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Spherical Sets Avoiding a Prescribed Set of Angles

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Let X be any subset of the interval $[-1, 1]$. A subset I of the unit sphere in $\mathbb{R}^n$ will be called *X-avoiding* if $\langle u, v\rangle \notin X$ for any $u, v \in I$. The problem of determining the maximum surface measure of a $\{0\}$-avoiding set was first stated in a 1974 note by Witsenhausen; there the upper bound of $1/n$ times the surface measure of the sphere is derived from a simple averaging argument. A consequence of the Frankl–Wilson theorem is that this fraction decreases exponentially, but until now the $1/3$ upper bound for the case $n = 3$ has not moved. We improve this bound to 0.313 using an approach inspired by Delsarte's linear programming bounds for codes, combined with some combinatorial reasoning. In the second part of the paper, we use harmonic analysis to show that, for $n \geq 3$, there always exists an X-avoiding set of maximum measure. We also show with an example that a maximizer need not exist when $n = 2$.

1 Introduction

Witsenhausen [19] presented the following problem: let S^{n-1} be the unit sphere in $\mathbb{R}^n$ and suppose $I \subset S^{n-1}$ is a Lebesgue measurable set such that no two vectors in I are orthogonal. What is the largest possible Lebesgue surface measure of I? Let $\alpha(n)$ denote the supremum of the measures of such sets I, divided by the total measure of S^{n-1}. The first

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upper bounds for $\alpha(n)$ appeared in [19], where Witsenhausen deduced that $\alpha(n) \le 1/n$. Frankl and Wilson [10] proved their powerful combinatorial result on intersecting set systems, and as an application they gave the first exponentially decreasing upper bound $\alpha(n) \le (1+o(1))(1.13)^{-n}$. Raigorodskii [15] improved the bound to $(1+o(1))(1.225)^{-n}$ using a refinement of the Frankl–Wilson method. Gil Kalai conjectured in his weblog [11] that an extremal example is to take two opposite caps, each of geodesic radius $\pi/4$; if true, this implies that $\alpha(n) = (\sqrt{2}+o(1))^{-n}$.

Besides being of independent interest, the above *Double Cap Conjecture* is also important because, if true, it would imply new lower bounds for the measurable chromatic number of Euclidean space, which we now discuss.

Let $c(n)$ be the smallest integer k such that $\mathbb{R}^n$ can be partitioned into sets $X_1, \dots, X_k$, with $\|x-y\|_2 \neq 1$ for each $x, y \in X_i$, $1 \le i \le k$. The number $c(n)$ is called the *chromatic number of* $\mathbb{R}^n$, since the sets $X_1, \dots, X_k$ can be thought of as color classes for a proper coloring of the graph on the vertex set $\mathbb{R}^n$, in which we join two points with an edge when they have distance 1. Frankl and Wilson [10, Theorem 3] showed that $c(n) \ge (1+o(1))(1.2)^n$, proving a conjecture of Erdős that $c(n)$ grows exponentially. Raigorodskii [16] improved the lower bound to $(1+o(1))(1.239)^n$. Requiring the classes $X_1, \dots, X_k$ to be Lebesgue measurable yields the *measurable chromatic number* $c_m(n)$. Clearly, $c_m(n) \ge c(n)$. Remarkably, it is still open if the inequality is strict for at least one n, although one can prove better lower bounds on $c_m(n)$. In particular, the exponent in Raigorodskii's bound was recently beaten by Bachoc et al. [3], who showed that $c_m(n) \ge (1.268+o(1))^n$. If the Double Cap Conjecture is true, then $c_m(n) \ge (\sqrt{2}+o(1))^n$ because, as it is not hard to show, $c_m(n) \ge 1/\alpha(n)$ for every $n \ge 2$. Note that the best known asymptotic upper bound on $c_m(n)$ (as well as on $c(n)$) is $(3+o(1))^n$, by Larman and Rogers [13].

Despite progress on the asymptotics of $\alpha(n)$, the upper bound of $1/3$ for $\alpha(3)$ has not been improved since the original statement of the problem in [19]. Note that the two-cap construction gives $\alpha(3) \ge 1 - 1/\sqrt{2} = 0.2928\cdots$. Our first main result is that $\alpha(3) < 0.313$. The proof involves tightening a Delsarte-type linear programming upper bound (see [2, 5, 7, 8]) by adding combinatorial constraints.

Let $\mathcal{L}$ be the σ-algebra of Lebesgue surface measurable subsets of S^{n-1}, and let λ be the surface measure, for simplicity normalized so that $\lambda\left(S^{n-1}\right) = 1$. For $X \subset [-1, 1]$, a subset $I \subset S^{n-1}$ will be called *X-avoiding* if $\langle \xi, \eta \rangle \notin X$ for all $\xi, \eta \in I$, where $\langle \xi, \eta \rangle$ denotes the standard inner product of the vectors ξ, η. The corresponding extremal problem is to determine

$$\alpha_X(n) := \sup\left\{\lambda(I) : I \in \mathcal{L},\ I \text{ is } X\text{-avoiding}\right\}. \tag{1}$$

For example, if $t \in (-1, 1)$ and $X = [-1, t)$, then $I \subset S^{n-1}$ is X-avoiding if and only if its geodesic diameter is at most $\arccos(t)$. Thus Levy's isodiametric inequality [14] shows that α_X is given by a spherical cap of the appropriate size.

A priori, it is not clear that the value of $\alpha_X(n)$ is actually attained by some measurable X-avoiding set I (so Witsenhausen [19] had to use supremum to define $\alpha(n)$). We prove in Theorem 8.6 that the supremum is attained as a maximum whenever $n \geq 3$. Remarkably, this result holds under no additional assumptions whatsoever on the set X. However, in a sense, only closed sets X matter: our Theorem 9.1 shows that $\alpha_X(n)$ does not change if we replace X by its closure. When $n = 2$, the conclusion of Theorem 8.6 fails; that is, the supremum in (1) need not be a maximum: an example is given in Theorem 3.2.

Besides also answering a natural question, the importance of the attainment result can also be seen through the historical lens: in 1838, Jakob Steiner tried to prove that a circle maximizes the area among all plane figures having some given perimeter. He showed that any non-circle could be improved, but he was not able to rule out the possibility that a sequence of ever-improving plane shapes of equal perimeter could have areas approaching some supremum which is not achieved as a maximum. Only 40 years later in 1879 was the proof completed, when Weierstrass showed that a maximizer must indeed exist.

The layout of the paper will be as follows. In Section 2, we make some general definitions and fix notation. In Section 3, we prove a simple and general proposition giving combinatorial upper bounds for $\alpha_X(n)$; this is basically a formalization of the method used by Witsenhausen [19] to obtain the $\alpha(n) \leq 1/n$ bound. We then apply the proposition to calculate $\alpha_X(2)$ when $|X| = 1$. In Section 5, we deduce linear programming upper bounds for $\alpha(n)$, in the spirit of the Delsarte bounds for binary [7] and spherical [8] codes. We then strengthen the linear programming bound in the $n = 3$ case in Section 6 to obtain the first main result. In Section 8, we prove that the supremum $\alpha_X(n)$ is a maximum when $n \geq 3$, and in Section 9, we show that $\alpha_X(n)$ remains unchanged when X is replaced with its topological closure. In Section 10, we formulate a conjecture generalizing the Double Cap Conjecture for the sphere in $\mathbb{R}^3$, in which other forbidden inner products are considered.

