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On the Cofinality of Infinite Partially Ordered Sets: Factoring a Poset into Lean Essential Subsets

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Abstract. We study which infinite posets have simple cofinal subsets such as chains, or decompose canonically into such subsets. The posets of countable cofinality admitting such a decomposition are characterized by a forbidden substructure; the corresponding problem for uncountable cofinality remains open.

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1. Introduction

A subset Q of a partially ordered set $(P, \leqslant)$ is *cofinal* in P if for every $x \in P$ there exists a $y \in Q$ with $x \leqslant y$. The least cardinality of a cofinal subset of P is the *cofinality* $\mathrm{cf}(P)$ of P.

In this paper, we shall work from the assumption that we 'understand' an infinite poset $(P, \leqslant)$ as soon as we understand one of its cofinal subsets, and our aim will be to either find in P a particularly simple cofinal subset Q (which may then be studied instead of P), or to decompose P into such simple subsets. This paradigm makes immediate sense, for example, if P is itself a down-closed subset of some larger poset, and 'understanding' P means being able to decide which elements of that larger poset it contains.*

So when is a cofinal subset Q of P 'particularly simple'? As a first approach, we might require that Q should be 'lean' in the sense that it contains no large amounts of junk not needed to make it cofinal in P. More precisely: we might require that every subset $Q' \subseteq Q$ of cardinality $|Q'| = |Q|$ is cofinal in Q (and, hence, in P). This works well for countable posets, and was the definition of leanness adopted

* In the context [2] from which this study arose, P would be a property of finite graphs, such as planarity with the graph minor relation. In this example, the grids form a cofinal subset of P (since every planar graph is a minor of some grid), and indeed for many minor-related questions it suffices to consider grids instead of arbitrary planar graphs.

in [1]. For arbitrary posets, it is clearly more appropriate to replace cardinality with cofinality: we shall call Q *lean* if every subset $Q' \subseteq Q$ of cardinality at least $\mathrm{cf}(Q)$ is cofinal in Q. Thus, in a lean poset we can pick any subset that is large enough to be cofinal, and can be sure that it is indeed cofinal.

Not every poset has a lean cofinal subset. For example, an unrelated disjoint union of two ω-chains does not, and neither does an infinite antichain. Indeed we shall see later that, trivial exceptions aside, any poset P with a lean cofinal subset must be *directed*: for every two points $x, y \in P$ there must be a $z \in P$ above both, i.e. such that $x \leqslant z$ and $y \leqslant z$. But also directed posets can fail to have lean cofinal subsets. A simple example was given in [1]: start with an uncountable antichain, and perform ω times the operation of adding for every pair of points a new point above both.

Our first aim, then, will be to see which posets have lean cofinal subsets. These posets will be characterized in Section 3; it turns out that they are precisely the posets that are *indivisible* in a sense defined in Section 2. For the remaining (divisible) posets we then try to find canonical decompositions into indivisible subsets (which will have lean cofinal subsets by the result from Section 3). We shall be able in Section 4 to characterize the posets admitting such canonical decompositions when their cofinality is countable. In general, it may still be possible to take a nontrivial step towards that goal: we show that decompositions of divisible posets into directed 'essential' subsets are canonical in the sense that any partition into indivisible sets would refine it. (A subset is *essential* if every cofinal subset meets it.) The problem of how to decompose those directed sets further into indivisible sets is addressed in Section 5 but remains open, as does the characterization problem from Section 4 for uncountable cofinality.

2. Terminology and Basic Facts

The ordering of any poset we consider will be denoted by $\leqslant$. Subsets will carry the induced ordering. The *direct product* $P \times Q$ of two posets P and Q is the set of pairs (p, q) with $p \in P$ and $q \in Q$ in which $(p, q) \leqslant (p', q')$ if and only if $p \leqslant p'$ and $q \leqslant q'$.

A *tree* is a poset in which the set of points less than any given point is well-ordered. A tree is *ever-branching* if it is nonempty and every point has at least two successors. If every point of a tree T has exactly two successors, T has a least element, and every other point has a predecessor, then T is the *binary tree* denoted by T_2.

To avoid trivialies we shall only be interested in posets P of infinite cofinality. We usually think of an ordering as vertical, and use freely expressions like 'x lies below y' to express that $x \leqslant y$. We call

$$\lfloor A \rfloor_P := \{x \in P \mid \exists a \in A\colon\ a \leqslant x\}$$

and

$$\lceil A \rceil_P := \{x \in P \mid \exists a \in A\colon\ x \leqslant a\}$$

the *up-closure* and the *down-closure* of $A \subseteq P$ in P, respectively. Two points of P are *compatible* if they have a common upper bound in P; if every two points of P are compatible then P is *directed*.

