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Local version of Vizing's theorem for multigraphs

Clinton T. Conley[1] | Jan Grebík[2,3] | Oleg Pikhurko[4]

[1]Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, Pennsylvania, USA

[2]Department of Mathematics, UCLA, Los Angeles, California, USA

[3]Faculty of Informatics, Masaryk University, Brno, Czech Republic

[4]Mathematics Institute and DIMAP, University of Warwick, Coventry, UK

Correspondence
Oleg Pikhurko, Mathematics Institute and DIMAP, University of Warwick, Coventry CV4 7AL, UK.
Email: o.pikhurko@warwick.ac.uk

Abstract

Extending a result of Christiansen, we prove that every multigraph $G = (V, E)$ admits a proper edge colouring $\phi : E \to \{1, 2, \dots\}$ which is *local*, that is, $\phi(e) \leqslant \max\{d(x) + \pi(x), d(y) + \pi(y)\}$ for every edge e with end-points $x, y \in V$, where $d(z)$ (resp. $\pi(z)$) denotes the degree of a vertex z (resp. the maximum edge multiplicity at z). This is derived from a local version of the Fan Equation.

KEYWORDS

edge colouring, local colouring, multigraphs, Vizing's theorem

1 | INTRODUCTION

Edge-colouring is an important and active area of graph theory; for an overview see, for example, the book by Stiebitz et al. [8]. One of the key results here is the remarkable theorem of Vizing [9], proved independently by Gupta [6], that every multigraph G admits a proper edge colouring with at most $\Delta(G) + \pi(G)$ colours, where $\Delta(G)$ and $\pi(G)$ denote respectively the maximum degree and the maximum edge multiplicity of G, and an edge colouring is called *proper* if no two different edges sharing a vertex get the same colour.

Some local versions of edge colouring problems (where a possible colour of an edge xy depends on local information such as the degrees of x and y rather than the global

maximum degree) were introduced already in the influential paper of Erdős et al. [5]. One of the strongest asymptotic results here is by Bonamy et al. [1] giving the following local list version of Vizing's theorem: for every $\varepsilon > 0$ if the maximum degree $\Delta(G)$ of a graph G is sufficiently large, the minimum degree $\delta(G)$ is at least $\ln^{25} \Delta(G)$ and we have an assignment of colour lists to edges $e \mapsto L(e)$ so that each edge $\{x, y\}$ gets at least $(1 + \varepsilon)\max\{d(x), d(y)\}$ colours then there is a proper edge colouring ϕ of G with $\phi(e) \in L(e)$ for each edge e, where $d(x)$ denotes the degree of a vertex x.

Resolving a conjecture posed in [1], Christiansen [3, 4] proved that every simple graph G admits a proper edge colouring into $\mathbb{N} := \{1, 2, 3, ...\}$ which is *local*, meaning that the colour of any edge $\{x, y\}$ is at most $\max\{d(x), d(y)\} + 1$. In the special case when the graph G is biparite, the stronger conclusion that the colour of $\{x, y\}$ is at most $\max\{d(x), d(y)\}$ (which may be called the *local König Theorem*) follows from the more general results of Borodin et al. [2].

The purpose of this note is to observe that the proof of Christiansen extends to multigraphs. The following definition seems to be the 'right' one in this context: let us call an $\mathbb{N}$-valued edge colouring ϕ of a multigraph *local* if for every edge e with endpoints x and y it holds that

$$\phi(e) \leqslant \max\{d(x) + \pi(x), d(y) + \pi(y)\}, \tag{1}$$

where $\pi(z)$ denotes the maximum edge multiplicity at a vertex z. Here is a local version of Vizing's theorem for multigraphs.

Theorem 1. *Every multigraph admits a proper local edge colouring.*

The standard example (a 3-vertex multigraph where every pair has the same multiplicity) shows that one cannot decrease the right-hand size of (1) in general. Note that, even without the extra locality restriction, we do not know the minimum number of colours as a function of Δ and π that suffices for proper edge colouring of every multigraph with maximum degree Δ and maximum edge multiplicity π; see Scheide and Stiebitz [7] for our current knowledge of this question.

The original Vizing's theorem as well as some other edge colouring results can be derived from the so-called Fan Equation of Vizing [10] (which is discussed in detail in [8, Section 2]). Here we present a local version of the Fan Equation (which, in this more general setting, becomes an inequality, see Theorem 4) and derive Theorem 1 from it. As another consequence of Theorem 4, we have the following result.

