)$ is either 8 or 9. As we are free to complement the rows, we may assume that each row of M contains exactly four 1's. Call $A \in \binom{[n]}{3}$ (and also $M(A,)$) *nearly complete* if $M(A,)$ has 7 distinct columns.

Claim 2 *Any nearly complete $M(A,)$ contains $(0,0,0)^T$ as a column.*

Proof of Claim Indeed, otherwise $M(A,) \supseteq T_3^{\geq 1}$ which already contains four 1's in each row; this implies that the (one or two) remaining columns must contain zeros only. Hence $M(A,) \supseteq K_3$, which is a contradiction. □

Claim 3 *Every nearly complete $M(A,)$ contains T_3^1 as a submatrix.*

Proof of Claim Indeed, if $(0,0,1)^T$ is the missing column of $M(A,)$, then some 7 columns contain a copy of $K_3 \setminus (0,0,1)^T$. By counting 1's in the rows we deduce that the remaining column(s) of $M(A,)$ must have exactly one non-zero entry, and moreover one of these columns equals $(0,0,1)^T$, which is a contradiction. □

By the K_3-saturation of M there exists some nearly complete $M(A,)$; choose one such. Assume without loss of generality that $A = [3]$ and that the first 7 columns of $M([3],)$ are distinct. We know that the 3-column missing from $M([3],[7])$ has at least two 1's.

If $(1,1,1)^T$ is missing, then $M([3],[7])$ contains exactly three ones in each row, so the remaining column(s) of M must contain an extra 1 in each row. As $(1,1,1)^T$ is the missing column, we conclude that $e(M) = 9$ and the 8th and 9th columns of $M([3],)$ are, up to a row permutation, $(0,0,1)^T$ and $(1,1,0)^T$. This implies that $M([3],)$ contains the column $(0,0,0)^T$ only once. Thus at least one of the columns $C_0 = ((0)^n)^T$ and $C_1 = ((0)^{n-1},1)^T$ is not in M and its addition creates a copy of K_3, say on the rows indexed by $B \in \binom{[n]}{3}$. The submatrix $M(B,)$ is nearly complete and, by Claims 2 and 3, contains $T_3^{\leq 1}$. But both $C_0(B)$ and $C_1(B)$ are columns of $T_3^{\leq 1} \subseteq M(B,)$, which is a contradiction.

Similarly, if $(1,1,0)^T$ is missing, then one can deduce that $e(M) = 9$ and, up to a row permutation, $M([3],\{8,9\})$ consists of the columns $(1,0,0)^T$ and $(0,1,0)^T$. Again, the column $(0,0,0)^T$ appears only once in $M([3],)$, which leads to a contradiction as above, completing the proof of the theorem. □

We do not have any non-trivial results concerning $K_k, k \geq 4$, except that a computer search [10] showed that $\mathrm{sat}(5, K_4) = 22$ and $\mathrm{sat}(6, K_4) \leq 24$ (we do not know if a K_4-saturated 6×24-matrix discovered by a partial search is minimum).

Problem 1 For which $k \geq 4$, is $\mathrm{sat}(n, K_k) = O(1)$?

4 Forbidding Small Matrices

In this final section we try to gain further insight into the sat-function by computing $\mathrm{sat}(n, F)$ for some forbidden matrices with up to three rows.

4.1 Forbidding 1-Row Matrices

For any given 1-row matrix F, we can determine $\mathrm{sat}(n, F)$ for all but finitely many values of n. The answer is unpleasantly intricate.

Proposition 1 *Let* $F = ((0)^m, (1)^l) = [mT_1^0, lT_1^1]$ *with* $l \geq m$. *Then, for* $n \geq \max(l-1, 1)$,

$$\mathrm{sat}(n, F) = \begin{cases} l, & \text{if } m = 0 \text{ and } l \leq 2 \text{ or if } m = 1 \text{ and } l \geq 1 \text{ is a power of } 2, \\ l+1, & \text{if } m = 0 \text{ and } l \geq 3 \text{ or if } m = 1 \text{ and } l \text{ is not a power of } 2, \\ l+m-1, & \text{if } m \geq 2 \text{ and } l \geq 2. \end{cases}$$

Proof Assume that $l \geq 3$, as the case $l \leq 2$ is trivial.

