Minimum number of additive tuples in groups of prime order

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Abstract

For a prime number p and a sequence of integers $a_0,\ldots,a_k \in \{0,1,\ldots,p\}$, let $s(a_0,\ldots,a_k)$ be the minimum number of $(k+1)$-tuples $(x_0,\ldots,x_k) \in A_0\times\cdots\times A_k$ with $x_0 = x_1+\cdots+x_k$, over subsets $A_0,\ldots,A_k \subseteq \mathbb{Z}_p$ of sizes $a_0,\ldots,a_k$ respectively. We observe that an elegant argument of Samotij and Sudakov can be extended to show that there exists an extremal configuration with all sets A_i being intervals of appropriate length. The same conclusion also holds for the related problem, posed by Bajnok, when $a_0=\cdots=a_k=: a$ and $A_0=\cdots=A_k$, provided k is not equal 1 modulo p. Finally, by applying basic Fourier analysis, we show for Bajnok's problem that if $p \geqslant 13$ and $a \in \{3,\ldots,p-3\}$ are fixed while $k \equiv 1 \pmod p$ tends to infinity, then the extremal configuration alternates between at least two affine non-equivalent sets.

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1 Introduction

Let Γ be a given finite Abelian group, with the group operation written additively.

For $A_0,\ldots,A_k \subseteq \Gamma$, let $s(A_0,\ldots,A_k)$ be the number of $(k+1)$-tuples $(x_0,\ldots,x_k) \in A_0\times\cdots\times A_k$ with $x_0 = x_1+\cdots+x_k$. If $A_0=\cdots=A_k := A$, then we use the shorthand $s_k(A) := S(A_0,\ldots,A_k)$. For example, $s_2(A)$ is the number of *Schur triples* in A, that is, ordered triples $(x_0,x_1,x_2)\in A^3$ with $x_0 = x_1+x_2$.

For integers $n \geqslant m \geqslant 0$, let $[m,n] := \{m,m+1,\ldots,n\}$ and $[n] := [0,n-1] = \{0,\ldots,n-1\}$. For a sequence $a_0,\ldots,a_k \in [\,|\Gamma|+1\,] = \{0,1,\ldots,|\Gamma|\}$, let $s(a_0,\ldots,a_k;\Gamma)$ be

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the minimum of $s(A_0, \dots, A_k)$ over subsets $A_0, \dots, A_k \subseteq \Gamma$ of sizes $a_0, \dots, a_k$ respectively. Additionally, for $a \in [0, p]$, let $s_k(a; \Gamma)$ be the minimum of $s_k(A)$ over all a-sets $A \subseteq \Gamma$.

The question of finding the maximal size of a sum-free subset of Γ (i.e. the maximum a such that $s_2(a; \Gamma) = 0$) originated in a paper of Erdős [2] in 1965 and took 40 years before it was resolved in full generality by Green and Ruzsa [3]. Huczynska, Mullen and Yucas [4], and later Samotij and Sudakov [7], introduced the problem of finding $s_2(a; \Gamma)$. This function has a resemblance to some classical questions in extremal combinatorics, where one has to minimise the number of forbidden configurations, see [7, Section 1] for more details.

Huczynska, Mullen and Yucas [4] were able to solve the s_2-problem for $\Gamma = \mathbb{Z}_p$, where p is prime and $\mathbb{Z}_p$ is the cyclic group of order p. Samotij and Sudakov [7] solved the s_2-problem for various groups, including a different proof of the $\mathbb{Z}_p$ case. Bajnok [1, Problem G.48] suggested the more general problem of considering $s_k(a; \Gamma)$. Since even the s_2-case is still wide open in full generality, Bajnok [1, Problem G.49] proposed, as a possible first step, to consider $s_k(a; \mathbb{Z}_p)$, where p is prime and $k \geqslant 3$.

This paper concentrates on the latter question of Bajnok. Therefore, let p be a fixed prime and let, by default, the underlying group be $\mathbb{Z}_p$, which we identify with the additive group of residues modulo p (also using the multiplicative structure on it when this is useful). In particular, we write $s(a_0, \dots, a_k) := s(a_0, \dots, a_k; \mathbb{Z}_p)$ and $s_k(a) := s_k(a; \mathbb{Z}_p)$. Since the case $p = 2$ is trivial, let us assume that $p \geqslant 3$. By an *m-term arithmetic progression* (or *m-AP* for short) we mean a set of the form $\{x, x + d, \dots, x + (m - 1)d\}$ for some $x, d \in \mathbb{Z}_p$ with $d \neq 0$. We call d the *difference*. For $I \subseteq \mathbb{Z}_p$ and $x, y \in \mathbb{Z}_p$, write $x \cdot I + y := \{x \cdot z + y \mid z \in I\}$.

As we already mentioned, the case $k = 2$ has been completely resolved: Huczynska, Mullen and Yucas determined $s_2(a)$, and Samotij and Sudakov [7] showed that, when $s_2(a) > 0$, then the a-sets that achieve the minimum are exactly those of the form $\xi \cdot I$ with $\xi \in \mathbb{Z}_p \setminus \{0\}$, where I consists of the residues modulo p of a integers closest to $\frac{p-1}{2} \in \mathbb{Z}$. Each such set is an arithmetic progression; its difference can be any non-zero value but the initial element has to be carefully chosen.

Here we propose a generalisation of Bajnok's question, namely to investigate the function $s(a_0, \dots, a_k)$. First, by adopting the elegant argument of Samotij and Sudakov [7], we show that at least one extremal configuration consists of $k+1$ arithmetic progressions with the same difference. Note that since

$$s(A_0, \dots, A_k) = s(\xi \cdot A_0 + \eta_0, \dots, \xi \cdot A_k + \eta_k), \quad \text{for } \xi \neq 0 \text{ and } \eta_0 = \eta_1 + \dots + \eta_k, \tag{1}$$

finding such arithmetic progressions reduces to finding progressions with difference 1 (and starting element 0 for some k of the sets).