Theorem 2. *Let $G = (V, E)$ be a simple graph and $k \in \mathbb{N}$. If the maximum degree of G is at most k and the set of vertices of degree exactly k spans no cycle in G, then there is a proper local colouring $E \to \{1, ..., k\}$.*

2 | NOTATION AND PRELIMINARIES

By $\mathbb{N} := \{1, 2, ...\}$ we denote the set of positive integers. For integers $k \geqslant \ell \geqslant 0$, we denote $[\ell, k] := \{\ell, ..., k\}$ and $[k] := \{1, ..., k\}$.

Let $G = (V, E, r)$ be a *multigraph*, that is, V and E are the sets of *vertices* and *edges* respectively, and r is a (not necessarily injective) function from E to $\binom{V}{2}$, the set of unordered

pairs of vertices. The *(edge) multiplicity* $\pi(x, y) := |r^{-1}(\{x, y\})|$ of a pair $\{x, y\} \in \binom{V}{2}$ is the number of edges whose end-points are x and y. For $x, y \in V$ and $e \in E$, we write $x \in e$ (resp. $e = xy$) to mean that $x \in r(e)$ (resp. $r(e) = \{x, y\}$).

The *degree* of $x \in V$ is $d(x) := \sum_{y \in V \setminus \{x\}} \pi(x, y)$ which is the number of edges incident to x. Also, let $\pi(x) := \max_{y \in V \setminus \{x\}} \pi(x, y)$ denote the maximum multiplicity at x.

As only edges will be coloured, we will usually say just 'colouring' instead of 'edge colouring'. A *partial colouring* of G is a function $\phi : \mathrm{dom}(\phi) \to \mathbb{N}$ where the domain $\mathrm{dom}(\phi)$ of ϕ is a subset of E. We may also write this as $\phi : E \rightharpoonup \mathbb{N}$. If $\mathrm{dom}(\phi) = E$ (i.e., every edge is coloured) then we call ϕ a *colouring* (and write $\phi : E \to \mathbb{N}$). A (partial) colouring ϕ is *proper* if no two distinct edges sharing at least one vertex get the same colour. It is called *local* if

$$\phi(e) \leqslant \max\{d(x) + \pi(x) : x \in e\}, \text{ for every } e \in \mathrm{dom}(\phi), \tag{2}$$

that is, for every coloured edge e there is $x \in e$ with $\phi(e) \leqslant d(x) + \pi(x)$.

Given a partial colouring ϕ, a *ϕ-chain* is a sequence of distinct edges $C = (e_1, ..., e_p)$ such that $\{e_1, ..., e_p\} \cap \mathrm{dom}(\phi) = \{e_2, ..., e_p\}$ (i.e., e_1 is the only uncoloured edge) and, for every $i \in [p-1]$, the edges e_i and e_{i+1} share exactly one vertex. The *C-shift* (or *shift along C*) of ϕ is the partial colouring ϕ' with $\mathrm{dom}(\phi') = (\mathrm{dom}(\phi) \setminus \{e_p\}) \cup \{e_1\}$ which coincides with ϕ, except $\phi'(e_i) := \phi(e_{i+1})$ for $i \in [p-1]$ (while e_p is uncoloured under ϕ'). Informally speaking, we shift colours one step down along C. A ϕ-chain (e, e') with $e = xy$ and $e' = xz$ (thus, e is uncoloured, e' is coloured and $y \neq z$) is called *ϕ-safe* if $\phi(e') \leqslant d(y) + \pi(y)$ and no edge incident to y has colour $\phi(e')$. A ϕ-chain $C = (e_1, ..., e_m)$ with $m \geqslant 3$ is *ϕ-safe* if, for every $i \in [m-1]$, the ϕ_i-chain (e_i, e_{i+1}) is ϕ_i-safe, where ϕ_i denotes the $(e_1, ..., e_i)$-shift of ϕ. In other words, C is ϕ-safe if each individual 1-edge step of the C-shift is safe with respect to the current colouring. Note that the sequence $C = (e_1)$ made of a single edge e_1 uncoloured under ϕ is a ϕ-chain (and its shift keeps the partial colouring ϕ unchanged); also, let us agree that every such single-edge chain is ϕ-safe. When the colouring ϕ is understood, we may say just 'chain' instead of 'ϕ-chain', and so on.