Given subsets $A, B \subseteq P$, we write $A \leqslant B$ if $A \subseteq \lceil B \rceil$. If both $A \leqslant B$ and $B \leqslant A$ we call A and B *cofinally equivalent*, or just *equivalent*, and write $A \sim B$; note that $A \sim P$ if and only if A is cofinal in P. We also write $A < B$ if $A \leqslant B$ but $A \not\geqslant B$. We shall usually be interested in subsets of P only up to cofinal equivalence.

LEMMA 2.1. *If* $A \sim B$ *then* $\mathrm{cf}(A) = \mathrm{cf}(B)$.

Proof. To show that $\mathrm{cf}(A) \leqslant \mathrm{cf}(B)$, choose a cofinal subset B' in B of cardinality $|B'| = \mathrm{cf}(B)$. For every $b \in B'$ pick an $a(b) \in A$ above it. Then $A' := \{a(b) \mid b \in B'\}$ is a set of cardinality at most $\mathrm{cf}(B)$ satisfying $A \leqslant B \leqslant B' \leqslant A'$, so A' is cofinal in A. □

If $A \subseteq P$ is not cofinal in P, we call A *small* in P. Complements in P of small sets are called *essential*; thus, every cofinal subset of P meets all its essential subsets. The canonical example of an essential set is the up-closure $\lfloor x \rfloor_P$ of a single point x: as is easily checked, a subset of P is essential if and only if it contains the up-closure in P of at least one of its points.

As a typical example of a poset that has no lean cofinal subset we mentioned an unrelated disjoint union of two ω-chains. The reason why no subset Q of this poset P can be both lean and cofinal in P is that cofinality will force it to contain infinitely many points from each of the two chains, which will prevent it from being lean because its $\aleph_0 = \mathrm{cf}(Q)$ points in one chain are not cofinal in the other. Posets with lean cofinal subsets thus cannot be divisible in this sense, which motivates the following definition:

DEFINITION 2.2. A partially ordered set $(P, \leqslant)$ is called *divisible* if it is a union of either two* or fewer than $\mathrm{cf}(P)$ small subsets.

Linear orders are easily seen to be indivisible, and we shall see that, conversely, all indivisible posets have equivalent linear suborders (Theorem 3.5).

Indivisible subsets** of a poset behave somewhat like primes:

LEMMA 2.3. *If* $A \subseteq P$ *is indivisible and* $A \leqslant B = \bigcup_{\beta<\alpha} B_\beta$ *with* $\alpha < \mathrm{cf}(A)$ *or* $\alpha = 2$*, then* $A \leqslant B_\beta$ *for some* $\beta < \alpha$.

Proof. Since $A \leqslant B$ we have $A = \bigcup_{\beta<\alpha} A_\beta$, where $A_\beta := A \cap \lceil B_\beta \rceil$. Since A is indivisible these A_β cannot all be small, so $A \leqslant A_\beta \leqslant B_\beta$ for some β. □

Unlike leanness, divisibility is invariant under cofinal equivalence:

* The 'either two' option is needed only to make posets of cofinality 2 divisible. For posets of infinite cofinality, which this paper is about, it is obviously redundant.

** Note that the divisibiblity or indivisibility of a subset $Q \subseteq P$ depends only on Q, not on P (except that Q inherits its ordering from P): the 'small' subsets referred to in the definition of divisibility for Q have to be small in Q, not just in P.

PROPOSITION 2.4. *Let $A, B \subseteq P$ and $A \sim B$. If B is divisible then so is A.*

Proof. Since B is divisible we have $B = \bigcup_{\beta<\alpha} B_\beta$, where either $\alpha = 2$ or $\alpha < \mathrm{cf}(B) = \mathrm{cf}(A)$ (cf. Lemma 2.1) and all the B_β are small in B. If A were indivisible, then Lemma 2.3 would imply that $B \leqslant A \leqslant B_\beta$ for some β, a contradiction. □

Recall that a poset is *directed* if every two points are *compatible*, i.e. have a common upper bound.

PROPOSITION 2.5. *P is directed if and only if P is not a union of two small subsets.*

Proof. Note that for every point $x \in P$ the set $A_x := P \setminus \lfloor x \rfloor$ is small in P, and that two points x, y are compatible in P if and only if $A_x \cup A_y \subsetneqq P$.

Now if $X, Y \subseteq P$ are two small subsets, there exist $x, y \in P$ such that $X \subseteq A_x$ and $Y \subseteq A_y$. If P is directed then x and y are compatible, so $X \cup Y \subseteq A_x \cup A_y \subsetneqq P$. Conversely if P is undirected then P has incompatible points x, y, and P is the union of the small sets A_x and A_y. □

COROLLARY 2.6. *Indivisible posets are directed.* □

The converse of Corollary 2.6 does not hold: the poset from [1] mentioned in the Introduction is directed and has only ω levels, but it has uncountable cofinality (and is hence divisible into its ω levels). In Section 5 we shall see that this structure is canonical: every divisible directed poset P has uncountable cofinality and can be viewed as a vertical stack of fewer than $\mathrm{cf}(P)$ horizontal layers.