2 Preliminaries

If $u, v \in \mathbb{R}^n$ are two vectors, their standard inner product will be denoted by $\langle u, v\rangle$. All vectors will be assumed to be column vectors. The transpose of a matrix A will be denoted A^t. We denote by $SO(n)$ the group of $n \times n$ matrices A over $\mathbb{R}$ having determinant

1, for which A^tA is equal to the identity matrix. We will think of $SO(n)$ as a compact topological group, and we will always assume its Haar measure is normalized so that $SO(n)$ has measure 1. We denote by S^{n-1} the set of unit vectors in $\mathbb{R}^n$,

$$S^{n-1} = \left\{x \in \mathbb{R}^n : \langle x, x\rangle = 1\right\},$$

equipped with its usual topology. The Lebesgue measure λ on S^{n-1} is always taken to be normalized so that $\lambda(S^{n-1}) = 1$. Recall that the standard surface measure of S^{n-1} is

$$\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \tag{2}$$

where Γ denotes Euler's gamma-function. The Lebesgue σ-algebra on S^{n-1} will be denoted by $\mathcal{L}$. When $(X, \mathcal{M}, \mu)$ is a measure space and $1 \le p < \infty$, we use

$$L^p(X) = \left\{ f : f \text{ is an } \mathbb{R}\text{-valued } \mathcal{M}\text{-measurable function and } \int |f|^p \, \mathrm{d}\mu < \infty \right\}.$$

For $f \in L^p(X)$, we define $\|f\|_p := \left(\int |f|^p \, \mathrm{d}\mu\right)^{1/p}$. Identifying two functions when they agree μ-almost everywhere, we make $L^p(X)$ a Banach space with the norm $\|\cdot\|_p$.

The expectation of a function f of a random variable X will be denoted by $\mathbb{E}_X[f(X)]$, or just $\mathbb{E}[f(X)]$. The probability of an event E will be denoted by $\mathbb{P}[E]$.

When X is a set, we use 1_X to denote its characteristic function; that is $1_X(x) = 1$ if $x \in X$, and $1_X(x) = 0$, otherwise.

If $G = (V, E)$ is a graph, a set I is called *independent* if $\{u, v\} \notin E$ for any $u, v \in I$. The *independence number* $\alpha(G)$ of G is the cardinality of a largest independent set in G. We define $\alpha_X(n)$ as in (1), and for brevity, we let $\alpha(n) = \alpha_{\{0\}}(n)$.

3 Combinatorial Upper Bound

Let us begin by deriving a simple "combinatorial" upper bound for the quantity $\alpha_X(n)$.

Proposition 3.1. Let $n \ge 2$ and $X \subset [-1, 1]$. For a finite subset $V \subset S^{n-1}$, we let $H = (V, E)$ be the graph on the vertex set V with edge set defined by putting $\{\xi, \eta\} \in E$ if and only if $\langle \xi, \eta\rangle \in X$. Then $\alpha_X(n) \le \alpha(H)/|V|$. □

Proof. Let $I \subset S^{n-1}$ be an X-avoiding set, and take a uniform $O \in SO(n)$. Let the random variable Y be the number of $\xi \in V$ with $O\xi \in I$. Since $O\xi \in S^{n-1}$ is uniformly distributed for every $\xi \in V$, we have by the linearity of expectation that $\mathbb{E}(Y) = |V|\lambda(I)$. On the other hand, $Y \le \alpha(H)$ for every outcome O. Thus $\lambda(I) \le \alpha(H)/|V|$. ■

We next use Proposition 3.1 to find the largest possible Lebesgue measure of a subset of the unit circle in $\mathbb{R}^2$ in which no two points lie at some fixed forbidden angle.

Theorem 3.2. Let $X = \{t\}$ and put $\gamma = \frac{\arccos t}{2\pi}$. If γ is rational and $\gamma = p/q$ with p and q coprime integers, then

$$\alpha_X(2) = \begin{cases} 1/2, & \text{if } q \text{ is even,} \\ (q-1)/(2q), & \text{if } q \text{ is odd.} \end{cases}$$

In this case, $\alpha_X(2)$ is attained as a maximum. If γ is irrational, then $\alpha_X(2) = 1/2$, but there exists no measurable X-avoiding set $I \subset S^1$ with $\lambda(I) = 1/2$. □

Proof. Write $\alpha = \alpha_X(2)$, and identify S^1 with the interval $[0, 1)$ via the map $(\cos x, \sin x) \mapsto x/2\pi$. We regard $[0, 1)$ as a group with the operation of addition modulo 1. Note that $I \subset [0, 1)$ is X-avoiding if and only if $I \cap (\gamma + I) = \emptyset$. This implies immediately that $\alpha \leq 1/2$ for all values of t.

Now suppose $\gamma = p/q$ with p and q coprime integers, and suppose that q is even. Let S be any open subinterval of $[0, 1)$ of length $1/q$, and define $T : [0, 1) \to [0, 1)$ by $Tx = x + \gamma \mod 1$. Using the fact that p and q are coprime, one easily verifies that $I = S \cup T^2 S \cup \cdots \cup T^{q-4} S \cup T^{q-2} S$ has measure 1/2. Also I is X-avoiding since $TI = TS \cup T^3 S \cup \cdots \cup T^{q-3} S \cup T^{q-1} S$ is disjoint from I. Therefore $\alpha = 1/2$.

Next suppose that q is odd. With notation as before, a similar argument shows that $I \cup T^2 I \cup \cdots \cup T^{q-3} I$ is an X-avoiding set of measure $(q-1)/(2q)$, and Proposition 3.1 shows that this is largest possible, since the points $x, Tx, T^2 x, \ldots, T^{q-1} x$ induce a q-cycle.

Finally, suppose that γ is irrational. By Dirichlet's approximation theorem, there exist infinitely many pairs of coprime integers p and q such that $|\gamma - p/q| < 1/q^2$. For each such pair, let $\varepsilon = \varepsilon(q) = |\gamma - p/q|$. Using an open interval S of length $\frac{1}{q} - \varepsilon$ and applying the same construction as above with T defined by $Tx = x + p/q$, one obtains an X-avoiding set of measure at least $((q-1)/2)(1/q - \varepsilon) = 1/2 - o(1)$. Alternatively, the lower bound $\alpha \geq 1/2$ follows from Rohlin's tower theorem [12, Theorem 169] applied to the ergodic transformation $Tx = x + \gamma$. Therefore $\alpha = 1/2$.

However, this supremum can never be attained. Indeed, if $I \subset [0, 1)$ is an X-avoiding set with $\lambda(I) = 1/2$ and T is defined by $Tx = x + \gamma$, then $I \cap TI = \emptyset$ and $TI \cap T^2 I = \emptyset$. Since $\lambda(I) = 1/2$, this implies that I and $T^2 I$ differ only on a nullset, contradicting the ergodicity of the irrational rotation T^2. ■

4 Gegenbauer Polynomials and Schoenberg's Theorem

Before proving the first main result, we recall the Gegenbauer polynomials and Schoenberg's theorem from the theory of spherical harmonics. For $\nu > -1/2$, define the *Gegenbauer weight function*

$$r_\nu(t) := \left(1-t^2\right)^{\nu-1/2}, \quad -1 < t < 1.$$

To motivate this definition, observe that if we take a uniformly distributed vector $\xi \in S^{n-1}$, $n \geq 2$, and project it to any given axis, then the density of the obtained random variable $X \in [-1, 1]$ is proportional to $r_{(n-2)/2}$, with the coefficient $\left(\int_{-1}^{1} r_{(n-2)/2}(x)\,\mathrm{d}x\right)^{-1} = \frac{\omega_{n-1}}{\omega_n}$, where ω_n is as in (2). (In particular, X is uniformly distributed in $[-1, 1]$ if $n = 3$.)

Applying the Gram–Schmidt process to the polynomials $1, t, t^2, \ldots$ with respect to the inner product $\langle f, g\rangle_\nu = \int_{-1}^{1} f(t)g(t)r_\nu(t)\,\mathrm{d}t$, one obtains the *Gegenbauer polynomials* $C_i^\nu(t)$, $i = 0, 1, 2, \ldots$, where C_i^ν is of degree i. For a concise overview of these polynomials, see, for example, [4, Section B.2]. Here, we always use the normalization $C_i^\nu(1) = 1$.