Theorem 1. *For arbitrary $k \geqslant 1$ and $a_0, \dots, a_k \in [0, p]$, there is $t \in \mathbb{Z}_p$ such that*

$$s(a_0, \dots, a_k) = s([a_0] + t, [a_1], \dots, [a_k]).$$

In particular, if $a_0 = \dots = a_k =: a$, then one extremal configuration consists of $A_1 = \dots = A_k = [a]$ and $A_0 = [t, t + a - 1]$ for some $t \in \mathbb{Z}_p$. Given this, one can write down some formulas for $s(a_0, \dots, a_k)$ in terms of $a_0, \dots, a_k$ involving summation (based on (3) or a version of (13)) but there does not seem to be a closed form in general.

If $k \not\equiv 1 \pmod p$, then by taking $\xi := 1$, $\eta_1 := \cdots := \eta_k := -t(k-1)^{-1}$, and $\eta_0 := -kt(k-1)^{-1}$ in (1), we can get another extremal configuration where all sets are the same: $A_0 + \eta_0 = \cdots = A_k + \eta_k$. Thus Theorem 1 directly implies the following corollary.

Corollary 2. *For every $k \geqslant 2$ with $k \not\equiv 1 \pmod p$ and $a \in [0, p]$, there is $t \in \mathbb{Z}_p$ such that $s_k(a) = s_k([t, t+a-1])$.* □

Unfortunately, if $k \geqslant 3$, then there may be sets A different from APs that attain equality in Corollary 2 with $s_k(|A|) > 0$ (which is in contrast to the case $k = 2$). For example, our (non-exhaustive) search showed that this happens already for $p = 17$, when

$$s_3(14) = 2255 = s_3([-1, 12]) = s_3([6, 18] \cup \{3\}).$$

Also, already the case $k = 2$ of the more general Theorem 1 exhibits extra solutions. Of course, by analysing the proof of Theorem 1 or Corollary 2 one can write a necessary and sufficient condition for the cases of equality. We do this in Section 2; in some cases this condition can be simplified.

However, by using basic Fourier analysis on $\mathbb{Z}_p$, we can describe the extremal sets for Corollary 2 when $k \not\equiv 1 \pmod p$ is sufficiently large.

Theorem 3. *Let a prime $p \geqslant 7$ and an integer $a \in [3, p-3]$ be fixed, and let $k \not\equiv 1 \pmod p$ be sufficiently large. Then there exists $t \in \mathbb{Z}_p$ for which the only $s_k(a)$-extremal sets are $\xi \cdot [t, t+a-1]$ for all non-zero $\xi \in \mathbb{Z}_p$.*

Problem 4. Find a 'good' description of all extremal families for Corollary 2 (or perhaps Theorem 1) for $k \geqslant 3$.

While Corollary 2 provides an example of an $s_k(a)$-extremal set for $k \not\equiv 1 \pmod p$, the case $k \equiv 1 \pmod p$ of the $s_k(a)$-problem turns out to be somewhat special. Here, translating a set A has no effect on the quantity $s_k(A)$. More generally, let $\mathcal{A}$ be the group of all invertible affine transformations of $\mathbb{Z}_p$, that is, it consists of maps $x \mapsto \xi \cdot x + \eta$, $x \in \mathbb{Z}_p$, for $\xi, \eta \in \mathbb{Z}_p$ with $\xi \neq 0$. Then

$$s_k(\alpha(A)) = s_k(A), \quad \text{for every } k \equiv 1 \pmod p \text{ and } \alpha \in \mathcal{A}. \tag{2}$$

Let us call two subsets $A, B \subseteq \mathbb{Z}_p$ *(affine) equivalent* if there is $\alpha \in \mathcal{A}$ with $\alpha(A) = B$. By (2), we need to consider sets only up to this equivalence. Trivially, any two subsets of $\mathbb{Z}_p$ of size a are equivalent if $a \leqslant 2$ or $a \geqslant p-2$.

Again using Fourier analysis on $\mathbb{Z}_p$, we show the following result.

Theorem 5. *Let a prime $p \geqslant 7$ and an integer $a \in [3, p-3]$ be fixed, and let $k \equiv 1 \pmod p$ be sufficiently large. Then the following statements hold for the $s_k(a)$-problem.*

1. *If a and k are both even, then $[a]$ is the unique (up to affine equivalence) extremal set.*

2. *If at least one of a and k is odd, define $I' := [a-1] \cup \{a\} = \{0, \dots, a-2, a\}$. Then*

   (a) *$s_k(a) < s_k([a])$ for all large k;*

   (b) *I' is the unique extremal set for infinitely many k;*

   (c) *$s_k(a) < s_k(I')$ for infinitely many k, provided there are at least three non-equivalent a-subsets of $\mathbb{Z}_p$.*

It is not hard to see that there are at least three non-equivalent a-subsets of $\mathbb{Z}_p$ if and only if $p \geqslant 13$ and $a \in [3, p-3]$, or $p \geqslant 11$ and $a \in [4, p-4]$. Thus Theorem 5 characterises pairs (p, a) for which there exists an a-subset A which is $s_k(a)$-extremal for *all* large $k \equiv 1 \pmod p$.

Corollary 6. *Let p be a prime and $a \in [0, p]$. There is an a-subset $A \subseteq \mathbb{Z}_p$ with $s_k(A) = s_k(a)$ for all large $k \equiv 1 \pmod p$ if and only if $a \leqslant 2$, or $a \geqslant p-2$, or $p \in \{7, 11\}$ and $a = 3$.* □

As is often the case in mathematics, a new result leads to further open problems.

Problem 7. Given $a \in [3, p-3]$, find a 'good' description of all a-subsets of $\mathbb{Z}_p$ that are $s_k(a)$-extremal for at least one (resp. infinitely many) values of $k \equiv 1 \pmod p$.

Problem 8. Is it true that for every $a \in [3, p-3]$ there is k_0 such that for all $k \geqslant k_0$ with $k \equiv 1 \pmod p$, any two $s_k(a)$-extremal sets are affine equivalent?

2 Proof of Theorem 1

Here we prove Theorem 1 by adopting the proof of Samotij and Sudakov [7].