Recall that every essential set $A \subseteq P$ contains the up-closure $\lfloor a \rfloor$ of one of its points a. If A is directed, then A is in fact equivalent to $\lfloor a \rfloor$ (and hence any two such sets $\lfloor a \rfloor$ are also equivalent):

LEMMA 2.7. *If $A \subseteq P$ is essential in P and directed, and if $a \in A$ is such that $\lfloor a \rfloor_P \subseteq A$, then $A \leqslant \lfloor a \rfloor_P$.*

Proof. Since A is directed, every $x \in A$ has a common upper bound with a in A, which is an element of $\lfloor a \rfloor_P$ above x. □

3. Posets with Lean Cofinal Subsets

In this section we characterize the posets that have lean cofinal subsets. Although the notion of a lean poset appears to be new, the results in this section could alternatively be derived from related results of Pouzet as presented in Fraïssé [4, §10 of Chapter 4]. Lemma 3.3 has also been noted by Galvin, Milner and Pouzet [3], who gave a different proof.

Let us start by recalling the definition of 'lean':

DEFINITION 3.1. A partially ordered set $(P, \leqslant)$ is called *lean* if every subset of cardinality at least $\mathrm{cf}(P)$ is cofinal in P.

Note that a poset of finite cofinality is lean if and only if it is an antichain. So we shall be interested in posets of infinite cofinality only.

We start with a couple of lemmas on the cofinality of lean posets. While the cofinality of an arbitrary poset P can be any cardinal (even singular) up to $|P|$, the cofinality of a lean poset always equals its cardinality, and it is always regular (or finite).

LEMMA 3.2. *If a poset P is lean then* $\mathrm{cf}(P) = |P|$.

Proof. Suppose not; then $|P| > \kappa := \mathrm{cf}(P) \geqslant \aleph_0$. Let $X = \{x_\alpha \mid \alpha < \kappa\}$ be a cofinal subset of P. Since $\bigcup_{\alpha<\kappa} \lceil x_\alpha \rceil_P = P$ and $\kappa^+ \leqslant |P|$ is regular, there exists an $\alpha < \kappa$ with $|\lceil x_\alpha \rceil| \geqslant \kappa^+$. Then $\lceil x_\alpha \rceil \setminus \{x_\alpha\}$ has a subset A of order κ, which contradicts the leanness of P because $x_\alpha \notin \lceil A \rceil$ and, hence, $P \not\leqslant A$. □

LEMMA 3.3. *Lean posets have finite or regular cofinality.*

Proof. Let P be a lean poset. By Lemma 3.2 we have $\kappa := \mathrm{cf}(P) = |P|$; let $P = \{p_\lambda \mid \lambda < \kappa\}$ be a well-ordering of P. Suppose κ is infinite and singular, say $\mu := \mathrm{cf}(\kappa) < \kappa$; let Λ be a cofinal subset of κ of order μ and consisting of regular cardinals.

We first prove that some $\lambda_0 \in \Lambda$ is such that $|\lceil x \rceil| \leqslant \lambda_0$ *for all* $x \in P$. Suppose not. For each $\lambda \in \Lambda$ choose an $x_\lambda \in P$ with $|\lceil x \rceil| > \lambda$, and put $X := \{x_\lambda \mid \lambda \in \Lambda\}$. Then $|\lceil X \rceil| = \kappa$, and hence $P \leqslant \lceil X \rceil \leqslant X$ because P is lean. This contradicts the fact that $|X| \leqslant \mu < \mathrm{cf}(P)$.

Now pick $\lambda_1 \in \Lambda$ with $\mu, \lambda_0 < \lambda_1$, and let $Y \subseteq P$ be a set of size λ_1. For each $y \in Y$ consider the set $P_y := P \setminus \lfloor y \rfloor$. Since P_y is not cofinal in P (as $y \notin \lceil P_y \rceil$) and P is lean, we have $|P_y| < \kappa$; pick $\lambda(y) \in \Lambda$ greater than $|P_y|$. Since $|Y| = \lambda_1$ is regular, one of the $\mu < \lambda_1$ sets $Y_\lambda := \{y \in Y \mid \lambda(y) = \lambda\}$ with $\lambda \in \Lambda$ (which have union Y) has size λ_1. Let this set be $Y' = Y_{\lambda_2}$; then $|Y'| = \lambda_1$ and $|P_y| < \lambda(y) = \lambda_2$ for all $y \in Y'$. Hence $|\bigcup_{y\in Y'} P_y| \leqslant \lambda_1\lambda_2 < \kappa$, so

$$\bigcap_{y\in Y'} \lfloor y \rfloor = P \setminus \bigcup_{y\in Y'} P_y \neq \emptyset.$$

But every $x \in \bigcap_{y\in Y'} \lfloor y \rfloor$ satisfies $\lceil x \rceil \supseteq Y'$ and hence

$$|\lceil x \rceil| \geqslant |Y'| = \lambda_1 > \lambda_0,$$

contradicting the definition of λ_0. □

COROLLARY 3.4. *Lean posets have regular cardinality.* □

The following theorem characterizes the posets that have lean cofinal subsets. But it has other aspects too. For example, it can be viewed as a characterization of (in)divisibility, or as a statement on how much stronger the property of being indivisible is than that of being directed (cf. condition (iv)).