For a vertex $x \in V$ and distinct colours $\alpha, \beta \in \mathbb{N}$ with β not present at x, let $P_x(\alpha, \beta, \phi)$ denote the sequence of edges on the maximal (α, β)-bichromatic path that starts at the vertex x. Thus the values of ϕ on $P_x(\alpha, \beta, \phi)$ alternate between α and β, starting with α; if the colour α is not present at x, then $P_x(\alpha, \beta, \phi)$ is the empty sequence. Note that we exclude the case that $P_x(\alpha, \beta, \phi)$ is a double edge by requiring that β is missing at x in this definition.

In some proofs, we will be given an integer k and the set of possible colours will be restricted to $[k]$. Then we will use the following definitions that implicitly depend on k. Given $\phi : E \rightharpoonup [k]$, we let for a vertex $x \in V$

$$\bar{\phi}(x) := [k] \setminus \{\phi(e) : e \ni x\},$$

to be the set of colours *missing* at x, and

$$\widetilde{\phi}(x) := \overline{\phi}(x) \cap [d(x) + \pi(x)],$$

to be the set of colours which are *safe* at x. Thus a colour is safe at x if it is missing at x and is at most $d(x) + \pi(x)$. Also, the *potential (function)* of $\phi : E \rightharpoonup [k]$ is

$$\Phi(\phi) := \sum_{x \in V} |\widetilde{\phi}(x)|.$$

Lemma 3. *Let $G = (V, E, ß)$ be a multigraph, $k \in \mathbb{N}$ be a positive integer, $\phi : E \rightharpoonup [k]$ be a proper local partial colouring, and ϕ' be the shift of ϕ along a ϕ-safe chain (e, e'), with $e = xy$ and $e' = xz$. Then ϕ' is a proper local partial colouring and $\Phi(\phi') \leqslant \Phi(\phi)$. Moreover, we have equality if and only if $\phi(e') \leqslant d(z) + \pi(z)$.*

Proof. The edge $e = xy$ gets coloured during the shift. The new colouring ϕ' is proper because the set of colours at x does not change while the new colour on $e = xy$ was missing at y. Also, the locality condition for e holds from the y-side, that is, $\phi'(e) = \phi(e') \leqslant d(y) + \pi(y)$.

Consider the difference $\Phi(\phi') - \Phi(\phi)$. Since the contributions of a vertex that sees the same sets of colours on incident edges in ϕ and ϕ' cancel each other, we have to look at y and z only. The contribution of y to $\Phi(\phi') - \Phi(\phi)$ is exactly -1, because the shift reduces the number of safe colours at y by 1 (namely, $\phi(e')$ is now gone from this list). The contribution of z to $\Phi(\phi') - \Phi(\phi)$ is at most 1, and it is equal to 1 if and only if the colour moved out from z is in $[d(z) + \pi(z)]$, that is, if and only if $\phi(e') \leqslant d(z) + \pi(z)$. This finishes the proof of the lemma. □

3 | MULTIFANS

Let $k \in \mathbb{N}$ and a multigraph $G = (V, E, ß)$ be given. Let $\phi : E \rightharpoonup [k]$ be any proper local partial colouring. Let $e \in E$ be an uncoloured edge and let $x \in e$.

A *multifan at x with respect to e and ϕ (and k)* is a sequence $F = (e_1, y_1, ..., e_p, y_p)$ with $p \geqslant 1$ consisting of edges $e_1, ..., e_p$ and vertices $y_1, ..., y_p$ satisfying the following conditions.

(F1) The edges $e_1, ..., e_p$ are distinct, $e_1 = e$, and $e_i = xy_i$ for $i \in [p]$.
(F2) For every $i \in [2, p]$ there is $j \in [i - 1]$ such that $\phi(e_i) \in \widetilde{\phi}(y_j)$; in particular, e_i is coloured by ϕ.

Also, we denote $V(F) := \{y_1, ..., y_p\}$. Note that $V(F)$ does not include x.

Usually, the Fan Equation is stated for an *edge critical* multigraph (when a desired colouring of the whole edge set exists when we remove any edge). Having in mind possible algorithmic and descriptive set theory applications, we state a version where the multigraph G we want to colour need not be edge critical and the presented result can be used to 'improve' the current partial colouring using chains of special kind. With this in mind, we make the following definitions.