Let $A_1, \dots, A_k$ be subsets of $\mathbb{Z}_p$. Define $\sigma(x; A_1, \dots, A_k)$ as the number of k-tuples $(x_1, \dots, x_k) \in A_1 \times \dots \times A_k$ with $x = x_1 + \dots + x_k$. Also, for an integer $r \geqslant 0$, let

$$\begin{aligned} N_r(A_1, \dots, A_k) &:= \{x \in \mathbb{Z}_p \mid \sigma(x; A_1, \dots, A_k) \geqslant r\}, \\ n_r(A_1, \dots, A_k) &:= |N_r(A_1, \dots, A_k)|. \end{aligned}$$

These notions are related to our problem because of the following easy identity:

$$s(A_0, \dots, A_k) = \sum_{r=1}^{\infty} |A_0 \cap N_r(A_1, \dots, A_k)|. \tag{3}$$

Let an *interval* mean an arithmetic progression with difference 1, i.e. a subset I of $\mathbb{Z}_p$ of form $\{x, x+1, \dots, x+y\}$. Its *centre* is $x + y/2 \in \mathbb{Z}_p$; it is unique if I is *proper* (that is, $0 < |I| < p$). Note the following easy properties of the sets N_r:

1. These sets are nested:

$$N_0(A_1, \dots, A_k) = \mathbb{Z}_p \supseteq N_1(A_1, \dots, A_k) \supseteq N_2(A_1, \dots, A_k) \supseteq \dots \tag{4}$$

2. If each A_i is an interval with centre c_i, then $N_r(A_1, \ldots, A_k)$ is an interval with centre $c_1 + \cdots + c_k$.

We will also need the following result of Pollard [6, Theorem 1].

Theorem 9. *Let p be a prime, $k \geqslant 1$, and $A_1, \ldots, A_k$ be subsets of $\mathbb{Z}_p$ of sizes $a_1, \ldots, a_k$. Then for every integer $r \geqslant 1$, we have*

$$\sum_{i=1}^{r} n_i(A_1, \ldots, A_k) \geqslant \sum_{i=1}^{r} n_i([a_1], \ldots, [a_k]).$$

Proof of Theorem 1. Let $A_0, \ldots, A_k$ be some extremal sets for the $s(a_0, \ldots, a_k)$-problem. We can assume that $0 < a_0 < p$, because $s(A_0, \ldots, A_k)$ is 0 if $a_0 = 0$ and $\prod_{i=1}^{k} a_i$ if $a_0 = p$, regardless of the choice of the sets A_i.

Since $n_0([a_1], \ldots, [a_k]) = p > p - a_0$ while $n_r([a_1], \ldots, [a_k]) = 0 < p - a_0$ when, for example, $r > \prod_{i=1}^{k-1} a_i$, there is a (unique) integer $r_0 \geqslant 0$ such that

$$\begin{aligned} n_r([a_1], \ldots, [a_k]) &> p - a_0, \quad \text{all } r \in [0, r_0], & (5)\\ n_r([a_1], \ldots, [a_k]) &\leqslant p - a_0, \quad \text{all integers } r \geqslant r_0 + 1. & (6) \end{aligned}$$

The nested intervals $N_1([a_1], \ldots, [a_k]) \supseteq N_2([a_1], \ldots, [a_k]) \supseteq \ldots$ have the same centre $c := ((a_1 - 1) + \cdots + (a_k - 1))/2$. Thus there is a translation $I := [a_0] + t$ of $[a_0]$, with t independent of r, which has as small as possible intersection with each N_r-interval above given their sizes, that is,

$$|I \cap N_r([a_1], \ldots, [a_k])| = \max\{\, 0,\, n_r([a_1], \ldots, [a_k]) + a_0 - p \,\}, \quad \text{for all } r \in \mathbb{N}. \qquad (7)$$

This and Pollard's theorem give the following chain of inequalities:

$$\begin{aligned} s(A_0, \ldots, A_k) &\overset{(3)}{=} \sum_{i=1}^{\infty} |A_0 \cap N_i(A_1, \ldots, A_k)| \\ &\geqslant \sum_{i=1}^{r_0} |A_0 \cap N_i(A_1, \ldots, A_k)| \\ &\geqslant \sum_{i=1}^{r_0} (n_i(A_1, \ldots, A_k) + a_0 - p) \\ &\overset{\text{Thm 9}}{\geqslant} \sum_{i=1}^{r_0} (n_i([a_1], \ldots, [a_k]) + a_0 - p) \\ &\overset{(5)-(6)}{=} \sum_{i=1}^{\infty} \max\{\, 0,\, n_i([a_1], \ldots, [a_k]) + a_0 - p \,\} \\ &\overset{(7)}{=} \sum_{i=1}^{\infty} |I \cap N_i([a_1], \ldots, [a_k])| \\ &\overset{(3)}{=} s(I, [a_1], \ldots, [a_k]), \end{aligned}$$

giving the required. □

Let us write a necessary and sufficient condition for equality in Theorem 1 in the case $a_0, \dots, a_k \in [1, p-1]$. Let $r_0 \geqslant 0$ be defined by (5)–(6). Then, by (4), a sequence $A_0, \dots, A_k \subseteq \mathbb{Z}_p$ of sets of sizes respectively $a_0, \dots, a_k$ is extremal if and only if

$$
\begin{aligned}
A_0 \cap N_{r_0+1}(A_1, \dots, A_k) &= \varnothing, & (8)\\
A_0 \cup N_{r_0}(A_1, \dots, A_k) &= \mathbb{Z}_p, & (9)\\
\sum_{i=1}^{r_0} n_i(A_1, \dots, A_k) &= \sum_{i=1}^{r_0} n_i([a_1], \dots, [a_k]). & (10)
\end{aligned}
$$

Let us now concentrate on the case $k = 2$, trying to simplify the above condition. We can assume that no a_i is equal to 0 or p (otherwise the choice of the other two sets has no effect on $s(A_0, A_1, A_2)$ and every triple of sets of sizes a_0, a_1 and a_2 is extremal). Also, as in [7], let us exclude the case $s(a_0, a_1, a_2) = 0$, as then there are in general many extremal configurations. Note that $s(a_0, a_1, a_2) = 0$ if and only if $r_0 = 0$; also, by the Cauchy-Davenport theorem (the special case $k = 2$ and $r = 1$ of Theorem 9), this is equivalent to $a_1 + a_2 - 1 \leqslant p - a_0$. Assume by symmetry that $a_1 \leqslant a_2$. Note that (5) implies that $r_0 \leqslant a_1$.