THEOREM 3.5. *Let $(P, \leqslant)$ be a partially ordered set of infinite cofinality. Then the following assertions are equivalent:*

(i) *P has a lean cofinal subset;*
(ii) *P is indivisible;*
(iii) *P contains a cofinal chain;*
(iv) *every set $A \subseteq P$ of order $|A| < \text{cf}(P)$ has an upper bound in P.*

Proof. We prove the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (i). Let $\text{cf}(P) =: \kappa$.

(i) $\Rightarrow$ (ii) Let Q be a lean cofinal subset of P, and suppose that P is divisible. Then Q is divisible too (Proposition 2.4), so it is a union of fewer than $\text{cf}(Q)$ small subsets. By Lemma 3.3 one of these has cardinality at least $\text{cf}(Q)$, which contradicts our assumption that Q is lean.

(ii) $\Rightarrow$ (iv) Let $A \subseteq P$ be given as in (iv), and for every $a \in A$ put $P_a := P \setminus \lfloor a \rfloor$. These P_a are small subsets of P. Hence if P is indivisible (as assumed in (ii)), their union cannot be all of P, as $|A| < \text{cf}(P)$. Thus $\emptyset \neq P \setminus \bigcup_{a \in A} P_a = \bigcap_{a \in A} \lfloor a \rfloor$, and any element of this set is the desired upper bound for A.

(iv) $\Rightarrow$ (iii) Let $X = \{x_\alpha \mid \alpha < \text{cf}(P)\}$ be cofinal in P, and assume that (iv) holds. Inductively for all $\alpha < \text{cf}(P)$ choose points $y_\alpha \in P$ so that y_α is an upper bound of $A := \{x_\alpha\} \cup \{y_\beta \mid \beta < \alpha\}$. Then $C := \{y_\alpha \mid \alpha < \text{cf}(P)\}$ is a chain, which is cofinal in P because X is cofinal in P and contained in $\lceil C \rceil$.

(iii) $\Rightarrow$ (i) Every chain C contains a lean cofinal subchain. (Any cofinal subchain of order type $\text{cf}(C)$ will do, because $\text{cf}(C)$ is a regular cardinal.) □

For completeness, we remark that the assertions of Theorem 3.5 for posets P of finite cofinality hold as follows: (i) is always true; (ii)–(iii) are true iff $\text{cf}(P) \leqslant 1$; (iv) is true iff $\text{cf}(P) \leqslant 2$.

We shall later be interested in the structure of divisible directed posets. Theorem 3.5 implies that these must have uncountable cofinality, which we note for future reference:

COROLLARY 3.6. *Directed posets of countable cofinality are indivisible.*

Proof. Directed posets of finite cofinality have cofinality 1, and the assertion is trivial. Directed posets of cofinality $\aleph_0$ satisfy condition (iv) of Theorem 3.5, and hence also condition (ii). □

We conclude with another corollary of Theorem 3.5:

COROLLARY 3.7. *Any directed divisible poset P contains a chain C such that $|C| < \text{cf}(P)$ and C has no upper bound in P.*

Proof. By Theorem 3.5(iv) there is a set $A \subseteq P$ such that $|A| < \text{cf}(P)$ and A has no upper bound in P. Well-order $A =: \{a_\beta \mid \beta < \alpha\}$. Inductively on $\beta < \alpha$ try to find a point c_β above both a_β and all of $C_\beta := \{c_\gamma \mid \gamma < \beta\}$. If we fail for some

$\beta < \alpha$, then the chain C_β has no upper bound in P. Otherwise C_α has no upper bound in P. □

4. Factoring Posets into Essential Directed Subsets

In Section 3 we asked which posets P have a lean cofinal subset (which could then replace P in any study of properties that depend only on the equivalence type of P), and found that these are precisely the indivisible posets. It remains to study the structure of divisible posets, if possible in terms of their indivisible subsets.

Of course, a poset can always be partitioned somehow into indivisible subsets, e.g. into single points. So one approach to the above question is to try to determine how few indivisible subsets will cover P. This approach was taken by Milner and Prikry [5], who found an upper bound on the number of *directed sets* (not necessarily indivisible) needed to cover P, in terms of the size of the antichains occurring in P.

We shall here take a slightly different tack. Instead of just trying to minimize the number of indivisible (or directed) sets covering P we shall ask whether there is some *canonical* such decomposition that makes sense. But like Milner and Prikry, we shall not at first insist that the parts of this decomposition must be indivisible, but be content with decompositions into directed sets.