For $m \in \{0, 1\}$ an example of $M \in \mathrm{SAT}(n, F)$ with $e(M) = l + 1$ can be built by taking T_n^0, T_n^n, $\chi_{[l-2]}$, and $\chi_{[n]\setminus\{i\}}$ for $i \in [l-2]$ as the columns. If $m = 1$ and $l = 2^k$, one can do slightly better by adding $n - k$ copies of the row $((1)^l)$ to K_k.

Let us prove the lower bound for $m \in \{0, 1\}$. Suppose that some F-saturated matrix M has $n \geq l-1$ rows and $c \leq l$ columns. First, let $m = 0$. As $c < 2^n$ and M contains the all-0 column, we have $c = l$ and some row $M(i,)$ contains exactly $l-1$ ones. As we are not allowed multiple columns in M, some other row, say $M(j,)$, has at most $l-2$ ones. Then $\chi_{\{j\}}$ is not a column of M but its addition does not create l ones in a row, a contradiction. Let $m = 1$. Trivially, $e(M) \geq e(F) - 1 = l$. It remains to show that l is a power of 2 if $e(M) = l$. Let C be the column whose ith entry is 1 if and only if $M(i,) = (1)^l$. Then the addition of the column C cannot create an F-submatrix, and so C is already a column of M. Let $C = M(, 1) = ((0)^i, (1)^{n-i})^T$. The last $n-i$ rows of M consist of 1's only. Since $l \geq 3$ and M has no multiple columns, we have that $i \geq 2$ and that $M([i], [2, l])$ must contain at least one 0, say $M(i, l) = 0$. Since the addition of $\chi_{[i,n]}$ cannot create F, it is already a column of M. Thus each row of $M([i],)$ has at least two 0's, and to avoid a contradiction we must have $M([i],) \cong K_i$ and $l = 2^i$. This completes the case when $m \leq 1$.

For $m \geq 2$, let M consist of T_n^n plus $\chi_{\{i\}}$, $i \in [m-2]$, plus $\chi_{[n]\setminus\{i\}}$, $i \in [l-1]$ and $\chi_{[m-1,l-1]}$. Clearly, each row of M contains l 1's and $m-1$ 0's, so the addition of any new column (which must contain at least one 0) creates an F-submatrix and the upper bound follows. The lower bound is trivial. □

Remark 2 The case when $n \leq l-2$ in Proposition 1 seems messy so we do not investigate it here.

4.2 Forbidding 2-Row Matrices

Now let us consider some particular 2-row matrices.

Let $F = lT_2^2$, that is, F consists of the column $(1, 1)^T$ taken l times. Trivially, for $l = 1$ or 2, $\mathrm{sat}(n, lT_2^2) = n + l$, with $T_n^{\leq 1}$ and $[T_n^{\leq 1}, T_n^n]$ being the only extremal matrices. For $l \geq 3$, we can only show the following lower bound. It is almost sharp for $l = 3$, when we can build a $3T_2^2$-saturated $n \times (2n+2)$-matrix by taking $T_n^{\leq 1}$, $\chi_{[n-1]}$, $\chi_{[n]}$, plus $\chi_{\{i,n\}}$ for $i \in [n-1]$.

Lemma 2 *For* $l \geq 3$ *and* $n \geq 3$, $\mathrm{sat}(n, lT_2^2) \geq 2n + 1$.