The condition in (10) states that we have equality in Pollard's theorem. A result of Nazarewicz, O'Brien, O'Neill and Staples [5, Theorem 3] characterises when this happens (for $k = 2$), which in our notation is the following.

Theorem 10. *For $k = 2$ and $1 \leqslant r_0 \leqslant a_1 \leqslant a_2 < p$, we have equality in (10) if and only if at least one of the following conditions holds:*

1. $r_0 = a_1$,
2. $a_1 + a_2 \geqslant p + r_0$,
3. $a_1 = a_2 = r_0 + 1$ *and* $A_2 = g - A_1$ *for some* $g \in \mathbb{Z}_p$,
4. A_1 *and* A_2 *are arithmetic progressions with the same difference.*

Let us try to write more explicitly each of these four cases, when combined with (8) and (9).

First, consider the case $r_0 = a_1$. We have $N_{a_1}([a_1], [a_2]) = [a_1 - 1, a_2 - 1]$ and thus $n_{a_1}([a_1], [a_2]) = a_2 - a_1 + 1 > p - a_0$, that is, $a_2 - a_1 \geqslant p - a_0$. The condition (8) holds automatically since $N_i(A_1, A_2) = \varnothing$ whenever $i > |A_1|$. The other condition (9) may be satisfied even when none of the sets A_i is an arithmetic progression (for example, take $p = 13$, $A_1 = \{0, 1, 3\}$, $A_2 = \{0, 2, 3, 5, 6, 7, 9, 10\}$ and let A_0 be the complement of $N_3(A_1, A_2) = \{3, 6, 10\}$). We do not see any better characterisation here, apart from stating that (9) holds.

Next, suppose that $a_1 + a_2 \geqslant p + r_0$. Then, for any two sets A_1 and A_2 of sizes a_1 and a_2, we have $N_{r_0}(A_1, A_2) = \mathbb{Z}_p$; thus (9) holds automatically. Similarly to the previous case, there does not seem to be a nice characterisation of (8). For example, (8) may hold even when none of the sets A_i is an AP: e.g. let $p = 11$, $A_1 = A_2 = \{0, 1, 2, 3, 4, 5, 7\}$, and let $A_0 = \{0, 2, 10\}$ be the complement of $N_4(A_1, A_2) = \{1, 3, 4, 5, 6, 7, 8, 9\}$ (here $r_0 = 3$).

Next, suppose that we are in the third case. The primality of p implies that $g \in \mathbb{Z}_p$ satisfying $A_2 = g - A_1$ is unique and thus $N_{r_0+1}(A_1, A_2) = \{g\}$. Therefore (8) is equivalent to $A_0 \not\ni g$. Also, note that if I_1 and I_2 are intervals of size $r_0 + 1$, then $n_{r_0}(I_1, I_2) = 3$. By the definition of r_0, we have $p - 2 \leqslant a_0 \leqslant p - 1$. Thus we can choose any integer $r_0 \in [1, p - 2]$ and $(r_0 + 1)$-sets $A_2 = g - A_1$, and then let A_0 be obtained from $\mathbb{Z}_p$ by removing g and at most one further element of $N_{r_0}(A_1, A_2)$. Here, A_0 is always an AP (as a subset of $\mathbb{Z}_p$ of size $a_0 \geqslant p - 2$) but A_1 and A_2 need not be.

Finally, let us show that if A_1 and A_2 are arithmetic progressions with the same difference d and we are not in Case 1 nor 2 of Theorem 10, then A_0 is also an arithmetic progression whose difference is d. By (1), it is enough to prove this when $A_1 = [a_1]$ and $A_2 = [a_2]$ (and $d = 1$). Since $a_1 + a_2 \leqslant p - 1 + r_0$ and $r_0 + 1 \leqslant a_1 \leqslant a_2$, we have that

$$\begin{aligned} N_{r_0}(A_1, A_2) &= [r_0 - 1, a_1 + a_2 - r_0 - 1] \\ N_{r_0+1}(A_1, A_2) &= [r_0, a_1 + a_2 - r_0 - 2] \end{aligned}$$

have sizes respectively $a_1 + a_2 - 2r_0 + 1 < p$ and $a_1 + a_2 - 2r_0 - 1 > 0$. We see that $N_{r_0+1}(A_1, A_2)$ is obtained from the proper interval $N_{r_0}(A_1, A_2)$ by removing its two endpoints. Thus A_0, which is sandwiched between the complements of these two intervals by (8)–(9), must be an interval too. (And, conversely, every such triple of intervals is extremal.)

3 The proof of Theorems 3 and 5

Let us recall the basic definitions and facts of Fourier analysis on $\mathbb{Z}_p$. For a more detailed treatment of this case, see e.g. [8, Chapter 2]. Write $\omega := e^{2\pi i/p}$ for the p^{th} root of unity. Given a function $f : \mathbb{Z}_p \to \mathbb{C}$, we define its *Fourier transform* to be the function $\widehat{f} : \mathbb{Z}_p \to \mathbb{C}$ given by

$$\widehat{f}(\gamma) := \sum_{x=0}^{p-1} f(x)\, \omega^{-x\gamma}, \qquad \text{for } \gamma \in \mathbb{Z}_p.$$

Parseval's identity states that

$$\sum_{x=0}^{p-1} f(x)\, \overline{g(x)} = \frac{1}{p} \sum_{\gamma=0}^{p-1} \widehat{f}(\gamma)\, \overline{\widehat{g}(\gamma)}. \tag{11}$$

The *convolution* of two functions $f, g : \mathbb{Z}_p \to \mathbb{C}$ is given by

$$(f * g)(x) := \sum_{y=0}^{p-1} f(y)\, g(x - y).$$

It is not hard to show that the Fourier transform of a convolution equals the product of Fourier transforms, i.e.