When the cofinality of those directed subsets is countable, they will in fact be indivisible by Corollary 3.6. But even in general, partitioning divisible posets into directed rather than indivisible subsets makes sense as a first step – especially if the directed partition sets are canonical in the sense that, as in factoring integers, any partition into indivisible subsets will refine this partition into directed sets. Our topic in this section is how to characterize the posets that admit such canonical decompositions into directed subsets; in Section 5 we shall address the question of how to partition those directed sets further into indivisible sets.

An arbitrary partition into directed sets will not, of course, be canonical in the above sense: for example, all partitions of a chain are partitions into both directed and indivisible sets, but not every two such partitions are compatible in the sense that one refines the other. To make our partitions canonical, we thus have to impose further conditions on the partitions allowed. Those conditions should ensure in particular that linear orders become indecomposable, i.e. have no nontrivial canonical partitions.

The following lemma and discussion show that partitions into *essential* directed subsets fit this bill:

LEMMA 4.1. *If $P = \bigcup_{\beta<\alpha} A_\beta$ is a partition of P into essential directed subsets, then every essential directed subset of P is cofinally equivalent in P to one of the A_β. In particular, there is at most one such partition up to cofinal equivalence.*

Proof. Let A be an essential directed subset of P, and let $a \in A$ be such that $\lfloor a \rfloor \subseteq A$. Let A_β be the partition set containing a, and let $b \in A_\beta$ be such that

$\lfloor b \rfloor \subseteq A_\beta$. Let c be a common upper bound of a and b in A_β. Then $\lfloor c \rfloor \subseteq \lfloor a \rfloor \cap \lfloor b \rfloor \subseteq A \cap A_\beta$. By Lemma 2.7 both A and A_β are equivalent to $\lfloor c \rfloor$ in P, and hence to each other. □

Decompositions into essential directed subsets are canonical not only in that they are unique up to cofinal equivalence. They are also (essentially) canonical in the sense discussed earlier, that no indivisible subset of P should meet more than one of their parts (and hence that this decomposition could be seen as a first step towards every possible decomposition of P into indivisible subsets). Indeed, every A_β as above has a point a_β with $A_\beta \leqslant \lfloor a_\beta \rfloor \subseteq A_\beta$ (Lemma 2.7), and we may think of $\lfloor a_\beta \rfloor$ as the 'essential part' of A_β. Now if B is any other directed subset of P (in particular, if B is indivisible), then B cannot meet the 'essential parts' $\lfloor a_\beta \rfloor$, $\lfloor a_\gamma \rfloor$ of two different partition sets A_β, A_γ: the common upper bound which a_β and a_γ would have in B would then lie in $\lfloor a_\beta \rfloor \cap \lfloor a_\gamma \rfloor \subseteq A_\beta \cap A_\gamma = \emptyset$.

The price we have to pay for obtaining uniqueness for the decompositions in Lemma 4.1 is that their existence is no longer trivial. Indeed there is an obvious poset that has no essential directed subsets at all, and hence no such partition: the binary tree T_2. In the remainder of this section we shall prove that T_2 must occur in every poset that has no partition into essential directed subsets, and investigate how exactly T_2 will be embedded in such a poset.

We begin with two lemmas. Call a point $x \in P$ *special* (in P) if its up-closure $\lfloor x \rfloor$ in P is directed. Our first lemma says that these points are cofinal in every essential directed subset:

LEMMA 4.2. *Every essential directed subset of P has a cofinal subset consisting of special points of P.*

Proof. Let $A \subseteq P$ be an essential directed subset, and let $x \in A$ be given; we have to find a point $y \in A$ above x that is special in P. Since A is essential, it contains the up-closure $\lfloor a \rfloor_P$ of one of its points. Since A is directed, the points x, a have a common upper bound y in A. Then $\lfloor y \rfloor_P \subseteq \lfloor a \rfloor_P \subseteq A$, so $\lfloor y \rfloor$ is directed because A is directed. Thus, y is special in P. □

Lemma 4.2 implies that if P has a partition into essential directed subsets then its special points are cofinal in P. The converse of this holds too:

LEMMA 4.3. *If the special points of P are cofinal in P, then P admits a partition into essential directed subsets.*

Proof. Let $A \subseteq P$ be a maximal set of points with disjoint directed up-closures. Then these up-closures are essential directed subsets of P, and $P \leqslant \bigcup_{a \in A} \lfloor a \rfloor$: given $x \in P$, pick a special point y above it, and notice that $\lfloor y \rfloor \cap \lfloor a \rfloor \neq \emptyset$ for some $a \in A$ by the maximality of A. Hence, some $z \in \lfloor a \rfloor$ satisfies $x \leqslant y \leqslant z$.