Proof Let $M = [T_n^{\leq 1}, M']$ be lK_2^2-saturated. Note that M' must have the property that every column χ_A, with $A \in \binom{[n]}{2}$, either belongs already to M', or its addition creates an F-submatrix; in the latter case, exactly $l-1$ columns of M' have ones in both positions of A. Therefore, by adding to M' some columns of T_n^2 (with possibly some columns being added more than once), we can obtain a new matrix M'' such that, for every $A \in \binom{[n]}{2}$, $M''(A,)$ contains the column $(1,1)^T$ exactly $l-1$ times. If we let the set X_i be encoded by the ith row of M'' as its characteristic vector, we have that $|X_i \cap X_j| = l-1$ for every $1 \le i < j \le n$. The result of Bose [8] (see [16, Theorem 14.6]), which can be viewed as an extension of the famous Fisher inequality [13], asserts that, either the rows of M'' are linearly independent over the reals, or M'' has two equal rows, say $X_i = X_j$. The second case is impossible here, because then $|X_i| = l-1$ and each other X_h contains X_i as a subset; this in turn implies that the column $((1)^n)^T$ appears at least $l-1 \ge 2$ times in M'' and (since $n \ge 3$) the same number of times in M', a contradiction. Thus the rank of M'' over the reals is n. Note that every column $C \in T_n^2$ added to M' during the construction of M'' was already present in M' (otherwise C contradicts the assumption that M is lT_2^2-saturated). Thus the matrices M' and M'' have the same rank over the reals. We conclude that M' has at least n columns and the lemma follows. □

Let us show that Lemma 2 is sharp for $l = 3$ and some n. Suppose there exists a *symmetric* $(n, k, 2)$*-design* (meaning we have n k-sets $X_1, \ldots, X_n \in \binom{[n]}{k}$ such that every pair $\{i, j\} \in \binom{[n]}{2}$ is covered by exactly two X_i's). Let M be the $n \times n$-matrix whose rows are the characteristic vectors of the sets X_i. Then $[T_n^{\le 1}, M]$ is a $3T_2^2$-saturated $n \times (2n+1)$-matrix. For $n = 4$, we can take all 3-subsets of $[n]$. For $n = 7$, we can take the family $\{[7] \setminus Y_i : i \in [7]\}$, where $Y_1, \ldots, Y_7 \in \binom{[7]}{3}$ form the Fano plane. Constructions of such designs for $n = 16$, 37, 56, and 79 can be found in [9, Table 6.47].

Of course, the non-existence of a symmetric $(n, k, 2)$-design does not directly imply anything about $\mathrm{sat}(n, 3T_2^2)$, since a minimum $3T_2^2$-saturated matrix $[T_n^{\le 1}, M]$ need not have the same number of ones in the rows of M.

Lemma 2 is not always optimal for $l = 3$. One trivial example is $n = 3$. Another one is $n = 5$.

Lemma 3 $\mathrm{sat}\left(5, 3T_2^2\right) = 12$.

Proof Suppose, on the contrary, that we have a $3T_2^2$-saturated $5 \times (s+6)$-matrix $M = [N, T_5^{\le 1}]$ with $s \le 5$. Let $X_1, \ldots, X_5$ be the subsets of $[s]$ encoded by the rows of N.

If, for example, $X_1 = [s]$, then every X_i with $i \ge 2$ has at most two elements. Let $C_1 = (0,1,1,0,0)^T$, $C_2 = (0,0,0,1,1)^T$ and $C_3 = (0,0,1,1,0)^T$. None of these columns is in M so the addition of any one of them creates a copy $3T_2^2$. So we may assume that $M(\{2,3\}, \{a,b\}) = M(\{4,5\}, \{c,d\}) = M(\{3,4\}, \{e,f\}) = 2T_2^2$. If $\{a,b\} = \{c,d\}$ then $M(, a)$ and $M(, b)$ are two equal columns with all 1's, a contradiction. Hence $\{a,b\} \ne \{c,d\}$, and so at least one of $\{e,f\} \ne \{a,b\}$ or $\{e,f\} \ne \{c,d\}$ holds: we may assume the former. But then $M(\{1,3\},)$ contains $3T_2^2$, a contradiction.

Thus we can assume that each X_i with $i \in [5]$ has at most $s-1$ elements. If $X_1 \subseteq \{1,2\}$, then by considering columns that begin with 1 and have one other entry 1, we conclude that $X_1 = \{1,2\}$ and that every X_i contains X_1 as a subset. Thus $M(, \{1,2\}) = 2T_5^5$, that is, M has two equal columns, a contradiction.