$$\widehat{f_1 * \ldots * f_k} = \widehat{f_1} \cdot \ldots \cdot \widehat{f_k}. \tag{12}$$

We write f^{*k} for the convolution of f with itself k times. (So, for example, $f^{*2} = f * f$.) Denote by $\mathbb{1}_A$ the *indicator function* of $A \subseteq \mathbb{Z}_p$ which assumes value 1 on A and 0 on $\mathbb{Z}_p \setminus A$. We will call $\widehat{\mathbb{1}}_A(0) = |A|$ the *trivial Fourier coefficient of* A. Since the Fourier transform behaves very nicely with respect to convolution, it is not surprising that our parameter of interest, $s_k(A)$, can be written as a simple function of the Fourier coefficients of $\mathbb{1}_A$. Indeed, let $A \subseteq \mathbb{Z}_p$ and $x \in \mathbb{Z}_p$. Then the number of tuples $(a_1, \ldots, a_k) \in A^k$ such that $a_1 + \ldots + a_k = x$ (which is $\sigma(x; A, \ldots, A)$ in the notation of Section 2) is precisely $\mathbb{1}_A^{*k}(x)$. The function $s_k(A)$ counts such a tuple if and only if its sum x also lies in A. Thus,

$$s_k(A) = \sum_{x=0}^{p-1} \mathbb{1}_A^{*k}(x)\, \mathbb{1}_A(x) \overset{(11)}{=} \frac{1}{p} \sum_{\gamma=0}^{p-1} \widehat{\mathbb{1}_A^{*k}}(\gamma)\, \overline{\widehat{\mathbb{1}_A}(\gamma)} \overset{(12)}{=} \frac{1}{p} \sum_{\gamma=0}^{p-1} \left(\widehat{\mathbb{1}_A}(\gamma)\right)^k \overline{\widehat{\mathbb{1}_A}(\gamma)}. \tag{13}$$

Since every set $A \subseteq \mathbb{Z}_p$ of size a has the same trivial Fourier coefficient (namely $\widehat{\mathbb{1}_A}(0) = a$), let us re-write (13) as

$$ps_k(A) - a^{k+1} = \sum_{\gamma=1}^{p-1} (\widehat{\mathbb{1}_A}(\gamma))^k\, \overline{\widehat{\mathbb{1}_A}(\gamma)} =: F(A). \tag{14}$$

Thus we need to minimise $F(A)$ (which is a real number for any A) over a-subsets $A \subseteq \mathbb{Z}_p$. To do this when k is sufficiently large, we will consider the largest in absolute value non-trivial Fourier coefficient $\widehat{\mathbb{1}_A}(\gamma)$ of an a-subset A. Indeed, the term $(\widehat{\mathbb{1}_A}(\gamma))^k \overline{\widehat{\mathbb{1}_A}(\gamma)}$ will dominate $F(A)$, so if it has strictly negative real part, then $F(A) < F(B)$ for all a-subsets $B \subseteq \mathbb{Z}_p$ with $\max_{\delta \neq 0} |\widehat{\mathbb{1}_B}(\delta)| < |\widehat{\mathbb{1}_A}(\gamma)|$.

Given $a \in [p-1]$, let

$$I := [a] = \{0, \ldots, a-1\} \quad \text{and} \quad I' := [a-1] \cup \{a\} = \{a, \ldots, a-2, a\}.$$

In order to prove Theorems 3 and 5, we will make some preliminary observations about these special sets. The set of a-subsets which are affine equivalent to I is precisely the set of a-APs.

Next we will show that

$$F(I) = 2 \sum_{\gamma=1}^{(p-1)/2} (-1)^{\gamma(a-1)(k-1)} \left|\widehat{\mathbb{1}_I}(\gamma)\right|^{k+1} \quad \text{if } k \equiv 1 \pmod{p}. \tag{15}$$

Note that $(-1)^{\gamma(a-1)(k-1)}$ equals $(-1)^\gamma$ if both a, k are even and 1 otherwise. To see (15), let $\gamma \in \{1, \ldots, \frac{p-1}{2}\}$ and write $\widehat{\mathbb{1}_I}(\gamma) = re^{\theta i}$ for some $r > 0$ and $0 \leqslant \theta < 2\pi$. Then θ is the midpoint of $0, -2\pi\gamma/p, \ldots, -2(a-1)\gamma\pi/p$, i.e. $\theta = -\pi(a-1)\gamma/p$. Choose $s \in \mathbb{N}$ such that $k = sp + 1$. Then

$$(\widehat{\mathbb{1}_I}(\gamma))^k \overline{\widehat{\mathbb{1}_I}(\gamma)} = \left(re^{-\pi i(a-1)\gamma/p}\right)^k re^{\pi i(a-1)\gamma/p} = r^{k+1} e^{-\pi i(a-1)\gamma s}, \tag{16}$$

and $e^{-\pi i(a-1)s}$ equals 1 if $(a-1)s$ is even, and -1 if $(a-1)s$ is odd. Note that, since p is an odd prime, $(a-1)s$ is odd if and only if a and k are both even. So (16) is real, and the fact that $\widehat{\mathbb{1}_I}(p-\gamma) = \overline{\widehat{\mathbb{1}_I}(\gamma)}$ implies that the corresponding term for $p-\gamma$ is the same as for γ. This gives (15). A very similar calculation to (16) shows that