For a fixed $n \geq 2$, a continuous function $f : [-1, 1] \to \mathbb{R}$ is called *positive definite*, if for every set of distinct points $\xi_1, \ldots, \xi_s \in S^{n-1}$, the matrix $(f(\langle \xi_i, \xi_j\rangle))_{i,j=1}^{s}$ is positive semidefinite. We will need the following result of Schoenberg [17]. For a modern presentation, see e.g. [4, Theorem 14.3.3].

Theorem 4.1 (Schoenberg's theorem)**.** For $n \geq 2$, a continuous function $f : [-1, 1] \to \mathbb{R}$ is positive definite if and only if there exist coefficients $a_i \geq 0$, for $i \geq 0$, such that

$$f(t) = \sum_{i=0}^{\infty} a_i C_i^{(n-2)/2}(t), \quad \text{for all } t \in [-1, 1].$$

Moreover, the convergence on the right-hand side is absolute and uniform for every positive definite function f. □

For a given positive definite function f, the coefficients a_i in Theorem 4.1 are unique and can be computed explicitly; a formula is given in [4, Equation (14.3.3)].

We are especially interested in the case $n = 3$. Then $\nu = 1/2$, and the first few Gegenbauer polynomials $C_i^{1/2}(x)$ are

$$C_0^{1/2}(x) = 1, \quad C_1^{1/2}(x) = x, \quad C_2^{1/2}(x) = \tfrac{1}{2}\left(3x^2 - 1\right),$$
$$C_3^{1/2}(x) = \tfrac{1}{2}\left(5x^3 - 3x\right), \quad C_4^{1/2}(x) = \tfrac{1}{8}\left(35x^4 - 30x^2 + 3\right).$$

5 Linear Programming Relaxation

Schoenberg's theorem allows us to set up a linear program whose value upper bounds $\alpha(n)$ for $n \geq 3$. The same result appears in [2, 5]; we present a self-contained (and slightly simpler) proof for the reader's convenience. In the next section, we strengthen the linear program, obtaining a better bound for $\alpha(3)$.

Lemma 5.1. Suppose $f, g \in L^2\left(S^{n-1}\right)$ and define $k : [-1, 1] \to \mathbb{R}$ by

$$k(t) := \mathbb{E}[f(O\xi)\, g(O\eta)], \tag{3}$$

where the expectation is taken over randomly chosen $O \in SO(n)$, and $\xi, \eta \in S^{n-1}$ are any two points satisfying $\langle \xi, \eta \rangle = t$. Then, $k(t)$ exists for every $-1 \leq t \leq 1$, and k is continuous. If $f = g$, then k is positive definite. □

Proof. The expectation in (3) clearly does not depend on the particular choice of $\xi, \eta \in S^{n-1}$. Fix any point $\xi_0 \in S^{n-1}$ and let $P : [-1, 1] \to SO(n)$ be any continuous function satisfying $\langle \xi_0, P(t)\xi_0 \rangle = t$ for each $-1 \leq t \leq 1$. We have

$$k(t) = \mathbb{E}[f(O\xi_0)\, g(OP(t)\,\xi_0)]. \tag{4}$$

The functions $O \mapsto f(O\xi_0)$ and $O \mapsto g(OP(t)\xi_0)$ on $SO(n)$ belong to $L^2(SO(n))$; being an inner product in $L^2(SO(n))$, the expectation (4), therefore, exists for every $t \in [-1, 1]$.

We next show that k is continuous. For each $O \in SO(n)$, let $R_O : L^2(SO(n)) \to L^2(SO(n))$ be the *right translation* operator defined by $(R_O F)(O') = F(O'O)$ for $F \in L^2(SO(n))$. For fixed F, the map $O \mapsto R_O F$ is continuous from $L^2(SO(n))$ to $L^2(SO(n))$; see e.g. [6, Lemma 1.4.2]. Therefore the function $t \mapsto R_{P(t)} F$ is continuous from $[-1, 1]$ to $L^2(SO(n))$. Using $F(O) = g(O\xi_0)$, the continuity of k follows.

Now suppose $f = g$; we show that k is positive definite. Let $\xi_1, \ldots, \xi_s \in S^{n-1}$. We need to show that the $s \times s$ matrix $K = (k(\xi_i, \xi_j))_{i,j=1}^s$ is positive semidefinite. But if $v = (v_1, \ldots, v_s)^{\mathrm{T}} \in \mathbb{R}^s$ is any column vector, then

$$v^{\mathrm{T}} K v = \sum_{i=1}^{s} \sum_{j=1}^{s} \mathbb{E}[f(O\xi_i)\, f\left(O\xi_j\right)] v_i v_j = \mathbb{E}\left[\left(\sum_{i=1}^{s} f(O\xi_i)\, v_i\right)^2\right] \geq 0. \qquad \blacksquare$$

Theorem 5.2. $\alpha(n)$ is no more than the value of the following infinite-dimensional linear program:

$$\begin{gathered}\max x_0\\ \sum_{i=0}^{\infty} x_i = 1\\ \sum_{i=0}^{\infty} x_i C_i^{(n-2)/2}(0) = 0\\ x_i \geq 0, \quad \text{for all } i = 0, 1, 2, \ldots.\end{gathered} \tag{5}$$

□

Proof. Let $I \in \mathcal{L}$ be a $\{0\}$-avoiding subset of S^{n-1} with $\lambda(I) > 0$. We construct a feasible solution to the linear program (5) having value $\lambda(I)$. Let $k: [-1, 1] \to \mathbb{R}$ be defined as in (3), with $f = g = 1_I$. Then k is a positive definite function satisfying $k(1) = \lambda(I)$ and $k(0) = 0$. By Theorem 4.1, k has an expansion in terms of the Gegenbauer polynomials:

$$k(t) = \sum_{i=0}^{\infty} a_i C_i^{(n-2)/2}(t), \tag{6}$$

where each $a_i \geq 0$ and the convergence of the right-hand side is uniform on $[-1, 1]$. Moreover, for each fixed $\xi_0 \in S^{n-1}$, we have by Fubini's theorem and (3) that

$$\begin{aligned}\int_{S^{n-1}} k(\langle \xi_0, \eta \rangle)\, \mathrm{d}\eta &= \int_{S^{n-1}} \int_{S^{n-1}} k(\langle \xi, \eta \rangle)\, \mathrm{d}\xi\, \mathrm{d}\eta\\ &= \mathbb{E}\left[\left(\int_{S^{n-1}} 1_I(O\xi)\, \mathrm{d}\xi\right)^2\right] = \lambda(I)^2.\end{aligned} \tag{7}$$

Note that

$$\int_{S^{n-1}} C_i^{(n-2)/2}(\langle \xi_0, \eta \rangle)\, \mathrm{d}\eta = \frac{\omega_{n-1}}{\omega_n} \int_{-1}^{1} C_i^{(n-2)/2}(t) \left(1 - t^2\right)^{(n-3)/2} \mathrm{d}t = 0$$

whenever $i \geq 1$ by the definition of the Gegenbauer polynomials. Putting (6) and (7) together and using that $C_0^{(n-2)/2} \equiv 1$, we conclude that $a_0 = \lambda(I)^2$.

Recalling that $C_i^{(n-2)/2}(1) = 1$ for $i \geq 0$, we find that setting $x_i = a_i/\lambda(I)$ for $i = 0, 1, 2, \ldots$ gives a feasible solution of value $\lambda(I)$ to the linear program (5). ■

Unfortunately, in the case $n=3$, the value of (5) is at least $1/3$, which is the same bound obtained when the problem was first stated in [19]. This can be seen from the feasible solution $x_0 = 1/3$, $x_2 = 2/3$ and $x_i = 0$ for all $i \neq 0, 2$.

6 Adding Combinatorial Constraints

For each $\xi \in S^{n-1}$ and $-1 < t < 1$, let $\sigma_{\xi,t}$ be the unique probability measure on the Borel subsets of S^{n-1} whose support is equal to the set

$$\xi^t := \left\{\eta \in S^{n-1} : \langle \eta, \xi \rangle = t\right\},$$

and which is invariant under all rotations fixing ξ.