So we can assume that each $|X_i| \geq 3$, which also implies that $s = 5$. If $X_1 = [4]$, then for each $i \in [2, 5]$ we have $5 \in X_i$ (because $|X_i| \geq 3$ and M is $3T_2^2$-free). Each two of the sets $X_2, \ldots, X_5$ have to intersect in exactly two elements, which is impossible.

Thus each $|X_i| = 3$. A simple case analysis gives a contradiction in this case as well. □

Problem 2 Determine $\text{sat}(n, 3T_2^2)$ for every n.

Remark 3 It is interesting to note that if we let $F = [lT_2^2, (0, 1)^T]$ then $\text{sat}(n, F)$-function is bounded. Indeed, complete $M' = [\chi_{[n]\setminus\{i\}}]_{i\in[l]}$ to an arbitrary F-saturated matrix M. Clearly, in any added column all entries after the lth position are either 0's or 1's; hence $\text{sat}(n, F) \leq 2 \cdot 2^l$.

It is easy to compute $\text{sat}(n, T_2^1)$ by observing that the n-row matrix M_Y whose columns encode $Y \subseteq 2^{[n]}$ is T_2^1-free if and only if Y is a chain—that is, for any two members of Y, one is a subset of the other. Thus M_Y is T_2^1-saturated if and only if Y is a maximal chain without repeated entries. As all maximal chains in $2^{[n]}$ have size $n + 1$, we conclude that

$$\text{sat}(n, T_2^1) = \text{forb}(n, T_2^1) = n + 1, \quad n \geq 2.$$

Theorem 5 *Let* $F = [T_2^0, T_2^2] = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. *Then* $\text{sat}(n, F) = 3$, $n \geq 2$.

Proof For $n \geq 3$, the matrix M consisting of the columns $(0, 1, (1)^{n-2})^T$, $(1, 0, (1)^{n-2})^T$ and $(0, 0, (1)^{n-2})^T$ can be easily verified to be F-saturated and the upper bound follows.

Since $n = 2$ is trivial, let $n \geq 3$. Any 2-column F-free matrix M is, without loss of generality, the following: we have n_{00} rows $(0, 0)$, followed by n_{11} rows $(1, 1)$, n_{10} rows $(1, 0)$ and n_{01} rows $(0, 1)$, where $n_{10} \leq 1$ and $n_{01} \leq 1$. Since (by taking complements if necessary) we may assume $n_{00} \leq n_{11}$, we have $n_{11} \geq 1$ because $n \geq 3$. But then the addition of a new column $((0)^{n_{00}+1}, 1, 1, \ldots)^T$ does not create an F-submatrix. □

Theorem 6 *Let* $F = T_2^{\geq 1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. *Then*

$$\text{sat}(n, F) = \text{forb}(n, F) = n + 1, \quad n \geq 2.$$

Proof Clearly, $\text{forb}(n, F) \leq \text{forb}(n, K_2) = n + 1$.

Suppose the theorem is false and that $\text{sat}(n, F) \leq n$ for some n. Since the rows of F are distinct, Theorem 1 shows that $\text{sat}(n, F)$ is bounded.

It follows that, if n is large enough, then $M \in \text{SAT}(n, F)$ has two equal rows, for example, $M(1,) = M(2,) = ((1)^l, (0)^m)$. By considering the column $(1, 0, \ldots, 0)^T$ that is not in M, we conclude that $l, m \geq 1$. Let $X = [l]$ and $Y = [l + 1, l + m]$. Define

$$A_i = \{j \in [l + m] : M(i, j) = 1\}, \quad i \in [n].$$

For example, $A_1 = A_2 = X$. As M is F-free, for every $i, j \in [n]$, the sets A_i and A_j are either disjoint or one is a subset of the other. For $i \in [3, n]$, let $b_i = 1$ if A_i strictly contains X or Y and let $b_i = 0$ otherwise (that is, when A_i is contained in X or Y). Let $b_1 = 1$ and $b_2 = 0$.