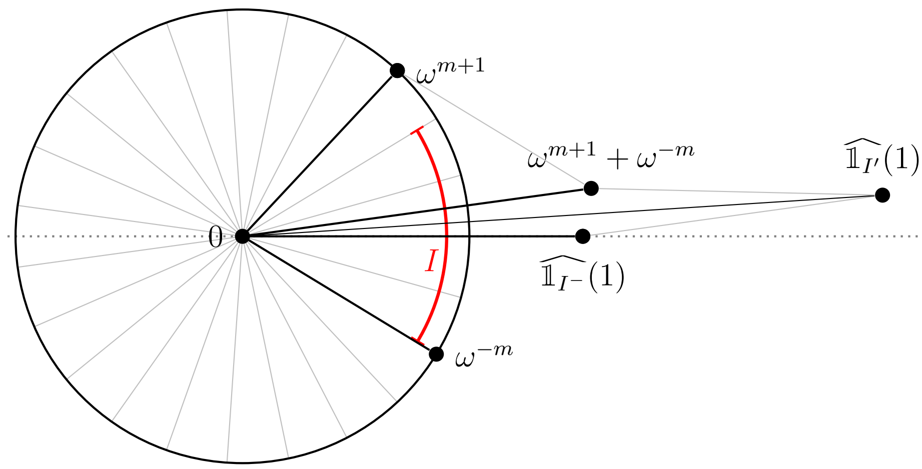
$$F(I+t) = \sum_{\gamma=1}^{p-1} e^{-\pi i(2t+a-1)(k-1)\gamma/p} |\widehat{\mathbb{1}_{I+t}}(\gamma)|^{k+1} \quad \text{for all } k \geqslant 3. \tag{17}$$

Given $r > 0$ and $0 \leqslant \theta < 2\pi$, we write $\arg(re^{\theta i}) := \theta$.

Proposition 11. *Suppose that $p \geqslant 7$ is prime and $a \in [3, p-3]$. Then $\arg\left(\widehat{\mathbb{1}_{I'}}(1)\right)$ is not an integer multiple of π/p.*

Proof. Since $\widehat{\mathbb{1}_A}(\gamma) = -\widehat{\mathbb{1}_{\mathbb{Z}_p \setminus A}}(\gamma)$ for all $A \subseteq \mathbb{Z}_p$ and non-zero $\gamma \in \mathbb{Z}_p$, we may assume without loss of generality that $a \leqslant p-a$. Since p is odd, we have $a \leqslant (p-1)/2$.

Suppose first that a is odd. Let $m := (a-1)/2$. Then $m \in [1, \frac{p-3}{4}]$. Observe that translating any $A \subseteq \mathbb{Z}_p$ changes the arguments of its Fourier coefficients by an integer multiple of $2\pi/p$. So, for convenience of angle calculations, here we may redefine $I := [-m, m]$ and $I' := \{-m-1\} \cup [-m+1, m]$. Also let $I^- := [-m+1, m-1]$, which is non-empty. The argument of $\widehat{\mathbb{1}_{I^-}}(1)$ is 0. Further, $\widehat{\mathbb{1}_{I'}}(1) = \widehat{\mathbb{1}_{I^-}}(1) + \omega^{m+1} + \omega^{-m}$. Since $\omega^{m+1}, \omega^{-m}$ lie on the unit circle, the argument of $\omega^{m+1} + \omega^{-m}$ is either π/p or $\pi + \pi/p$. But the bounds on m imply that it has positive real part, so $\arg(\omega^{m+1} + \omega^{-m}) = \pi/p$. By looking at the non-degenerate parallelogram in the complex plane with vertices $0, \widehat{\mathbb{1}_{I^-}}(1), \omega^{m+1} + \omega^{-m}, \widehat{\mathbb{1}_{I'}}(1)$, we see that the argument of $\widehat{\mathbb{1}_{I'}}(1)$ lies strictly between that of $\widehat{\mathbb{1}_{I^-}}(1)$ and $\omega^{m+1} + \omega^{-m}$, i.e. strictly between 0 and π/p, giving the required.



Suppose now that a is even and let $m := (a-2)/2 \in [1, \frac{p-5}{4}]$. Again without loss of generality we may redefine $I := [-m, m+1]$ and $I' := \{-m-1\} \cup [-m+1, m+1]$. Let also $I^- := [-m+1, m]$, which is non-empty. The argument of $\widehat{\mathbb{1}_{I^-}}(1)$ is $-\pi/p$. Further, $\widehat{\mathbb{1}_{I'}}(1) = \widehat{\mathbb{1}_{I^-}}(1) + \omega^{m+1} + \omega^{-(m+1)}$. The argument of $\omega^{m+1} + \omega^{-(m+1)}$ is 0, so as before the argument of $\widehat{\mathbb{1}_{I'}}(1)$ is strictly between $-\pi/p$ and 0, as required. □

We say that an a-subset A is a *punctured interval* if $A = I' + t$ or $A = -I' + t$ for some $t \in \mathbb{Z}_p$. That is, A can be obtained from an interval of length $a+1$ by removing a penultimate point.

Lemma 12. *Let $p \geqslant 7$ be prime and let $a \in \{3, \ldots, p-3\}$. Then the sets $I, I' \subseteq \mathbb{Z}_p$ are not affine equivalent. Thus no punctured interval is affine equivalent to an interval.*

Proof. Suppose on the contrary that there is $\alpha \in \mathcal{A}$ with $\alpha(I') = I$. Let a *reflection* mean an affine map R_c with $c \in \mathbb{Z}_p$ that maps x to $-x + c$. Clearly, $I = [a]$ is invariant under the reflection $R := R_{a-1}$. Thus I' is invariant under the map $R' := \alpha^{-1} \circ R \circ \alpha$. As is easy to see, R' is also some reflection and thus preserves the cyclic distances in $\mathbb{Z}_p$. So R' has to fix a, the unique element of I' with both distance-1 neighbours lying outside of I'. Furthermore, R' has to fix $a-2$, the unique element of I' at distance 2 from a. However, no reflection can fix two distinct elements of $\mathbb{Z}_p$, a contradiction. $\square$

We remark that the previous lemma can also be deduced from Proposition 11. Indeed, for any $A \subseteq \mathbb{Z}_p$, the multiset of Fourier coefficients of A is the same as that of $x \cdot A$ for $x \in \mathbb{Z}_p \setminus \{0\}$, and translating a subset changes the argument of Fourier coefficients by an integer multiple of $2\pi/p$. Thus for every subset which is affine equivalent to I, the argument of each of its Fourier coefficients is an integer multiple of π/p.