Now let $n=3$. As before, let $I \in \mathcal{L}$ be a $\{0\}$-avoiding subset of S^2 and define $k : [-1, 1] \to \mathbb{R}$ as in (3) with $f = g = 1_I$; that is,

$$k(t) = \mathbb{E}[1_I(O\xi)\, 1_I(O\eta)],$$

where $\xi, \eta \in S^2$ satisfy $\langle \xi, \eta \rangle = t$.

Our aim now is to strengthen (5) for the case $n=3$ by adding combinatorial inequalities coming from Proposition 3.1 applied to the sections of S^2 by affine planes. We proceed as follows. Let p and q be coprime integers with $1/4 \le p/q \le 1/2$, and let

$$t_{p,q} = \sqrt{\frac{-\cos(2\pi p/q)}{1 - \cos(2\pi p/q)}}.$$

Let $\xi \in S^2$ be arbitrary. If we take two orthogonal unit vectors with endpoints in $\xi^{t_{p,q}}$ and the center $\xi_0 = t_{p,q}\xi$ of this circle, then we get an isosceles triangle with side lengths $\left(1 - t_{p,q}^2\right)^{1/2}$ and base $\sqrt{2}$; by the Cosine Theorem, the angle at ξ_0 is $2\pi p/q$.

Let $\xi_0, \eta_0 \in S^2$ be arbitrary points satisfying $\langle \xi_0, \eta_0 \rangle = t_{p,q}$. By Fubini's theorem, we have

$$\begin{aligned} k\left(t_{p,q}\right) &= \mathbb{E}[1_I(O\xi_0)\, 1_I(O\eta_0)] = \int_{\xi_0^{t_{p,q}}} \mathbb{E}[1_I(O\xi_0)\, 1_I(O\eta)] \mathrm{d}\sigma_{\xi_0, t_{p,q}}(\eta) \\ &= \mathbb{E}\left[1_I(O\xi_0) \int_{\xi_0^{t_{p,q}}} 1_I(O\eta)\, \mathrm{d}\sigma_{\xi_0, t_{p,q}}(\eta)\right]. \end{aligned}$$

But if q is odd, then $\int_{\xi_0^{t_{p,q}}} 1_I(O\eta)\, \mathrm{d}\sigma_{\xi_0,t_{p,q}}(\eta) \le \frac{q-1}{2q}$ for all $O \in SO(n)$ by Proposition 3.1 applied to the circle $(O\xi_0)^{t_{p,q}} \cong S^1$, since the subgraph it induces contains a cycle of length q. Therefore $k(t_{p,q}) \le \lambda(I)\frac{q-1}{2q}$.

It follows that the inequalities

$$\sum_{i=0}^{\infty} x_i C_i^{1/2}\left(t_{p,q}\right) \leq (q-1)/2q, \tag{8}$$

are valid for the relaxation and can be added to (5). The same holds for the inequalities $\sum_{i=0}^{\infty} x_i C_i^{1/2}(-t_{p,q}) \leq (q-1)/2q$.

So we have just proved the following result.

Theorem 6.1. $\alpha(3)$ is no more than the value of the following infinite-dimensional linear program.

$$\begin{gathered}
\max x_0 \\
\sum_{i=0}^{\infty} x_i = 1 \\
\sum_{i=0}^{\infty} x_i C_i^{1/2}(0) = 0 \\
\sum_{i=0}^{\infty} x_i C_i^{1/2}\left(\pm t_{p,q}\right) \leq (q-1)/2q, \quad \text{for } q \text{ odd, } p, q \text{ coprime} \\
x_i \geq 0, \quad \text{for all } i = 0, 1, 2, \ldots.
\end{gathered} \tag{9}$$

□

Rather than attempting to find the exact value of the linear program (9), the idea will be to discard all but finitely many of the combinatorial constraints, and then to apply the weak duality theorem of linear programming. The dual linear program has only finitely many variables, and any feasible solution gives an upper bound for the value of program (9), and therefore also for $\alpha(3)$. At the heart of the proof is the verification of the feasibility of a particular dual solution which we give explicitly. While part of the verification has been carried out by computer in order to deal with the large numbers that appear, it can be done using only rational arithmetic and can therefore be considered rigorous.

Theorem 6.2. $\alpha(3) < 0.313$. □

Proof. Consider the following linear program:

$$\max\left\{x_0 : \sum_{i=0}^{\infty} x_i = 1,\ \sum_{i=0}^{\infty} x_i C_i^{1/2}(0) = 0,\ \sum_{i=0}^{\infty} x_i C_i^{1/2}\left(t_{1,3}\right) \le 1/3,\right.$$
$$\left.\sum_{i=0}^{\infty} x_i C_i^{1/2}\left(t_{2,5}\right) \le 2/5,\ \sum_{i=0}^{\infty} x_i C_i^{1/2}\left(-t_{2,5}\right) \le 2/5,\ x_i \ge 0, \text{ for all } i = 0, 1, 2, \ldots\right\}. \tag{10}$$

The linear programming dual of (10) is the following.

$$\min b_1 + \frac{1}{3} b_{1,3} + \frac{2}{5} b_{2,5} + \frac{2}{5} b_{2,5-}$$
$$b_1 + b_0 + b_{1,3} + b_{2,5} + b_{2,5-} \ge 1$$
$$b_1 + C_i^{1/2}(0)\, b_0 + C_i^{1/2}\left(t_{1,3}\right) b_{1,3} + C_i^{1/2}\left(t_{2,5}\right) b_{2,5} + C_i^{1/2}\left(-t_{2,5}\right) b_{2,5-} \ge 0 \quad \text{for } i = 1, 2, \ldots$$
$$b_1, b_0 \in \mathbb{R},\ b_{1,3}, b_{2,5}, b_{2,5-} \ge 0. \tag{11}$$

By linear programming duality, any feasible solution for program (6) gives an upper bound for (10), and therefore also for $\alpha(3)$. So in order to prove the claim $\alpha(3) < 0.313$, it suffices to give a feasible solution to (6) having objective value no more than 0.313. Let

$$b = \left(b_1, b_0, b_{1,3}, b_{2,5}, b_{2,5-}\right) = \tfrac{1}{10^6}\,(128614, 404413, 36149, 103647, 327177)\,.$$

It is easily verified that b satisfies the first constraint of (6) and that its objective value is less than 0.313. To verify the infinite family of constraints

$$b_1 + C_i^{1/2}(0)\, b_0 + C_i^{1/2}\left(t_{1,3}\right) b_{1,3} + C_i^{1/2}\left(t_{2,5}\right) b_{2,5} + C_i^{1/2}\left(-t_{2,5}\right) b_{2,5-} \ge 0 \tag{12}$$

for $i = 1, 2, \ldots$, we apply [18, Theorem 8.21.11] (where C_i^{λ} is denoted as $P_i^{(\lambda)}$), which implies

$$|C_i^{1/2}(\cos\theta)| \le \frac{\sqrt{2}}{\sqrt{\pi}\sqrt{\sin\theta}} \frac{\Gamma(i+1)}{\Gamma(i+3/2)} + \frac{1}{\sqrt{\pi}2^{3/2}(\sin\theta)^{3/2}} \frac{\Gamma(i+1)}{\Gamma(i+5/2)} \tag{13}$$

for each $0 < \theta < \pi$. Note that $t_{1,3} = 1/\sqrt{3}$ and $t_{2,5} = 5^{-1/4}$. When $\theta \in A := \{\pi/2, \arccos t_{1,3}, \arccos t_{2,5}, \arccos(-t_{2,5})\}$, we have $\sin\theta \in \left\{1, \sqrt{\frac{2}{3}}, \gamma\right\}$, where $\gamma = \frac{2}{\sqrt{5+\sqrt{5}}}$. The right-hand side of Equation (13) is maximized over $\theta \in A$ at $\sin\theta = \gamma$ for each fixed i, and since the right-hand side is decreasing in i, one can verify using rational arithmetic only that it is no greater than $128614/871386 = b_1/(b_0 + b_{1,3} + b_{2,5} + b_{2,5-})$ when $i \ge 40$, by evaluating at $i = 40$. Therefore,

$$\begin{aligned} & b_1 + C_i^{1/2}(0)\, b_0 + C_i^{1/2}\left(t_{1,3}\right) b_{1,3} + C_i^{1/2}\left(t_{2,5}\right) b_{2,5} + C_i^{1/2}\left(-t_{2,5}\right) b_{2,5-} \\ & \quad \geq b_1 - \left(b_0 + b_{1,3} + b_{2,5} + b_{2,5-}\right) \max_{\theta \in A} \left\{\left|C_i^{1/2}(\cos\theta)\right|\right\} \\ & \quad \geq 0 \end{aligned}$$

when $i \geq 40$. It now suffices to check that b satisfies the constraints (12) for $i = 0, 1, \ldots, 39$. This can also be accomplished using rational arithmetic only. ■