Clearly, $C = (b_1, \ldots, b_n)^T$ is not a column of M so its addition creates a forbidden submatrix, say $F \subseteq [M, C](\{i, j\},)$. Of course, $b_i = b_j = 0$ is impossible because $(0, 0)^T \not\subseteq F$. If $b_i = b_j = 1$ then necessarily $A_i \cap A_j \neq \emptyset$, and $M(\{i, j\},) \supseteq (1, 1)^T$ contains F, a contradiction. Finally, if $b_i \neq b_j$, e.g., $b_i = 0$, $b_j = 1$ and $i < j$, then $A_i \supseteq A_j$ (as $(0, 1)^T$ cannot be a column of $M(\{i, j\},))$, which implies $A_i = A_j$; but then we do not have a copy of F as $(1, 0)^T$ is missing. This contradiction completes the proof. □

Remark 4 It is trivial that

$$\text{sat}(n, [(0,1)^T, (1,1)^T]) = \text{sat}(n, [(0,0)^T, (0,1)^T, (1,1)^T]) = 2.$$

We have thus determined the sat-function for every simple 2-row matrix.

4.3 Forbidding 3-Row Matrices

Here we consider some particular 3-row matrices. First we solve completely the case when $F = [T_3^0, T_3^3]$.

Theorem 7 *Let* $F = [T_3^0, T_3^3] = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$. *Then*

$$\text{sat}(n, F) = \begin{cases} 7, & \text{if } n = 3 \text{ or } n \geq 6, \\ 10, & \text{if } n = 4 \text{ or } 5. \end{cases}$$

Proof For the upper bound we define the following family of matrices:

$$M_4 = \begin{bmatrix} 1&0&1&0&1&0&1&1&0&0 \\ 0&1&1&0&0&1&1&0&1&0 \\ 0&0&0&1&1&1&1&0&0&1 \\ 0&0&0&0&0&0&0&1&1&1 \end{bmatrix},$$

$$M_5 = \begin{bmatrix} 1&1&0&1&1&0&1&0&1&0 \\ 1&0&1&1&0&1&0&1&1&0 \\ 0&1&1&1&0&0&1&1&0&1 \\ 0&0&0&0&1&1&1&1&0&0 \\ 0&0&0&0&0&0&0&0&1&1 \end{bmatrix},$$

$$M_6 = \begin{bmatrix} 1&0&1&0&0&1&0 \\ 1&0&0&1&1&0&0 \\ 0&1&1&0&1&0&0 \\ 0&1&0&1&0&1&0 \\ 0&0&1&1&0&0&1 \\ 0&0&0&0&1&1&1 \end{bmatrix}.$$

For any $n \geq 7$ define the $(n \times 7)$-matrix M_n by $M_n([6],) = M_6$ and $M_n(i,) = \begin{bmatrix} 0\,0\,0\,0\,0\,0\,0 \end{bmatrix}$ for every $7 \leq i \leq n$. A computer search [10] showed that M_n is a minimum F-saturated matrix for $3 \leq n \leq 10$. This implies that each M_n with $n \geq 11$ is F-saturated. It remains to show that

$$\mathrm{sat}(n, F) \geq 7$$

for $n \geq 11$. In order to see this, we show the following result first.

Claim *If M is an F-saturated $n \times m$-matrix with $n \geq 11$ and $m \leq 6$ then M contains a row with all zero entries or with all one entries.*

Proof of Claim Suppose, on the contrary, that we have a counterexample M. We may assume that the first 6 entries of the first column of M are equal to 0. Consider a matrix $A = M([6], \{2, \ldots, m\})$. Note that every column of A contains at most two entries equal to 1, otherwise $M([6],) \supseteq F$. Hence, the number of 1's in A is at most $2(m-1)$. By our assumption, each row of A has at least one 1. Since $2(m-1) < 12$, A has a row with precisely one 1. We may assume that $A(1, 1) = 1$ and $A(1, i) = 0$ for $2 \leq i \leq m-1$. Let C_2 be the second column of M (remember that $C_2(1) = A(1, 1) = 1$).

Consider the n-column $C_3 = [0, C_2(\{2, \ldots, n\})^T]^T$ which is obtained from C_2 by changing the first entry to 0. If it is not in M, then $F \subseteq [M, C_3]$. This copy of F has to contain the entry in which C_3 differs from C_2. But the only non-zero entry in Row 1 is $M(1, 2)$; thus $F \subseteq [C_2, C_3]$, which is an obvious contradiction. Thus we may assume that C_3 is the third column of M.