It remains to extend the sets $\lfloor a \rfloor$ so as to cover all of P, while keeping them disjoint and directed. This is easily done: just add any point of P not in any of these sets to some set $\lfloor a \rfloor$ in whose down-closure it lies. This covers all of P

since $P \leqslant \bigcup_{a \in A} \lfloor a \rfloor$, and the enlarged sets will be directed because the $\lfloor a \rfloor$ are directed. □

Lemmas 4.2 and 4.3 imply that a decomposition into essential directed subsets exists as soon as every point lies in such a set:

COROLLARY 4.4. *If P is a union of essential directed subsets, then it admits a partition into such sets.* □

Call a subset X of P *cofinally faithful* if any two incomparable points of X are incompatible in P, i.e. have no common upper bound.

PROPOSITION 4.5. *If P admits no partition into essential directed subsets then P contains a cofinally faithful copy of T_2.*

Proof. Suppose that P has no partition into essential directed subsets. By Lemma 4.3 there is a point $x \in P$ such that $\lfloor x \rfloor$ contains no special points. Starting with x as the root, we can construct a cofinally faithful copy of T_2 in $\lfloor x \rfloor$ inductively: we first find incompatible points $x_0, x_1 \in \lfloor x \rfloor$ because x is not special (and, hence, $\lfloor x \rfloor$ is undirected), and let these be the two successors of x in our copy of T_2; we then continue inductively inside the undirected disjoint sets $\lfloor x_0 \rfloor$ and $\lfloor x_1 \rfloor$, finding incompatible successors x_{00} and x_{01} of x_0 in $\lfloor x_0 \rfloor$ and incompatible successors x_{10} and x_{11} of x_1 in $\lfloor x_1 \rfloor$, and so on. □

It is not difficult to see that the converse of Proposition 4.5 is false. For example, we can add above each point x of T_2 a new maximal point x', so that the new poset P contains the original T_2 as a cofinally faithful subset but partitions into the essential directed sets $\{x, x'\}$. Thus in order to preclude the existence of a partition of P into essential directed subsets, T_2 has to be contained in P 'more cofinally' than in this example.

More precisely, we have the following definition and converse of Proposition 4.5. Call a set $X \subseteq P$ *essentially cofinal* in P if, for some essential subset A of P, the set $X \cap A$ is cofinal in A. (Requiring A to be of the form $\lfloor a \rfloor$ would yield an equivalent definition. Another equivalent requirement is simply that $\lceil X \rceil$ be essential in P.)

LEMMA 4.6. *If P admits a partition into essential directed subsets, then P has no essentially cofinal subset isomorphic to an ever-branching tree* (*cofinally faithful or not*).

Proof. Suppose that $T \subseteq P$ is an isomorphic copy in P of an ever-branching tree whose points are cofinal in some essential subset B of P. Let $b \in B$ be such that $\lfloor b \rfloor \subseteq B$. We show that P has no essential directed subset A containing b.

Suppose A is such a set, and let $a \in A$ be such that $\lfloor a \rfloor \subseteq A$. Let c be a common upper bound of a and b in A. Then $\lfloor c \rfloor \subseteq \lfloor a \rfloor \cap \lfloor b \rfloor$, so $\lfloor c \rfloor$ is again directed and $T \cap \lfloor c \rfloor$ is cofinal in $\lfloor c \rfloor$. Pick $t \in T \cap \lfloor c \rfloor$, and let t', t'' be distinct successors of t

in T. As these lie in $\lfloor c \rfloor$ they have a common upper bound in $\lfloor c \rfloor$, which lies below some $t^* \in T \cap \lfloor c \rfloor$. Then $t', t'' \in \lceil t^* \rceil_T$, which is linear since T is a tree. But t', t'' are incomparable in T, so we have a contradiction. □

We believed for a while that the converse of Lemma 4.6 was true too, i.e. that containing an ever-branching tree as an essentially cofinal subset characterized the posets without our desired partition. In fact it does not, and we shall give a counterexample at the end of this section. For posets of countable cofinality, however, we do have an exact characterization:

THEOREM 4.7. *For a partially ordered set* $(P, \leqslant)$ *of countable cofinality the following assertions are equivalent:*

(i) *P admits a partition into essential directed sets;*
(ii) *P admits a partition into essential indivisible sets;*
(iii) *P has no essentially cofinal subset isomorphic to an ever-branching tree;*
(iv) *P has no essentially cofinal subset that is isomorphic to an ever-branching tree and cofinally faithful in P.*

If these assertions hold, then the partitions in (i) *and* (ii) *are unique up to cofinal equivalence of their parts.*

Proof. The uniqueness statement at the end of the theorem follows from Lemma 4.1. Assertions (i) and (ii) are equivalent by Corollaries 2.6 and 3.6. Since (i) implies (iii) by Lemma 4.6, and (iii) trivially implies (iv), it remains to prove (iv) $\Rightarrow$ (i).