The rational arithmetic calculations required in the above proof were carried out with *Mathematica*. When verifying the upper bound for the right-hand side of (13), it is helpful to recall the identity $\Gamma(i + 1/2) = (i - 1/2)(i - 3/2) \cdots (1/2)\sqrt{\pi}$. When verifying the constraints (12) for $i = 0, 1, \ldots, 39$, it can be helpful to observe that $t_{1,3}$ and $\pm t_{2,5}$ are roots of the polynomials $x^2 - 1/3$ and $x^4 - 1/5$, respectively; this can be used to cut down the degree of the polynomials $C_i^{1/2}(x)$ to at most 3 before evaluating them. The ancillary folder of the `arxiv.org` version of this paper contains a *Mathematica* notebook that verifies all calculations.

The combinatorial inequalities of the form (8) we chose to include in the strengthened linear program (10) were found as follows: let L_0 denote the linear program (5). We first find an optimal solution σ_0 to L_0. We then proceed recursively; having defined the linear program L_{i-1} and found an optimal solution σ_{i-1}, we search through the inequalities (8) until one is found which is violated by σ_{i-1}, and we strengthen L_{i-1} with that inequality to produce L_i. At each stage, an optimal solution to L_i is found by first solving the dual minimization problem, and then applying the complementary slackness theorem from linear programming to reduce L_i to a linear programming maximization problem with just a finite number of variables.

Adding more inequalities of the form (8) appears to give no improvement on the upper bound. Also adding the constraints $\sum_{i=0}^{\infty} x_i C_i^{1/2}(t) \geq 0$ for $-1 \leq t \leq 1$ appears to give no improvement. A small (basically insignificant) improvement can be achieved by allowing the odd cycles to embed into S^2 in more general ways, for instance with the points lying on two different latitudes rather than just one.

7 Adjacency Operator

Let $n \geq 3$. For $\xi \in S^{n-1}$ and $-1 < t < 1$, we use the notations ξ^t and $\sigma_{\xi,t}$ from Section 6. For $f \in L^2(S^{n-1})$ define $A_t f : S^{n-1} \to \mathbb{R}$ by

$$(A_t f)(\xi) := \int_{\xi^t} f(\eta)\, \mathrm{d}\sigma_{\xi,t}(\eta), \quad \xi \in S^{n-1}. \tag{14}$$

Here we establish some basic properties of A_t which will be helpful later. The operator A_t can be thought of as an adjacency operator for the graph with vertex set S^{n-1}, in which we join two points with an edge when their inner product is t. Adjacency operators for infinite graphs are explored in greater detail and generality in [1].

Lemma 7.1. For every $t \in (-1, 1)$, A_t is a bounded linear operator from $L^2(S^{n-1})$ to $L^2(S^{n-1})$ having operator norm equal to 1. □

Proof. The right-hand side of (14) involves integration over nullsets of a function $f \in L^2(S^{n-1})$ which is only defined almost everywhere, and so strictly speaking one should argue that (14) really makes sense. In other words, given a particular representative f from its L^2-equivalence class, we need to check that the integral on the right-hand side of (14) is defined for almost all $\xi \in S^{n-1}$, and that the L^2-equivalence class of $A_t f$ does not depend on the particular choice of representative f.

Our main tool will be Minkowski's integral inequality; see e.g. [9, Theorem 6.19].

Let $e_n = (0, \ldots, 0, 1)$ be the nth basis vector in $\mathbb{R}^n$ and let

$$S = \left\{(x_1, x_2, \ldots, x_n) : x_n = 0,\, x_1^2 + \cdots + x_{n-1}^2 = 1\right\}$$

be a copy of S^{n-2} inside $\mathbb{R}^n$. Considering f as a particular measurable function (not an L^2-equivalence class), we define $F : SO(n) \times S \to \mathbb{R}$ by

$$F(\rho, \eta) = f\left(\rho\left(te_n + \sqrt{1 - t^2}\eta\right)\right), \quad \rho \in SO(n), \quad \eta \in S.$$

Let us formally check all the hypotheses of Minkowski's integral inequality applied to F, where $SO(n)$ is equipped with the Haar measure, and where S is equipped with the normalized Lebesgue measure; this will show that the function $\tilde{F} : SO(n) \to \mathbb{R}$ defined by $\tilde{F}(\rho) = \int_S F(\rho, \eta)\, \mathrm{d}\eta$ belongs to $L^2(SO(n))$.

Clearly, the function F is measurable. To see that the function $\rho \mapsto F(\rho, \eta)$ belongs to $L^2(SO(n))$ for each fixed $\eta \in S$, simply note that

$$\int_{SO(n)} |F(\rho, \eta)|^2\, \mathrm{d}\rho = \int_{SO(n)} \left| f\left(\rho\left(te_n + \sqrt{1 - t^2}\eta\right)\right)\right|^2 \mathrm{d}\rho = \|f\|_2^2.$$

That the function $\eta \mapsto \|F(\cdot,\eta)\|_2$ belongs to $L^1(S)$ then also follows easily (in fact, this function is constant):

$$\int_S \left(\int_{SO(n)} |F(\rho,\eta)|^2 \, d\rho \right)^{1/2} d\eta = \int_S \|f\|_2 \, d\eta = \|f\|_2.$$

Minkowski's integral inequality now gives that the function $\eta \mapsto F(\rho,\eta)$ is in $L^1(S)$ for a.e. ρ, the function $\tilde{F}$ is in $L^2(SO(n))$, and its norm can be bounded as follows:

$$\begin{aligned} \|\tilde{F}\|_2 &= \left(\int_{SO(n)} \left| \int_S F(\rho,\eta) \, d\eta \right|^2 d\rho \right)^{1/2} \\ &\le \int_S \left(\int_{SO(n)} |F(\rho,\eta)|^2 d\rho \right)^{1/2} d\eta = \|f\|_2. \end{aligned} \tag{15}$$

Applying (15) to $f-g$, where g is a.e. equal to f, we conclude that the L^2-equivalence class of $\tilde{F}$ does not depend on the particular choice of representative f from its equivalence class.

Now $(A_t f)(\xi)$ is simply $\tilde{F}(\rho)$, where $\rho \in SO(n)$ can be any rotation such that $\rho e_n = \xi$. This shows that the integral in (14) makes sense for almost all $\xi \in S^{n-1}$.

We have $\|A_t\| \le 1$ since, for any $f \in L^2(S^{n-1})$,

$$\begin{aligned} \|A_t f\|_2 = \left(\int_{S^{n-1}} |(A_t f)(\xi)|^2 \, d\xi \right)^{1/2} &= \left(\int_{SO(n)} |(A_t f)(\rho e_n)|^2 \, d\rho \right)^{1/2} \\ &= \left(\int_{SO(n)} \left| \tilde{F}(\rho) \right|^2 d\rho \right)^{1/2} \le \|f\|_2, \end{aligned}$$

by (15).