We have to consider two cases. First, suppose that $C_2(\{2, \ldots, 6\})$ has at least one entry equal to 1. Without loss of generality, assume that $C_2(2) = C_3(2) = 1$.

It follows that $C_2(i) = C_3(i) = 0$ for $3 \leq i \leq 6$ (otherwise the first and the second columns of M would contain F). Let

$$B = M(\{3, 4, 5, 6\}, \{4, \ldots, m\}). \tag{2}$$

By our assumption, each row of B has at least one 1; in particular $m \geq 5$. Clearly, B contains at most $2(m-3) < 8$ ones. Thus, by permuting Rows $3, \ldots, 6$ and Columns $4, \ldots, m$, we can assume that $B(1, 1) = 1$ while $B(1, i) = 0$ for $2 \leq i \leq m-3$. Let C_4 be the fourth column of M and C_5 be such that C_4 and C_5 differ at the third position only, i.e., $C_4(3) = 1$ and $C_5(3) = 0$. As before, C_5 must be in M, say it is the fifth column. Since $C_4(\{4, 5, 6\})$ has at most one 1, assume that $C_4(5) = C_4(6) = C_5(5) = C_5(6) = 0$. We need another column C_6 with $C_6(5) = C_6(6) = 1$ (otherwise the fifth or the sixth row of M would consist of all zero entries). In particular, $m = 6$. But now the new column C_7 which differs from C_6 at the fifth position only (i.e., $C_7(5) = 0$ and $C_7(i) = C_6(i)$ for $i \neq 5$) should be also in M, since M is F-saturated. This contradicts $e(M) = 6$. Thus the first case does not hold.

In the second case, we have $C_2(i) = C_3(i) = 0$ for every $2 \leq i \leq 6$. We may define B as in (2) and get a contradiction in the same way as above. This proves the claim. □

Suppose, contrary to the theorem, that we can find an F-saturated matrix M with $n \geq 11$ rows and $m \leq 6$ columns. By the claim, M has a constant row; we may assume that the final row of M is all zero, and let $N = M([n-1],)$. If C is an $(n-1)$-column missing from N, then the column $Q = (C^T, 0)^T$ is missing in M. Moreover, a copy of F in $[M, Q]$ cannot use the n-th row. Thus $F \subseteq [N, C]$, which means that $N \in \text{SAT}(n-1, F)$ and $\text{sat}(n-1, F) \leq m \leq 6$. Repeating this argument, we eventually conclude that $\text{sat}(10, F) \leq 6$, a contradiction to the results of our computer search. The theorem is proved. □

Theorem 8 *Let* $F = [T_3^0, T_3^2, T_3^3] = \begin{bmatrix} 0&0&1&1&1 \\ 0&1&0&1&1 \\ 0&1&1&0&1 \end{bmatrix}$. *Then*

$$\text{sat}(n, F) = \begin{cases} 7, & \text{if} \quad n = 3, 6 \text{ or } 7, \\ 9, & \text{if} \quad n = 4 \text{ or } 5. \end{cases}$$

Moreover, for any $n \geq 8$, $\text{sat}(n, F) \leq 7$.

Proof We define the following matrices:

$$M_4 = \begin{bmatrix} 1&0&1&0&1&0&0&0&1 \\ 0&1&1&0&0&1&0&1&1 \\ 0&0&0&1&1&0&1&1&1 \\ 0&0&0&0&0&1&1&1&1 \end{bmatrix},$$

$$M_5 = \begin{bmatrix} 1&1&1&0&1&0&1&0&1 \\ 0&1&0&1&0&1&0&1&1 \\ 0&0&1&0&1&0&1&1&1 \\ 0&0&0&1&1&0&0&1&1 \\ 0&0&0&0&0&1&1&1&1 \end{bmatrix},$$

$$M_6 = \begin{bmatrix} 1&1&0&0&1&1&0 \\ 1&0&1&1&0&1&0 \\ 1&0&1&0&1&0&1 \\ 0&1&1&1&1&0&0 \\ 0&1&1&0&0&1&1 \\ 0&0&0&1&1&1&1 \end{bmatrix}.$$