Suppose that (i) fails. Then by Lemma 4.3 the special points of P are not cofinal in P; choose $x_0 \in P$ so that $\lfloor x_0 \rfloor$ contains no special point. Let $X = \{x_0, x_1, \ldots\}$ be a countable cofinal subset of $\lfloor x_0 \rfloor$. Starting with $T_0 := \{x_0\}$, we shall construct a sequence $T_0 \subseteq T_1 \subseteq \cdots$ of finite trees in X that are cofinally faithful in P, and such that $T := \bigcup_{n<\omega} T_n$ is cofinal in X and hence in $\lfloor x_0 \rfloor$. Since $\lfloor x_0 \rfloor$ contains no special points, every point of T will have two incompatible points of P above it, which we can find in T because T is cofinal in $\lfloor x_0 \rfloor$. Deleting from T every point that has only one successor, we thus obtain an ever-branching subtree of T that is cofinal in T and hence in $\lfloor x_0 \rfloor$, and is cofinally faithful in P (because all the T_n are).

Supposing that $T_0, \ldots, T_{n-1}$ have already been constructed, let us construct T_n. If $x_n \in \lceil T_{n-1} \rceil$, let $T_n := T_{n-1}$. If not, then for every $x \in X \cap \lfloor x_n \rfloor$ the points of T_{n-1} to which x is related lie below x, and they form a chain in P because T_{n-1} is cofinally faithful. (The chain is nonempty, because it contains x_0.) Choose $x \in X \cap \lfloor x_n \rfloor$ with this chain maximal; call it C_n and its greatest element t_n. To form T_n we add x to T_{n-1} directly above t_n, i.e. making it greater than every point of C_n but incomparable to every other point of T_{n-1}. Then $x_n \in \lceil T_n \rceil$.

Let us show that T_n is cofinally faithful in P, i.e. that any two incomparable points $t, t' \in T_n$ are incompatible in P. Assuming inductively that T_{n-1} is cofinally faithful, we may assume that t' is our new point x. Then $t \notin C_n$. Now if t and x

have a common upper bound r in P, then there exists an $x' \in X$ with $x' \geqslant r \geqslant x \geqslant x_n$, and the chain C'_n of points in T_{n-1} comparable to (and hence below) x' includes C_n as well as t. This contradicts the maximality of C_n assumed in the choice of x. □

We finish this section with a counterexample to the statement of Theorem 4.7 for posets of uncountable cofinality.

PROPOSITION 4.8. *The direct product $P = T_2 \times \omega_1$ admits no partition into essential directed subsets, and no essentially cofinal subset of P is isomorphic to an ever-branching tree.*

Proof. For the first claim, note that every set of the form $\lfloor(t, \lambda)\rfloor_P$ contains the incompatible points (t', λ) and (t'', λ), where t', t'' are the successors of t in T_2. So no set of this form, and hence no essential subset of P, can be directed.

To prove the second claim, suppose that $T \subseteq P$ is isomorphic to an ever-branching tree and $T \cap A$ is cofinal in A for some essential subset A of P. Pick $a \in A$ with $\lfloor a \rfloor \subseteq A$. Then $T \cap \lfloor a \rfloor$ is again an ever-branching tree cofinal in $\lfloor a \rfloor$. As $\lfloor a \rfloor$ is isomorphic to P, we may thus assume from the start that T is cofinal in P.

Since the projection of T onto the second coordinate of P must be cofinal in ω_1, T must be uncountable. So for some $t_0 \in T_2$ there is an uncountable set $\Lambda \subseteq \omega_1$ such that $(t_0, \lambda) \in T$ for all $\lambda \in \Lambda$. Pick $\lambda_0 \in \Lambda$, and let (t_1, λ_1) and (t_2, λ_2) be distinct successors of (t_0, λ_0) in T. As these successors must be incomparable, we cannot have $t_1 = t_0 = t_2$; we assume that $t_1 > t_0$. Now pick $\lambda_3 \in \Lambda$ with $\lambda_3 > \lambda_1$. Then (t_0, λ_3) and (t_1, λ_1) are incomparable points in T that have the common upper bound (t_1, λ_3) in P, which lies below some $(t^*, \lambda^*) \in T$ because T is cofinal in P. But then (t_0, λ_3) and (t_1, λ_1) lie below (t^*, λ^*) in T and should therefore be comparable, a contradiction. □

The problem of how to characterize the posets of uncountable cofinality that admit a partition into essential directed or indivisible subsets thus remains open. It may be helped however by the following observation, which is more of a reformulation than a material characterization.

PROPOSITION 4.9. *A poset P admits a partition into essential directed subsets if and only if P has an up-closed cofinal subset Q on which compatibility in P is an equivalence relation.*

Proof. Suppose first that P has an up-closed cofinal subset Q on which compatibility in P is an equivalence relation. Then the equivalence classes of Q are essential directed subsets of P whose union is cofinal in P. Applying Corollary 4.4 to their down-closures (which are again essential and directed), we deduce that P decomposes into essential directed subsets.