Finally, applying A_t to the constant function 1 shows that $\|A_t\| = 1$. ■

Lemma 7.2. Let f and g be functions in $L^2\left(S^{n-1}\right)$, let $\xi, \eta \in S^{n-1}$ be arbitrary points, and write $t = \langle \xi, \eta \rangle$. If $O \in SO(n)$ is chosen uniformly at random with respect to the Haar measure on $SO(n)$, then

$$\int_{S^{n-1}} f(\zeta)\,(A_t g)(\zeta)\, d\zeta = \mathbb{E}[f(O\xi)\, g(O\eta)], \tag{16}$$

which is exactly the definition of $k(t)$ from (3). □

Proof. We have

$$\begin{aligned}\int_{S^{n-1}} f(\zeta)\,(A_t g)(\zeta)\;\mathrm{d}\zeta &= \int_{SO(n)} f(O\xi)\,(A_t g)(O\xi)\,\mathrm{d}O\\ &= \int_{SO(n)} f(O\xi)\int_{(O\xi)^t} g(\psi)\,\mathrm{d}\sigma_{O\xi,t}(\psi)\,\mathrm{d}O,\end{aligned}$$

If H is the subgroup of all elements in $SO(n)$ which fix ξ, then the above integral can be rewritten as

$$\int_{SO(n)} f(O\xi)\int_H g(Oh\eta)\,\mathrm{d}h\,\mathrm{d}O.$$

By Fubini's theorem, this integral is equal to

$$\begin{aligned}\int_H\int_{SO(n)} f(O\xi)\,g(Oh\eta)\,\mathrm{d}O\;\mathrm{d}h &= \int_H\int_{SO(n)} f\left(Oh^{-1}\xi\right)g(O\eta)\,\mathrm{d}O\;\mathrm{d}h\\ &= \int_{SO(n)} f(O\xi)\,g(O\eta)\,\mathrm{d}O,\end{aligned}$$

where we use the right-translation invariance of the Haar integral on $SO(n)$ at the first equality, and the second equality follows by noting that the integrand is constant with respect to h. ■

Lemma 7.3. For every $t\in(-1,1)$, the operator $A_t : L^2\left(S^{n-1}\right)\to L^2\left(S^{n-1}\right)$ is self-adjoint. □

Proof. Fix $\xi,\eta\in S^{n-1}$ that satisfy $\langle\xi,\eta\rangle = t$. Lemma 7.2 implies that, for any f, $g\in L^2\left(S^{n-1}\right)$, we have

$$\langle A_t f, g\rangle = \mathbb{E}_{O\in SO(n)}[f(O\xi)\,g(O\eta)] = \langle f, A_t g\rangle,$$

giving the required. ■

8 Existence of a Measurable Maximum Independent Set

Let $n\geq 2$ and $X\subset[-1,1]$. From Theorem 3.2, we know that the supremum $\alpha_X(n)$ is sometimes attained as a maximum, and sometimes not. It is therefore interesting to ask when a maximizer exists. The main positive result in this direction is Theorem 8.6, which says that a largest measurable X-avoiding set always exists when $n\geq 3$. Remarkably, this result holds under no additional restrictions (not even Lebesgue measurability) on the set X of forbidden inner products. Before arriving at this theorem, we shall need to establish a number of technical results. For the remainder of this section, we suppose $n\geq 3$.

For $d \geq 0$, let $\mathscr{H}_d^n$ be the vector space of homogeneous polynomials $p(x_1, \ldots, x_n)$ of degree d in n variables belonging to the kernel of the Laplace operator; that is,

$$\frac{\partial^2 p}{\partial x_1^2} + \cdots + \frac{\partial^2 p}{\partial x_n^2} = 0.$$

Note that each $\mathscr{H}_d^n$ is finite-dimensional. The restrictions of the elements of $\mathscr{H}_d^n$ to the surface of the unit sphere are called the *spherical harmonics*. For fixed n, we have $L^2(S^{n-1}) = \bigoplus_{d=0}^{\infty} \mathscr{H}_d^n$ [4, Theorem 2.2.2]; that is, each function in $L^2\left(S^{n-1}\right)$ can be written uniquely as an infinite sum of elements from $\mathscr{H}_d^n$, $d = 0, 1, 2, \ldots$, with convergence in the L^2-norm.

Recall the definition (14) of the adjacency operator from Section 7:

$$(A_t f)(\xi) := \int_{\xi^t} f(\eta)\, \mathrm{d}\sigma_{\xi,t}(\eta), \qquad f \in L^2\left(S^{n-1}\right).$$

The next lemma states that each spherical harmonic is an eigenfunction of the operator A_t. It extends the Funk–Hecke formula [4, Theorem 1.2.9] to the Dirac measures, obtaining the eigenvalues of A_t explicitly. The proof relies on the fact that integral kernel operators K having the form $(Kf)(\xi) = \int f(\zeta) k(\langle \zeta, \xi \rangle)\, \mathrm{d}\zeta$ for some function $k : [-1, 1] \to \mathbb{R}$ are diagonalized by the spherical harmonics, and moreover that the eigenvalue of a specific spherical harmonic depends only on its degree.

Proposition 8.1. Let $t \in (-1, 1)$. Then, for every spherical harmonic Y_d of degree d,

$$(A_t Y_d)(\xi) = \int_{\xi^t} Y_d(\eta)\, \mathrm{d}\sigma_{\xi,t}(\eta) = \mu_d(t)\, Y_d(\xi), \qquad \xi \in S^{n-1},$$

where $\mu_d(t)$ is the constant

$$\mu_d(t) = C_d^{(n-2)/2}(t) \left(1 - t^2\right)^{(n-3)/2}. \qquad \square$$

Proof. Let $\mathrm{d}s$ be the Lebesgue measure on $[-1, 1]$ and let $\{f_\alpha\}_\alpha$ be a net of functions in $L^1([-1, 1])$ such that $\{f_\alpha\, \mathrm{d}s\}$ converges to the Dirac point mass δ_t at t in the weak-* topology on the set of Borel measures on $[-1, 1]$. By [4, Theorem 1.2.9], we have

$$\int_{S^{n-1}} Y_d(\eta)\, f_\alpha(\langle \xi, \eta \rangle)\, \mathrm{d}\eta = \mu_{d,\alpha} Y_d(\xi),$$

where

$$\mu_{d,\alpha} = \int_{-1}^{1} C_d^{(n-2)/2}(s) \left(1 - s^2\right)^{(n-3)/2} f_\alpha(s)\, \mathrm{d}s.$$

By taking limits, we complete the proof. ■

Lemma 8.2 is a general fact about weakly convergent sequences in a Hilbert space.

Lemma 8.2. Let $\mathcal{H}$ be a Hilbert space and let $K : \mathcal{H} \to \mathcal{H}$ be a compact operator. Suppose $\{x_i\}_{i=1}^{\infty}$ is a sequence in $\mathcal{H}$ converging weakly to $x \in \mathcal{H}$. Then

$$\lim_{i\to\infty} \langle Kx_i, x_i\rangle = \langle Kx, x\rangle. \qquad \square$$

Proof. Let C be the maximum of $\|x\|$ and $\sup_{i\geq 1} \|x_i\|$, which is finite by the Principle of Uniform Boundedness. Let $\{K_m\}_{m=1}^{\infty}$ be a sequence of finite rank operators such that $K_m \to K$ in the operator norm as $m \to \infty$. Clearly,

$$\lim_{i\to\infty} \langle K_m x_i, x_i\rangle = \langle K_m x, x\rangle$$

for each, $m = 1, 2, \ldots$. Let $\varepsilon > 0$ be given and choose m_0 so that $\|K - K_{m_0}\| < \varepsilon/(3C^2)$. Choosing i_0 so that $|\langle K_{m_0} x_i, x_i\rangle - \langle K_{m_0} x, x\rangle| < \varepsilon/3$ whenever $i \geq i_0$, we have

$$\begin{aligned}
&|\langle Kx_i, x_i\rangle - \langle Kx, x\rangle| \\
&\quad \leq |\langle Kx_i, x_i\rangle - \langle K_{m_0} x_i, x_i\rangle| + |\langle K_{m_0} x_i, x_i\rangle - \langle K_{m_0} x, x\rangle| + |\langle K_{m_0} x, x\rangle - \langle Kx, x\rangle| \\
&\quad \leq \|K - K_{m_0}\| C^2 + \varepsilon/3 + \|K - K_{m_0}\| C^2 < \varepsilon,
\end{aligned}$$

and the lemma follows. ∎

Corollary 8.3, which is also a result stated in [1], says that the adjacency operators A_t are compact when $n \geq 3$.