Let

$$\rho(A) := \max_{\gamma \in \mathbb{Z}_p \setminus \{0\}} |\widehat{\mathbb{1}}_A(\gamma)| \quad \text{and} \quad R(a) := \left\{ \rho(A) : A \in \binom{\mathbb{Z}_p}{a} \right\} = \{m_1(a) > m_2(a) > \ldots\}.$$

Given $j \geqslant 1$, we say that A *attains* $m_j(a)$, and specifically that A *attains* $m_j(a)$ *at* γ if $m_j(a) = \rho(A) = |\widehat{\mathbb{1}_A}(\gamma)|$. Notice that, since $\widehat{\mathbb{1}_A}(-\gamma) = \overline{\widehat{\mathbb{1}_A}(\gamma)}$, the set A attains $m_j(a)$ at γ if and only if A attains $m_j(a)$ at $-\gamma$ (and $\gamma, -\gamma \neq 0$ are distinct values).

As we show in the next lemma, the a-subsets which attain $m_1(a)$ are precisely the affine images of I (i.e. arithmetic progressions), and the a-subsets which attain $m_2(a)$ are the affine images of the punctured interval I'.

Lemma 13. *Let $p \geqslant 7$ be prime and let $a \in [3, p-3]$. Then $|R(a)| \geqslant 2$ and*

(i) $A \in \binom{\mathbb{Z}_p}{a}$ attains $m_1(a)$ if and only if A is affine equivalent to I, and every interval attains $m_1(a)$ at 1 and -1 only;

(ii) $B \in \binom{\mathbb{Z}_p}{a}$ attains $m_2(a)$ if and only if B is affine equivalent to I', and every punctured interval attains $m_2(a)$ at 1 and -1 only.

Proof. Given $D \in \binom{\mathbb{Z}_p}{a}$, we claim that there is some $D_{\text{pri}} \in \binom{\mathbb{Z}_p}{a}$ with the following properties:

- D_{pri} is affine equivalent to D;
- $\rho(D) = |\widehat{\mathbb{1}_{D_{\text{pri}}}}(1)|$; and

- $-\pi/p < \arg\left(\widehat{\mathbb{1}_{D_{\mathrm{pri}}}}(1)\right) \leqslant \pi/p$.

Call such a D_{pri} a *primary image* of D. Indeed, suppose that $\rho(D) = |\widehat{\mathbb{1}_D}(\gamma)|$ for some non-zero $\gamma \in \mathbb{Z}_p$, and let $\widehat{\mathbb{1}_D}(\gamma) = r'e^{\theta' i}$ for some $r' > 0$ and $0 \leqslant \theta' < 2\pi$. (Note that we have $r' > 0$ since p is prime.) Choose $\ell \in \{0, \ldots, p-1\}$ and $-\pi/p < \phi \leqslant \pi/p$ such that $\theta' = 2\pi\ell/p + \phi$. Let $D_{\mathrm{pri}} := \gamma \cdot D + \ell$. Then

$$|\widehat{\mathbb{1}_{D_{\mathrm{pri}}}}(1)| = \left|\sum_{x \in D} \omega^{-\gamma x - \ell}\right| = |\omega^{-\ell}\widehat{\mathbb{1}_D}(\gamma)| = |\widehat{\mathbb{1}_D}(\gamma)| = \rho(D),$$

and

$$\arg\left(\widehat{\mathbb{1}_{D_{\mathrm{pri}}}}(1)\right) = \arg(e^{\theta' i}\omega^{-\ell}) = 2\pi\ell/p + \phi - 2\pi\ell/p = \phi,$$

as required.

Let $D \subseteq \mathbb{Z}_p$ have size a and write $\widehat{\mathbb{1}_D}(1) = re^{\theta i}$. Assume by the above that $-\pi/p < \theta \leqslant \pi/p$. For all $j \in \mathbb{Z}_p$, let

$$h(j) := \Re(\omega^{-j}e^{-\theta i}) = \cos\left(\frac{2\pi j}{p} + \theta\right),$$

where $\Re(z)$ denotes the real part of $z \in \mathbb{C}$. Given any a-subset E of $\mathbb{Z}_p$, we have

$$H_D(E) := \sum_{j \in E} h(j) = \Re\left(e^{-\theta i}\sum_{j \in E}\omega^{-j}\right) = \Re\left(e^{-\theta i}\widehat{\mathbb{1}_E}(1)\right) \leqslant |\widehat{\mathbb{1}_E}(1)|. \tag{18}$$

Then

$$H_D(D) = \sum_{j \in D} h(j) = \Re(e^{-\theta i}\widehat{\mathbb{1}_D}(1)) = r = |\widehat{\mathbb{1}_D}(1)|. \tag{19}$$

Note that $H_D(E)$ is the (signed) length of the orthogonal projection of $\widehat{\mathbb{1}_E}(1) \in \mathbb{C}$ on the 1-dimensional line $\{xe^{i\theta} \mid x \in \mathbb{R}\}$. As stated in (18) and (19), $H_D(E) \leqslant |\widehat{\mathbb{1}_E}(1)|$ and this is equality for $E = D$. (Both of these facts are geometrically obvious.) If $|\widehat{\mathbb{1}_D}(1)| = m_1(a)$ is maximum, then no $H_D(E)$ for an a-set E can exceed $m_1(a) = H_D(D)$. Informally speaking, the main idea of the proof is that if we fix the direction $e^{i\theta}$, then the projection length is maximised if we take a distinct elements $j \in \mathbb{Z}_p$ with the a largest values of $h(j)$, that is, if we take some interval (with the runner-up being a punctured interval).

Let us provide a formal statement and proof of this now.

Claim 14. *Let $\mathcal{I}_a$ be the set of length-a intervals in $\mathbb{Z}_p$.*

(i) Let $M_1(D) \subseteq \binom{\mathbb{Z}_p}{a}$ consist of a-sets $E \subseteq \mathbb{Z}_p$ such that $H_D(E) \geqslant H_D(C)$ for all $C \in \binom{\mathbb{Z}_p}{a}$. Then $M_1(D) \subseteq \mathcal{I}_a$.

(ii) Let $M_2(D) \subseteq \binom{\mathbb{Z}_p}{a}$ be the set of $E \notin \mathcal{I}_a$ for which $H_D(E) \geqslant H_D(C)$ for all $C \in \binom{\mathbb{Z}_p}{a} \setminus \mathcal{I}_a$. Then every $E \in M_2(A)$ is a punctured interval.