For any $n \geq 7$ let $M_n([6],) = M_6$ and $M_n(i,) = \begin{bmatrix} 0&0&0&1&1&1&1 \end{bmatrix}$ for every $7 \leq i \leq n$ (i.e., the last row of M_6 is repeated $(n-6)$ times). For $n = 3, \ldots, 7$ the theorem (with M_n being a minimum F-saturated matrix) follows from a computer search [10]. It remains to show that M_n, $n \geq 8$, is F-saturated. Clearly, this is the case, since M_7 is F-saturated and F contains no pair of equal rows. □

Conjecture 1 *Let* $F = [T_3^0, T_3^2, T_3^3]$. *Then* $\text{sat}(n, F) = 7$ *for every* $n \geq 8$.

Theorem 9 *Let* $F = T_3^{\leq 2} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$. *Then*

$$\mathrm{sat}(n, F) = \begin{cases} 7, & \text{if } n = 3, \\ 10, & \text{if } 4 \leq n \leq 6. \end{cases}$$

Moreover, for any $n \geq 7$, $\mathrm{sat}(n, F) \leq 10$.

Proof For $n = 3, \ldots, 6$ the statement follows from a computer search [10] with the following F-saturated matrices:

$$M_4 = \begin{bmatrix} 0\,1\,0\,1\,0\,1\,1\,0\,0\,1 \\ 0\,0\,1\,1\,0\,0\,1\,1\,0\,1 \\ 0\,0\,0\,0\,1\,1\,1\,0\,1\,1 \\ 0\,0\,0\,0\,0\,0\,0\,1\,1\,1 \end{bmatrix},$$

$$M_5 = \begin{bmatrix} 1\,0\,1\,0\,1\,0\,0\,0\,1\,1 \\ 0\,1\,0\,1\,1\,0\,1\,0\,0\,1 \\ 0\,0\,1\,0\,1\,1\,1\,0\,0\,1 \\ 0\,0\,0\,1\,1\,0\,0\,1\,1\,1 \\ 0\,0\,0\,0\,0\,1\,1\,1\,1\,1 \end{bmatrix}.$$

For any $n \geq 6$ let $M_n([5],) = M_5$ and $M_n(i,) = \begin{bmatrix} 1\,1\,0\,0\,0\,0\,1\,0\,1\,1 \end{bmatrix}$ for every $6 \leq i \leq n$. It remains to show that M_n, $n \geq 7$, is F-saturated. Clearly, this is the case, since M_6 is F-saturated and F contains no pair of equal rows. □

Conjecture 2 *Let* $F = T_3^{\leq 2}$. *Then* $\mathrm{sat}(n, F) = 10$ *for every* $n \geq 7$.

Theorem 10 *Let* $F_1 = T_3^2 = \begin{bmatrix} 0\,1\,1 \\ 1\,0\,1 \\ 1\,1\,0 \end{bmatrix}$, *and* $F_2 = [T_3^2, T_3^3] = \begin{bmatrix} 0\,1\,1\,1 \\ 1\,0\,1\,1 \\ 1\,1\,0\,1 \end{bmatrix}$. *Then* $\mathrm{sat}(n, F_1) = \mathrm{sat}(n, F_2) = 3n - 2$ *for any* $3 \leq n \leq 6$. *Moreover, for any* $n \geq 7$, $\mathrm{sat}(n, F_1) \leq 3n - 2$ *and* $\mathrm{sat}(n, F_2) \leq 3n - 2$ *as well.*