A chain $C = (e_1, ..., e_p)$ is *improving* if it is ϕ-safe and the C-shift ϕ' of ϕ has strictly smaller potential, or there is a way to extend ϕ' to e_p keeping ϕ' proper and local (in particular, strictly decreasing the potential). Note that the new proper local colouring ϕ' satisfies $\Phi(\phi') < \Phi(\phi)$ in either case. We say that a chain C is *Vizing* if it starts with some edges $f_1, ..., f_q, q \geqslant 1$, all containing the same vertex z and then continues with a (possibly empty) initial segment of a bichromatic path starting at y, where $y \neq z$ is the other endpoint of f_q. (Some further restrictions can be put on possible chains arising from the proof of Theorem 4 but we would like to keep this definition simple and short.) Also, for $z \in V$ let $d_\phi(z) := |\{e \in \text{dom}(\phi) : e \ni z\}|$ denote the number of edges at z that are assigned a colour by ϕ.

Theorem 4 (Local Fan Inequality). *Let $k \geqslant 2$, let $G = (V, E, r)$ be a multigraph with $\Delta(G) \leqslant k$, and let $\phi : E \rightharpoonup [k]$ be a partial proper local colouring that admits no improving*

Vizing chain. Suppose that $e \in E$ is uncoloured. Let $x \in e$ and let $F = (e_1, y_1, \ldots, e_p, y_p)$ be a multifan at x with respect to e and ϕ. Then all of the following properties hold.

(a) *For every $i \in [p]$ there are $m \geq 0$ and a sequence $i_0 > i_1 > \cdots > i_m$ such that $i_0 = i$, $i_m = 1$ and $(e_{i_m}, e_{i_{m-1}}, \ldots, e_{i_0})$ is a ϕ-safe chain.*

(b) *For every $i \in [p]$, we have*

$$\overline{\phi}(x) \cap \overline{\phi}(y_i) \cap [\max\{d(x) + \pi(x), d(y_i) + \pi(y_i)\}] = \varnothing. \tag{3}$$

(c) *For all choices of $i \in [p], \alpha \in \widetilde{\phi}(x)$, and $\beta \in \widetilde{\phi}(y_i)$, the (α, β)-bichromatic path $P_{y_i}(\alpha, \beta, \phi)$ from y_i ends in x.*

(d) *For every $i, j \in [p]$, if $y_i \neq y_j$ then*

$$\widetilde{\phi}(y_i) \cap \widetilde{\phi}(y_j) = \varnothing. \tag{4}$$

(e) *If the multifan F is maximal then $|\{y_1, \ldots, y_p\}| \geq 2$ and*

$$\sum_{z \in V(F)} (\pi_F(x, z) - \min\{k, d(z) + \pi(z)\} + d_\phi(z)) \geq 1. \tag{5}$$

(Here π_F denotes the edge multiplicity in the multigraph F, while $\pi = \pi_G$ is taken with respect to the whole multigraph G.)

Proof. To prove Part (a) for any given $i \in [p]$, we construct a required sequence as follows. Initially, let $i_0 := i$ and $m := 0$. If the current i_m is equal to 1 then stop. Otherwise let $i_{m+1} \in [i_m - 1]$ be any index satisfying (F2) for i_m (i.e., $\phi(e_{i_m}) \in \widetilde{\phi}(e_{i_{m+1}})$), increase m by 1, and repeat. Let $i_0 > i_1 > \cdots > i_m$ be the final sequence (with $i_0 = i$, $i_m = 1$, and $m \geq 0$). Clearly, $(e_{i_m}, \ldots, e_{i_0})$ is a chain.