Conversely, suppose that $P = \bigcup_{i \in I} A_i$ where the A_i are disjoint essential directed subsets of P. By Lemma 2.7, there are points $a_i \in A_i$ such that $A_i \leqslant$

$\lfloor a_i \rfloor_P \subseteq A_i$ for all i. Then the union $Q = \bigcup_{i \in I} \lfloor a_i \rfloor$ of these sets is cofinal in P, and compatibility in P is an equivalence relation on Q with classes $\lfloor a_i \rfloor$, because $\lfloor a_i \rfloor \cap \lfloor a_j \rfloor \subseteq A_i \cap A_j = \emptyset$ for distinct $i, j \in I$. □

5. Decomposing Directed Posets

Having decomposed a given poset into essential directed subsets, one will ask next what can be said about the internal structure of those subsets.

As we observed earlier, directed posets of countable cofinality are indivisible (and equivalent to chains) by Theorem 3.5. A directed poset P of uncountable cofinality, however, may be divisible. (The poset from [1] cited in the Introduction is one example, the direct product $\omega \times \omega_1$ is another.) If we wish to decompose P further into indivisible subsets but rule out trivial decompositions (e.g., into singletons), we have to impose some restrictions; compare the opening paragraphs of Section 4. Requiring the parts to be essential subsets is not an option this time: since our directed poset P cannot contain disjoint subsets $\lfloor a \rfloor$ and $\lfloor b \rfloor$, it cannot contain disjoint essential sets.

One reasonable restriction one might impose on the desired decompositions of P is that the number of parts should be strictly smaller than $\mathrm{cf}(P)$. If P has no infinite antichain, then such a decomposition is indeed possible. As a lemma, we need the following structure theorem of Pouzet [6] for such posets; a proof can be found in [4, Ch. 7].

LEMMA 5.1. *Every directed poset P without infinite antichains has a cofinal subset that is isomorphic to the direct product of finitely many distinct regular cardinals, the largest of which is* $\mathrm{cf}(P)$. (*In particular,* $\mathrm{cf}(P)$ *is regular.*)

The requirement that P contain no infinite antichain even enables us to drop our assumption that P be directed:

THEOREM 5.2. *Let P be a poset of infinite cofinality.*

(i) *If every antichain in P is finite, then P admits a partition into fewer than* $\mathrm{cf}(P)$ *indivisible subsets.*

(ii) *If every antichain in P is countable* (*but not necessarily finite*) *then P need not have such a partition, even if P is directed.*

Proof. (i) By another result of Pouzet (proved in [4, Ch. 4], and independently in [1]), every poset with no infinite antichain can be partitioned into finitely many essential directed subsets. Any part in this decomposition that has finite cofinality has a greatest element, and is therefore indivisible. Replacing our poset P with each of the other parts in turn, we may therefore assume that P is directed.

By Pouzet's theorem (Lemma 5.1), P has a cofinal subset

$$Q = \kappa_1 \times \cdots \times \kappa_n$$

with suitable cardinals $\kappa_1 > \cdots > \kappa_n$. If $n = 1$, then P is indivisible by Theorem 3.5. If $n \geqslant 2$, then Q admits a partition into $\kappa_2 < \kappa_1 = \mathrm{cf}(Q)$ chains:

$$Q = \bigcup_{x \in \kappa_2 \times \cdots \times \kappa_n} \kappa_1 \times \{x\}.$$

Omitting overlaps inductively, we can make the down-closures in P of these chains disjoint. Since Q is cofinal in P, the sets thus obtained (which need no longer be down-closed) partition P. Moreover, each still contains a chain that is cofinal in it, which makes it indivisible by Theorem 3.5.

(ii) Let P be a subset of $\mathbb{R}$ of cardinality $\aleph_1$ in which every point has an uncountable up-closure (in the natural ordering on $\mathbb{R}$). Let $\preccurlyeq$ be the Sierpinski ordering [7] on P, in which $x \preccurlyeq y$ if and only if $x \leqslant y$ both in the natural ordering of $\mathbb{R}$ and in some fixed well-ordering of P of order type ω_1. It is well known and easy to see that every antichain in this ordering is countable, and similarly our assumption on the choice of P ensures that P is directed. Now every chain in P has countable cofinality (consider the natural ordering) and, hence, has a countable down-closure in P (consider the well-ordering). So P is not contained in the down-closure of any countable union of chains in P. (In particular, $\mathrm{cf}(P) = \aleph_1$.) Hence, by Theorem 3.5, P cannot be partitioned into countably many indivisible sets. □

Milner and Prikry [5] showed that directed posets with no antichain of *arbitrary* given cardinality cannot be described as in Pouzet's theorem, i.e. in terms of just a few types of subset one of which would necessarily occur cofinally in every such poset. (But they do obtain such results under stronger set-theoretic assumptions.) Todorcevic [8] studies such 'cofinality types' for posets of cofinality $\aleph_1$; it turns out that, depending on the axioms of set theory assumed, there can be as few as three or as many as $2^{\aleph_1}$ such types.

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