Corollary 8.3. If $n \geq 3$ and $t \in (-1, 1)$, then A_t is compact. □

Proof. The operator A_t is diagonalizable by Proposition 8.1, since the spherical harmonics form an orthonormal basis for $L^2\left(S^{n-1}\right)$. It therefore suffices to show that its eigenvalues cluster only at 0.

By [18, Theorem 8.21.8] and Proposition 8.1, the eigenvalues $\mu_d(t)$ tend to zero as $d \to \infty$. The eigenspace corresponding to the eigenvalue $\mu_d(t)$ is precisely the vector space of spherical harmonics of degree d, which is finite-dimensional. Therefore A_t is compact. ∎

For each $\xi \in S^{n-1}$, let $C_h(\xi)$ be the open spherical cap of height h in S^{n-1} centered at ξ. Recall that $C_h(\xi)$ has volume proportional to $\int_{1-h}^{1} (1 - t^2)^{(n-3)/2}\, \mathrm{d}t$.

Lemma 8.4. For each $\xi \in S^{n-1}$, we have $\lambda(C_h(\xi)) = \Theta(h^{(n-1)/2})$ and $\lambda(C_{h/2}(\xi)) \geq \lambda(C_h(\xi))/2^{(n-1)/2} - o(h^{(n-1)/2})$ as $h \to 0^+$. □

Proof. If $f(h) = \int_{1-h}^{1} \left(1-t^2\right)^{(n-3)/2} \mathrm{d}t$, then we have $\frac{df}{dh}(h) = \left(2h - h^2\right)^{(n-3)/2}$. Since $f(0) = 0$, the smallest power of h occurring in $f(h)$ is of order $(n-1)/2$. This gives the first result. For the second, note that the coefficient of the lowest order term in $f(h)$ is $2^{(n-1)/2}$ times that of $f(h/2)$. ■

Lemma 8.5. Suppose $n \geq 3$ and let $I \subset S^{n-1}$ be a Lebesgue measurable set with $\lambda(I) > 0$. Define $k\colon [-1,1] \to \mathbb{R}$ by

$$k(t) := \int_{S^{n-1}} 1_I(\zeta)\,(A_t 1_I)(\zeta)\,\mathrm{d}\zeta,$$

which, by Lemma 7.2, is the same as Definition (3) applied with $f = g = 1_I$. If $\xi_1, \xi_2 \in S^{n-1}$ are Lebesgue density points of I, then $k(\langle \xi_1, \xi_2 \rangle) > 0$. □

Proof. Let $t = \langle \xi_1, \xi_2 \rangle$. If $t = 1$, then the conclusion holds since $k(1) = \lambda(I) > 0$. If $t = -1$, then $\xi_2 = -\xi_1$, and by the Lebesgue density theorem we can choose $h > 0$ small enough that $\lambda(C_h(\xi_i) \cap I) > \frac{2}{3}\lambda(C_h(\xi_i))$ for $i = 1, 2$. By Lemma 7.2, we have

$$\begin{aligned} k(-1) &= \mathbb{E}[1_I(O\xi_1)\,1_I(O(-\xi_1))] \\ &\geq \mathbb{E}[1_{I\cap C_h(\xi_1)}(O\xi_1)\,1_{I\cap C_h(\xi_2)}(O(-\xi_1))] \geq \tfrac{1}{3}\lambda\left(C_h(\xi_1)\right). \end{aligned}$$

From now on, we may therefore assume $-1 < t < 1$. Let $h > 0$ be a small number that will be determined later. Suppose $x \in C_h(\xi_1)$. The intersection $x^t \cap C_h(\xi_2)$ is a spherical cap in the $(n-2)$-dimensional sphere x^t having height proportional to h; this is because $C_h(\xi_2)$ is the intersection of S^{n-1} with a certain half-space H, and $x^t \cap C_h(\xi_2) = x^t \cap H$. We have $\sigma_{x,t}\left(x^t \cap C_h(\xi_2)\right) = \Theta\left(h^{(n-2)/2}\right)$ by Lemma 8.4, and it follows that there exists $D > 0$ such that $\sigma_{x,t}\left(x^t \cap C_h(\xi_2)\right) \leq D h^{(n-2)/2}$ for sufficiently small $h > 0$.

If $x \in C_{h/2}(\xi_1)$, then $x^t \cap C_{h/2}(\xi_2) \neq \emptyset$ since x^t is just a rotation of the hyperplane ξ_1^t through an angle equal to the angle between x and ξ_1. Therefore $x^t \cap C_h(\xi_2)$ is a spherical cap in x^t having height at least $h/2$.

Thus there exists $D' > 0$ such that $\sigma_{x,t}\left(x^t \cap C_h(\xi_2)\right) \geq D' h^{(n-2)/2}$ for all $x \in C_{h/2}(\xi_1)$, by Lemma 8.4.

Now choose $h > 0$ small enough that $\lambda(C_h(\xi_i) \cap I) \geq \left(1 - \frac{D'}{2^n D}\right)\lambda(C_h(\xi_i))$ for $i = 1, 2$; this is possible by the Lebesgue density theorem since ξ_1 and ξ_2 are density points. We have by Lemma 7.2 that

$$k(t) = \mathbb{P}[\eta_1 \in I, \eta_2 \in I],$$

if η_1 is chosen uniformly at random from S^{n-1}, and if η_2 is chosen uniformly at random from η_1^t. Then

$$\begin{aligned} k(t) &\geq \mathbb{P}[\eta_1 \in I \cap C_h(\xi_1), \eta_2 \in I \cap C_h(\xi_2)] \\ &\geq \mathbb{P}[\eta_1 \in C_h(\xi_1), \eta_2 \in C_h(\xi_2)] - \mathbb{P}[\eta_1 \in C_h(\xi_1) \setminus I, \eta_2 \in C_h(\xi_2)] \\ &\quad - \mathbb{P}[\eta_1 \in C_h(\xi_1), \eta_2 \in C_h(\xi_2) \setminus I]. \end{aligned}$$

The first probability is at least

$$D' h^{(n-2)/2} \lambda\left(C_{h/2}(\xi_1)\right) \geq \frac{D'}{2^{(n-1)/2}} h^{(n-2)/2} \lambda\left(C_h(\xi_1)\right) - o\left(h^{(2n-3)/2}\right)$$

by Lemma 8.4. The second and third probabilities are each no more than

$$\frac{D'}{2^n D} \lambda\left(C_h(\xi_1)\right) D h^{(n-2)/2} = \frac{D'}{2^n} \lambda\left(C_h(\xi_1)\right) h^{(n-2)/2}$$

for sufficiently small $h > 0$, and therefore, by the first part of Lemma 8.4,

$$k(t) \geq \frac{D'}{2^{(n-1)/2}} \lambda\left(C_h(\xi_1)\right) h^{(n-2)/2} - o\left(h^{(2n-3)/2}\right) - \frac{D'}{2^{n-1}} \lambda\left(C_h(\xi_1)\right) h^{(n-2)/2},$$

and this is strictly positive for sufficiently small $h > 0$. ■

We are now in a position to prove the second main result of this paper.

Theorem 8.6. Suppose $n \geq 3$ and let X be any subset of $[-1, 1]$. Then there exists an X-avoiding set $I \in \mathcal{L}$ such that $\lambda(I) = \alpha_X(n)$. □

Proof. We may suppose that $1 \notin X$ for otherwise every X-avoiding set is empty and the theorem holds with $I = \emptyset$.