Although this is intuitively obvious, let us formally check that the chain $(e_{i_m}, \ldots, e_{i_0})$ is ϕ-safe. This vacuously holds if $m = 0$, so assume that $m \geq 1$. Let $\phi_1 := \phi$ and, inductively for $j \in [m]$, let ϕ_{j+1} be the $(e_{i_{m-j+1}}, e_{i_{m-j}})$-shift of ϕ_j; equivalently, ϕ_{j+1} is obtained from ϕ by shifting along $(e_{i_m}, \ldots, e_{i_{m-j}})$. We show by induction on $j \in [m]$ that the ϕ_j-chain $(e_{i_{m-j+1}}, e_{i_{m-j}})$ is ϕ_j-safe and the partial colouring ϕ_{j+1} is proper and local. So take any $j \in [m]$. We know that ϕ_j is proper and local (by induction if $j \geq 2$ and by $\phi_1 = \phi$ if $j = 1$). Note that $\phi_j(e_{i_{m-j}}) = \phi(e_{i_{m-j}})$ (since, by (F1), the edge $e_{i_{m-j}}$ is distinct from any edge whose colour changes when we construct ϕ_j from $\phi_1 = \phi$). By the choice of i_{m-j+1} (i.e., by (F2)), $\phi(e_{i_{m-j}})$ is in $\widetilde{\phi}(y_{i_{m-j+1}})$. Since all colours at x are pairwise distinct, the colour $\phi(e_{i_{m-j}})$ cannot appear at $y_{i_{m-j+1}}$ when we pass from $\phi_1 = \phi$ to ϕ_j. Thus $\phi_j(e_{i_{m-j}}) = \phi(e_{i_{m-j}})$ is also in $\widetilde{\phi}_j(y_{i_{m-j+1}})$ and the ϕ_j-chain $(e_{i_{m-j+1}}, e_{i_{m-j}})$ is ϕ_j-safe. It follows that the partial colouring ϕ_{j+1} is proper and local by Lemma 3, as required.

Suppose that Part (b) is false. Let $i \in [p]$ be the smallest index violating it. Let α be a colour present in the left-hand side of (3). Let $(i_0, \ldots, i_m)$ be the sequence returned by Part (a) for this index i. Let ϕ' be the shift of ϕ along $(e_{i_m}, e_{i_{m-1}}, \ldots, e_{i_0})$. By Lemma 3, the new colouring ϕ' is proper and local (and satisfies $\Phi(\phi') \leq \Phi(\phi)$). Under ϕ', the edge $e_i = xy_i$ is uncoloured; also, the colour α is missing at both x and y_i. Indeed, α was missing at both x

and y_i before the shift (i.e., under ϕ) while the shift, that affects only edges at x, does not move colour α at all.

Obtain ϕ'' from ϕ' by colouring e_i with the colour α. By the above arguments, the new colouring ϕ'' is still proper. By definition, α is at most $\max\{d(x) + \pi(x), d(y_i) + \pi(y_i)\}$, so ϕ'' is also local. Thus $(e_{i_m}, ..., e_{i_0})$ is an improving Vizing chain for ϕ. This contradiction proves Part (b).

Let us turn to Part (c). Suppose that the claim is false for some vertex y_i and colours α and β. Choose the smallest possible such index $i \in [p]$. Recall that β is missing at y_i, $P := P_{y_i}(\alpha, \beta, \phi)$ is the (α, β)-bichromatic path starting at y_i, and we assumed on the contrary that the final endpoint of P is a vertex $x' \neq x$. The vertex set $V(P)$ of P does not contain x, as otherwise x would be an endpoint (since α is missing at x). It follows that none of the edges incident to x (in particular, none of $e_1, ..., e_p$) can belong to P. Let $i_0 = i > \cdots > i_m = 1$ be the sequence returned by Part (a) for this i and let ϕ' be the shift of ϕ along the chain $(e_{i_m}, ..., e_{i_0})$. By Lemma 3, the partial colouring ϕ' is proper and local, and satisfies $\Phi(\phi') \leqslant \Phi(\phi)$. Note that the edge e_i is not coloured by ϕ'.

Let the path P traverse edges $(f_1, ..., f_\ell)$ and vertices $(u_1, u_2, ..., u_{\ell+1})$ in this order. Thus $u_1 = y_i$ and $f_s = u_s u_{s+1}$ for every $s \in [\ell]$. By Part (b), the colour α is present at y_i under ϕ and also under ϕ' (since the other endpoint of the colour-α edge at y cannot be x). Thus $\ell \geqslant 1$. Let $f_0 := e_i$ denote the edge coming before the path P in our shifting procedure. Let $P' := (f_0, ..., f_\ell)$. It is a path (since $V(P) \not\ni x$) and a ϕ'-chain (since $f_0 \notin \text{dom}(\phi')$). Also, let $u_0 := x$ (so that $f_0 = u_0 u_1$) and let C be the concatenation of $(e_{i_m}, ..., e_{i_0})$ and P. Clearly, C is a Vizing chain.