Proof Let $M_n = [T_n^0, T_n^1, T_n^n, \tilde{T}_n^2]$, where $\tilde{T}_n^2 \subseteq T_n^2$ consists of all those columns of T_n^2 which have precisely one entry equal to 1 either in the first or in the nth row (but not in both), e.g., for $n = 5$ we obtain

$$M_5 = \begin{bmatrix} 0\,1\,0\,0\,0\,0\,1\,1\,1\,1\,0\,0\,0 \\ 0\,0\,1\,0\,0\,0\,1\,1\,0\,0\,1\,0\,0 \\ 0\,0\,0\,1\,0\,0\,1\,0\,1\,0\,0\,1\,0 \\ 0\,0\,0\,0\,1\,0\,1\,0\,0\,1\,0\,0\,1 \\ 0\,0\,0\,0\,0\,1\,1\,0\,0\,0\,1\,1\,1 \end{bmatrix}.$$

Clearly, $e(M_n) = e(T_n^0) + e(T_n^1) + e(T_n^n) + e(\tilde{T}_n^2) = 1 + n + 1 + 2n - 4 = 3n - 2$. Moreover, since $\tilde{T}_n^2$ is F_1-admissible we get that M_n is both F_1 and F_2 admissible. Now we show that M_n is F_1-saturated. Indeed, pick any column $C = (c_1, \ldots, c_n)^T$

which is not present in M_n. Such a column must contain at least 2 ones and 1 zero. Let $1 \le i, j, k \le n$ be the indices such that $c_i = 0, c_j = c_k = 1$. If $i = 1$ or $i = n$, then the matrix $[M_n, C](\{i, j, k\},)$ contains F_1. Otherwise, $c_1 = c_n = 1$, and there also exists $1 < i < n$ such that $c_i = 0$. Here $[M_n, C](\{1, i, n\},)$ contains F_1. Thus M_n is F_1 saturated and, since it must contain T_n^n is a column, M_n is also F_2-saturated. We conclude that $\text{sat}(n, F_1) \le 3n-2$ and $\text{sat}(n, F_2) \le 3n-2$ for any $n \ge 3$. A computer search [10] yields that these inequalities are equalities when $n = 3, \ldots, 6$. □

Conjecture 3 *Let* $F_1 = T_3^2$ *and* $F_2 = [T_3^2, T_3^3]$. *Then* $\text{sat}(n, F_1) = \text{sat}(n, F_2) = 3n - 2$ *for every* $n \ge 7$.

Remark 5 It is not hard to see that $\text{sat}(n, F_1) \ge n + c\sqrt{n}$ for some absolute constant c and all $n \ge 3$. Indeed, let M be an $n \times (n+2+\lambda)$ F_1-saturated matrix of size $\text{sat}(n, F_1)$ for some $\lambda = \lambda(n)$. We may assume that $M(, [n+2]) = [T_n^0, T_n^1, T_n^n]$. Suppose that $\lambda \le n$ for otherwise we are done. Moreover, we assume that every column of matrix $M([\lambda], \{n+3, \ldots, n+2+\lambda\})$ contains at least one entry equal to 1 (trivially, there must be a permutation of the rows of M satisfying this requirement). We claim that all rows of $M(\{\lambda+1, \ldots, n\}, \{n+3, \ldots, n+2+\lambda\})$ are different. Suppose not. Then, there are indices $\lambda + 1 \le i, j \le n$ such that $M(i, \{n+3, \ldots, n+2+\lambda\}) = M(j, \{n+3, \ldots, n+2+\lambda\})$. Now consider a column C in which the only nonzero entries correspond to i and j. Clearly, C is not present in M, since the first λ entries of C equal 0. Moreover, since M is F_1-saturated, the matrix $[M, C]$ contains F_1. In other words, there are three rows in M which form F_1 as a submatrix. Note that the ith and jth row must be among them. But this is not possible since F_1 has no pair of equal rows.

Let $M_0 = M(\{\lambda+1, \ldots, n\}, \{n+3, \ldots, n+2+\lambda\})^T$. Clearly, M_0 is F_1-admissible. Anstee and Sali showed (see Theorem 1.3 in [5]) that $\text{forb}(\lambda, F_1) = O(\lambda^2)$. That means that $n - \lambda = O(\lambda^2)$, and consequently, $\lambda = \Omega(\sqrt{n})$. Hence, $\text{sat}(n, F_1) = e(M) \ge n + \Omega(\sqrt{n})$, as required.

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