Let $\{I_i\}_{i=1}^{\infty}$ be a sequence of measurable X-avoiding sets such that $\lim_{i \to \infty} \lambda(I_i) = \alpha_X(n)$. Passing to a subsequence if necessary, we may suppose that the sequence $\{1_{I_i}\}$ of characteristic functions converges weakly in $L^2\left(S^{n-1}\right)$; let h be its limit. Then $0 \leq h \leq 1$ almost everywhere since $0 \leq 1_{I_i} \leq 1$ for every i.

Denote by I' the set $h^{-1}((0, 1])$, and let I be the set of Lebesgue density points of I'. We claim that I is X-avoiding.

For all $t \in X \setminus \{-1\}$, the operator $A_t : L^2(S^{n-1}) \to L^2(S^{n-1})$ is self-adjoint and compact by Lemma 7.3 and Corollary 8.3. Since $\langle A_t 1_{I_i}, 1_{I_i} \rangle = 0$ for each i, Lemma 8.2 implies

$\langle A_t h, h\rangle = 0$. Since $h \geq 0$, it follows from the definition of A_t that $\langle A_t 1_{I'}, 1_{I'}\rangle = 0$, and therefore also that $\langle A_t 1_I, 1_I\rangle = 0$. But if there exist points $\xi, \eta \in I$ with $t_0 = \langle \xi, \eta\rangle \in X \setminus \{-1\}$, then $\langle A_{t_0} 1_I, 1_I\rangle > 0$ by Lemma 8.5.

Thus, in order to show that I is X-avoiding, it remains to derive a contradiction from assuming that $-1 \in X$ and $-\xi, \xi \in I$ for some $\xi \in S^{n-1}$. Since ξ and $-\xi$ are Lebesgue density points of I, there is a spherical cap C centered at ξ such that $\lambda(I \cap C) > \frac{2}{3}\lambda(C)$ and $\lambda(I \cap (-C)) > \frac{2}{3}\lambda(C)$. The same applies to I_i for all large i (since a cap is a continuity set). But this contradicts the fact that I_i and its reflection $-I_i$ are disjoint for every i. Thus I is X-avoiding.

Finally, we have

$$\lambda(I) = \lambda\left(I'\right) \geq \langle 1_{S^{n-1}}, h\rangle = \lim_{i\to\infty} \langle 1_{S^{n-1}}, 1_{I_i}\rangle = \lim_{i\to\infty} \lambda(I_i) = \alpha_X(n),$$

whence $\lambda(I) = \alpha_X(n)$ since $\lambda(I) \leq \alpha_X(n)$. ■

Note that the proof of Theorem 8.6 would fail for $n=2$, because the adjacency operators A_t need not be compact; the reason for this is that the eigenvalues $\mu_d(t)$ of Proposition 8.1 do not tend to zero as $d \to \infty$.

9 Invariance of $\alpha_X(n)$ Under Taking the Closure of X

Again let $n \geq 2$ and $X \subset [-1, 1]$. We will use $\bar{X}$ to denote the topological closure of X in $[-1, 1]$. In general it is false that X-avoiding sets are $\bar{X}$-avoiding. In spite of this, we have the following result.

Theorem 9.1. Let X be an arbitrary subset of $[-1, 1]$. Then $\alpha_X(n) = \alpha_{\bar{X}}(n)$. In particular $\alpha_X(n) = 0$ if $1 \in \bar{X}$. □

Proof. Clearly $\alpha_X(n) \geq \alpha_{\bar{X}}(n)$. For the reverse inequality, let $I' \subset S^{n-1}$ be any measurable X-avoiding set. Let $I \subset I'$ be the set of Lebesgue density points of I', and define $k\colon [-1, 1] \to \mathbb{R}$ by $k(t) = \int_{S^{n-1}} 1_I(\zeta)(A_t 1_I)(\zeta)\,\mathrm{d}\zeta$. Then k is continuous by Lemmas 5.1 and 7.2, and since $k(t) = 0$ for every $t \in X$, it follows that $k(t) = 0$ for every $t \in \bar{X}$. Lemma 8.5 now implies that I is $\bar{X}$-avoiding. The theorem now follows since I' was arbitrary, and $\lambda(I) = \lambda(I')$ by the Lebesgue density theorem. ■

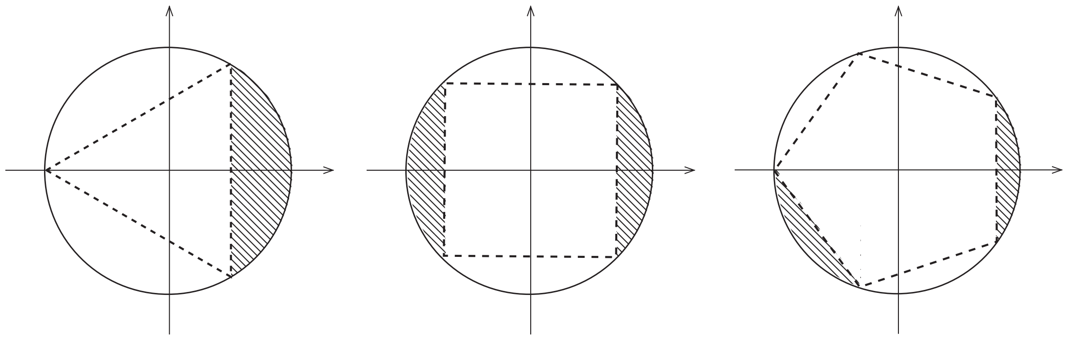

Fig. 1. $\{t\}$-Avoiding set for $t = -\frac{1}{2}$, 0 and $\cos\frac{2\pi}{5}$.

10 Single Forbidden Inner Product

An interesting case to consider is when $|X| = 1$, motivated by the fact that $1/\alpha_{\{t\}}(n)$ is a lower bound on the measurable chromatic number of R^n for any $t \in (-1, 1)$ and this freedom of choosing t may lead to better bounds.

Let us restrict ourselves to the special case when $n = 3$ (that is, we look at the two-dimensional sphere). For a range of $t \in [-1, \cos\frac{2\pi}{5}]$, the best construction that we could find consists of one or two spherical caps as follows. Given t, let h be the maximum height of an open spherical cap that is $\{t\}$-avoiding. A simple calculation shows that $h = 1 - \sqrt{(t+1)/2}$. If $t \leq -1/2$, then we just take a single cap C of height h, which gives that $\alpha_{\{t\}}(3) \geq h/2$ then. When $-1/2 < t \leq 0$, we can add another cap C' whose center is opposite to that of C. When t reaches 0, the caps C and C' have the same height (and we get the two-cap construction from Kalai's conjecture). When $0 < t \leq \frac{2\pi}{5}$, we can form a $\{t\}$-avoiding set by taking two caps of the same height h. (Note that the last construction cannot be optimal for $t > \frac{2\pi}{5}$, as then the two caps can be arranged, so that a set of positive measure can be added; see the third picture of Figure 1.)

Calculations show that the above construction gives the following lower bound (where $h = 1 - \sqrt{(t+1)/2}$):

$$\alpha_{\{t\}}(3) \geq \begin{cases} \dfrac{h}{2}, & -1 \leq t \leq -\dfrac{1}{2}, \\ h + t - ht, & -\dfrac{1}{2} \leq t \leq 0, \\ h, & 0 \leq t \leq \cos\dfrac{2\pi}{5}. \end{cases} \tag{17}$$

We conjecture that the bounds in (17) are all equalities. In particular, our conjecture states that, for $t \leq -1/2$, we can strengthen Levy's isodiametric inequality by forbidding a single inner product t instead of the whole interval $[-1, t]$.

As in Section 6, one can write an infinite linear program that gives an upper bound on $\alpha_{\{t\}}(3)$. Although our numerical experiments indicate that the upper bound given by the LP exceeds the lower bound in (17) by at most 0.062 for all $-1 \leq t \leq 0.3$, we were not able to determine the exact value of $\alpha_{\{t\}}(3)$ for any single $t \in (-1, \cos \frac{2\pi}{5}]$.

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