Observe that if we were interested in only proper edge-colourings (i.e., not requiring that the extra locality property in (2) holds), then we could have taken the P'-shift of ϕ' and assign one of α or β to the (now uncoloured) edge f_ℓ, thus finding a colouring with strictly larger domain than ϕ. Instead, we proceed as follows.

Starting with ϕ', we iteratively apply the (f_s, f_{s+1})-shifts for $s = 0, 1, ..., \ell - 1$ as long as each of them is safe. Lemma 3 implies that the potential Φ cannot increase at any step. Thus Φ stays constant by our assumptions on ϕ.

Suppose first that we cannot perform the above shift for some $s \leqslant \ell - 1$, that is, before doing the whole chain P'. Denote the obtained colouring ϕ''. Thus ϕ'' is the $(f_0, ..., f_s)$-shift of ϕ', where s is the maximum index (assumed to be at most $\ell - 1$) such that $(f_0, ..., f_s)$ is a ϕ'-safe chain. Note that $s \geqslant 1$ since the first shift (along (e_i, f_1)) moves the colour α to $e_i = xy_i$ and cannot violate the locality condition at x by $\alpha \leqslant d(x) + \pi(x)$. Since the (f_s, f_{s+1})-shift is not safe in the current colouring ϕ'' but results in a proper colouring (since we shift along a bichromatic path), we have that

$$\gamma := \phi''(f_{s+1}) > d(u_s) + \pi(u_s). \tag{6}$$

Suppose first that $s \geqslant 2$. Consider the (f_{s-2}, f_{s-1})-shift (i.e., the penultimate one before we obtained ϕ''). This shift moves the colour from $f_{s-1} = u_{s-1} u_s$ to $f_{s-2} = u_{s-2} u_{s-1}$, with the moved colour being γ. Thus by (6) and the last claim of Lemma 3, the (f_{s-2}, f_{s-1})-shift strictly decreased the value of the potential, while the chain C truncated at f_{s-1} is safe and thus improving, a contradiction. So, suppose that $s = 1$. Here when we shift the colouring ϕ' along the path P', we perform the first safe shift (f_0, f_1) but we cannot proceed with the second shift, namely along (f_1, f_2). However, then we have that $u_s = y_i$ and $\gamma = \beta$ in (6), contradicting the assumption of Part (c) that β lies in $\widetilde{\phi}(y_i) \subseteq [d(y_i) + \pi(y_i)]$.

Thus, we can assume that the whole chain P' is ϕ'-safe (and thus C is ϕ-safe). We shift ϕ' all way along it to obtain ϕ''; equivalently, ϕ'' is the C-shift of ϕ. The edge f_ℓ is uncoloured by ϕ''. Let γ be the element of $\{\alpha, \beta\}$ different from $\phi''(f_{\ell-1}) = \phi'(f_\ell)$. Recall that f_ℓ is the last edge of the maximal α/β-bichromatic path P with respect to ϕ that starts with y_i. Clearly, the colour γ is missing at the penultimate vertex u_ℓ under ϕ''.

Let us show that γ cannot occur at $u_{\ell+1}$ under ϕ''. Suppose otherwise. As γ is not present at $u_{\ell+1}$ under ϕ (by the maximality of P) but is present at $u_{\ell+1}$ under ϕ', it must be the case that $\gamma = \beta$ and $u_{\ell+1} = y_{i_j}$ for some $j \in [m]$ (and the colour β appeared at y_{i_j} because it was shifted to it from $y_{i_{j-1}}$ during the $(e_{i_m}, ..., e_{i_0})$-shift of ϕ). We conclude that i_j is strictly less than $i = i_0$ and the reversal of the path P is the maximal α/β-bichromatic path starting at y_{i_j} under ϕ. This means that $P_{y_{i_j}}(\alpha, \beta, \phi)$ ends at $y_i \neq x$ and the index i_j contradicts the minimality of i. This contradiction proves that γ is not present at $u_{\ell+1}$ under ϕ'', as claimed.

We conclude that, under ϕ'', the colour γ is missing at both u_ℓ and $u_{\ell+1}$. Next, one can show similarly to above that $\gamma \leqslant d(u_\ell) + \pi(u_\ell)$: otherwise, for $\ell \geqslant 2$, the shift for $s = \ell - 2$ would strictly decrease Φ while, for $\ell = 1$, this would contradict the choice of β by $\gamma = \beta$ and $u_\ell = y_i$. Thus if we colour f_ℓ by γ then we obtain a proper local colouring. Hence, C is an improving Vizing chain. This contradiction proves Part (c).

Let us turn to Part (d). Suppose on the contrary there are $i, j \in [p]$ such that $y_i \neq y_j$ and there is a colour $\beta \in \widetilde{\phi}(y_i) \cap \widetilde{\phi}(y_j)$. Let α be any element in $\widetilde{\phi}(x)$. The last set is nonempty since at most $d(x) - 1 < k$ edges at x are coloured. Part (c) applies to i (resp. j) and gives that the α/β-alternating chain starting at y_i (resp. y_j) ends at x. This means that, in the multigraph formed by the edges coloured α or β, the three distinct vertices y_i, y_j and x lie in the same connectivity component and each has degree 1. This is contradiction, since the $\{\alpha, \beta\}$-bichromatic multigraph has maximum degree at most 2.

For Part (e), suppose that the multifan F is maximal. Since at most $d(y_1) - 1$ edges are coloured at y_1, there is $\beta \in \widetilde{\phi}(y_1)$. The colour β must be present at x for otherwise, by colouring e_1 with colour β, we obtain a larger proper local colouring and thus (e_1) is an improving chain, a contradiction. Let $e' = xy'$ be the edge of colour β. By maximality, the fan F contains the edge e'. Thus $V(F)$ contains $y' \neq y$ and has size at least 2, as desired. It remains to prove (5). For this, define $\Gamma := \{\phi(e_2), ..., \phi(e_p)\}$ and $\Gamma' := \bigcup_{z \in V(F)} \widetilde{\phi}(z)$. (Recall that $V(F) = \{y_1, ..., y_p\}$.)

Claim 1. The sets Γ and Γ' are the same.

Proof of Claim. Let us show that $\Gamma \subseteq \Gamma'$. Take any index $i \in [2, p]$. By (F2), there is $j \in [i-1]$ such that $\phi(e_i) \in \widetilde{\phi}(y_j)$. Thus $\phi(e_i) \in \Gamma'$, as desired.

Conversely, pick any $i \in [p]$ and $\beta \in \widetilde{\phi}(y_i)$. By Part (b), we have that $\beta \notin \overline{\phi}(x)$. Thus there is $e' = xz$ with $\phi(e') = \beta$. By the maximality of F, there is $j \in [2, p]$ with $e' = e_j$. Thus $\beta = \phi(e_j)$ belongs to Γ. We conclude that $\Gamma \supseteq \Gamma'$, finishing the proof of the claim. $\qquad\square$

By Claim 1 and Part (d), we have that

$$p - 1 = |\Gamma| = |\Gamma'| = \sum_{z \in V(F)} |\widetilde{\phi}(z)| \geqslant \sum_{z \in V(F)} (\min\{k, d(z) + \pi(z)\} - d_\phi(z)).$$

Also, we have that $p = \sum_{z \in V(F)} \pi_F(x, z)$. Putting these two identities together, we obtain (5). This finishes the proof of the lemma. $\qquad\square$

Note that, in Theorem 4, if e is the only edge of G not coloured by ϕ then $d_\phi(y_i) = d(y_i)$ except $d_\phi(y_1) = d(y_1) - 1$ and the Local Fan Inequality simplifies to

$$\sum_{z \in V(F)} (\pi_F(x, z) - \min\{k - d(z), \pi(z)\}) \geqslant 2.$$

Now we are ready to derive the promised local version of Vizing's theorem.

Proof of Theorem 1.   Let $k := \max\{d(x) + \pi(x) : x \in V(G)\}$. Starting with the empty colouring, iteratively apply improving Vizing chains until none exists. (We stop since the potential strictly decreases each time.) Suppose that some edge e is not coloured by the final colouring ϕ. Let $x \in e$ and let $F = (e_1, y_1, ..., e_p, y_p)$ be a maximal multifan at x with respect to e and ϕ. Then the Local Fan Inequality (5) holds by Theorem 4. Since $k \geqslant d(z) + \pi(z)$ for every $z \in V(G)$, the inequality states that

$$\sum_{z \in V(F)} (\pi_F(x, z) - \pi(z) + d_\phi(z) - d(z)) \geqslant 1.$$

This is impossible as each summand in the left-hand side is clearly nonpositive. $\qquad\square$

It easy to convert the proof of Theorem 1 into an algorithm that finds a local colouring of input multigraph $G = (V, E, r)$ without isolated vertices with running time polynomial in $|E|$. The algorithm starts with the empty partial colouring $\phi_\varnothing$ and iteratively finds an improving Vizing chain and changes the current partial colouring accordingly. Each step decreases the potential. Thus we make at most $\Phi(\phi_\varnothing) = O(|E|)$ improvements. Given an uncoloured edge, an improving Vizing chain as in the proof of Theorem 4 can be easily found in polynomial in $|E|$ time.

Note that if G is a simple graph, then the Local Fan Inequality (5) states that

$$\sum_{z \in V(F)} (d_\phi(z) - \min\{k - 1, d(z)\}) \geqslant 1. \tag{7}$$

Proof of Theorem 2.   Let $\mathcal{C}$ be the set of pairs (ϕ, e) such that ϕ is a proper local partial colouring $E(G) \rightharpoonup [k]$ with the smallest possible potential, while $e \in E$ an edge is not coloured by ϕ. Suppose that the theorem is false. Thus $\mathcal{C}$ is nonempty.

Claim 2.   For every $(\phi, e) \in \mathcal{C}$ and $x \in e$ there are at least two different choices of $f \in E$ such that $f = xy$ for some $y \in V$ with $d(y) = k$ and $(\phi', f) \in \mathcal{C}$ for some ϕ'.

Proof of Claim.   Let $F = (e_1, y_1, ..., e_p, y_p)$ be a maximal multi-fan at x with respect to $e_1 = e$ and ϕ. By Theorem 4, the Local Fan Inequality (7) holds. Since $d_\phi(y_1) \leqslant d(y_1) - 1$, the contribution of y_1 to the left-hand size of (7) is at most 0 if $d(y_1) = k$ and at most -1 if $d(y_1) \leqslant k - 1$. We conclude by $k \geqslant d(z)$ for every $z \in V(G)$ that there are at least two indices $i \in [p]$ such that $d(y_i) = k$.

Let us show that, for each such i, the edge $f := e_i$ satisfies the claim. Indeed, let $i_0 = i > i_1 > \cdots > i_m = 1$ be the sequence of indices returned by Part (a) of Theorem 4. Let ϕ' be the shift of ϕ along the ϕ-safe sequence $(e_{i_m}, e_{i_{m-1}}, ..., e_{i_0})$. By Lemma 3, $(\phi', f) \in \mathcal{C}$, as required. $\qquad\square$

Start with any $(\phi_1, e_1) \in \mathcal{C}$ with $e_1 = x_1 x_2$. By applying Claim 2 (twice) and changing the pair (ϕ_1, e_1) we can assume first that $d(x_1) = k$ and then that also $d(x_2) = k$ (and thus $k = \Delta(G)$). Starting with ϕ_1 and $e_1 = x_1 x_2$, we inductively construct an infinite sequence $\phi_1, \phi_2, ...$ and an infinite path visiting edges $e_1, e_2, ...$ and vertices $x_1, x_2, x_3, ...$ in the stated order such that $(\phi_i, e_i) \in \mathcal{C}$ and $d(x_i) = k$ for every $i \geqslant 1$ as follows. Let $i \geqslant 1$ and suppose that we already have $\phi_1, ..., \phi_i$ and $e_1, ..., e_i$ as above. Claim 2 when applied to (ϕ_i, e_i) with $x = x_{i+1}$ gives at least two different potential choices of the next edge $e_{i+1} = x_{i+1} x_{i+2}$ and the next partial colouring ϕ_{i+1} (i.e., such that $(\phi_{i+1}, e_{i+1}) \in \mathcal{C}$ and $d(x_{i+2}) = k$). For at least one of these choices we have $x_{i+2} \notin \{x_1, ..., x_i\}$ as otherwise this would create a cycle on vertices of degree k. Thus we can always extend the path by a new edge. However, this contradicts the finiteness of our graph G, thus proving Theorem 2. $\qquad\square$

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