Proof. Suppose that $0 < \theta < \pi/p$. Then $h(0) > h(1) > h(-1) > h(2) > h(-2) > \ldots > h(\frac{p-1}{2}) > h(-\frac{p-1}{2})$. In other words, $h(j_\ell) > h(j_k)$ if and only if $\ell < k$, where $j_m := (-1)^{m-1}\lceil m/2 \rceil$. Letting $J_{a-1} := \{j_0, \ldots, j_{a-2}\}$, we see that

$$H_D(J_{a-1} \cup \{j_{a-1}\}) > H_D(J_{a-1} \cup \{j_a\}) > H_D(J_{a-1} \cup \{j_{a+1}\}), H_D(J_{a-2} \cup \{j_{a-1}, j_a\}) > H_D(J)$$

for all other a-subsets J. But $J_{a-1} \cup \{j_{a-1}\}$ and $J_{a-1} \cup \{j_a\}$ are both intervals, and $J_{a-1} \cup \{j_{a+1}\}$ and $J_{a-2} \cup \{j_{a-1}, j_a\}$ are both punctured intervals. So in this case $M_1(D) := \{J_{a-1} \cup \{j_{a-1}\}\}$ and $M_2(D) \subseteq \{J_{a-1} \cup \{j_{a+1}\}, J_{a-2} \cup \{j_{a-1}, j_a\}\}$, as required.

The case when $-\pi/p < \theta < 0$ is almost identical except now $j_\ell := (-1)^\ell \lceil \ell/2 \rceil$ for all $0 \leqslant \ell \leqslant p-1$. If $\theta = 0$ then $h(0) > h(1) = h(-1) > h(2) = h(-2) > \ldots > h(\frac{p-1}{2}) = h(-\frac{p-1}{2})$. If $\theta = -\pi/p$ then $h(0) = h(-1) > h(1) = h(-2) > \ldots = h(-\frac{p-1}{2}) > h(\frac{p-1}{2})$. □

We can now prove part (i) of the lemma. Suppose $A \in \binom{\mathbb{Z}_p}{a}$ attains $m_1(a)$ at $\gamma \in \mathbb{Z}_p \setminus \{0\}$. Then the primary image D of A satisfies $|\widehat{\mathbb{1}_D}(1)| = m_1(a) = |\widehat{\mathbb{1}_A}(\gamma)|$. So, for any $E \in M_1(D)$,

$$|\widehat{\mathbb{1}_A}(\gamma)| = |\widehat{\mathbb{1}_D}(1)| \overset{(19)}{=} H_D(D) \leqslant H_D(E) \overset{(18)}{\leqslant} |\widehat{\mathbb{1}_E}(1)|,$$

with equality in the first inequality if and only if $D \in M_1(D)$. Thus, by Claim 14(i), D is an interval, and so A is affine equivalent to an interval, as required. Further, if A is an interval then D is an interval if and only if $\gamma = \pm 1$. This completes the proof of (i).

For (ii), note that $m_2(a)$ exists since by Lemma 12, there is a subset (namely I') which is not affine equivalent to I. By (i), it does not attain $m_1(a)$, so $\rho(I') \leqslant m_2(a)$. Suppose now that B is an a-subset of $\mathbb{Z}_p$ which attains $m_2(a)$ at $\gamma \in \mathbb{Z}_p \setminus \{0\}$. Let D be the primary image of B. Then D is not an interval. This together with Claim 14(i) implies that $H_D(D) < H_D(E)$ for any $E \in M_1(D)$. Thus, for any $C \in M_2(D)$, we have

$$m_2(a) = |\widehat{\mathbb{1}_B}(\gamma)| = |\widehat{\mathbb{1}_D}(1)| = H_D(D) \leqslant H_D(C) \leqslant |\widehat{\mathbb{1}_C}(1)|.$$

with equality in the first inequality if and only if $D \in M_2(D)$. Since C is a punctured interval, it is not affine equivalent to an interval. So the first part of the lemma implies that $|\widehat{\mathbb{1}_C}(1)| \leqslant m_2(a)$. Thus we have equality everywhere and so $D \in M_2(D)$. Therefore B is the affine image of a punctured interval, as required. Further, if B is a punctured interval, then D is a punctured interval if and only if $\gamma = \pm 1$. This completes the proof of (ii). □

We will now prove Theorem 3.

Proof of Theorem 3. Recall that $p \geqslant 7$, $a \in [3, p-3]$ and $k > k_0(a,p)$ is sufficiently large with $k \not\equiv 1 \pmod p$. Let $I = [a]$. Given $t \in \mathbb{Z}_p$, write $\rho_t := (\widehat{\mathbb{1}_{I+t}}(1))^k \overline{\widehat{\mathbb{1}_{I+t}}(1)}$ as $r_t e^{\theta_t i}$, where $\theta_t \in [0, 2\pi)$ and $r_t > 0$. Then (17) says that θ_t equals $-\pi(2t+a-1)(k-1)/p$ modulo 2π. Increasing t by 1 rotates ρ_t by $-2\pi(k-1)/p$. Using the fact that $k-1$ is invertible modulo p, we have the following. If $(a-1)(k-1)$ is even, then the set of θ_t for $t \in \mathbb{Z}_p$ is precisely $0, 2\pi/p, \ldots, (2p-2)\pi/p$, so there is a unique t (resp. a unique t')

in $\mathbb{Z}_p$ for which $\theta_t = \pi + \pi/p$ (resp. $\theta_{t'} = \pi - \pi/p$). Furthermore, $t' = -(a-1) - t$ and $I + t' = -(I + t)$; thus $I + t$ and $I + t'$ have the same set of dilations. If $(a-1)(k-1)$ is odd, then the set of θ_t for $t \in \mathbb{Z}_p$ is precisely $\pi/p, 3\pi/p, \ldots, (2p-1)\pi/p$, so there is a unique $t \in \mathbb{Z}_p$ for which $\theta_t = \pi$. We call t (and t', if it exists) *optimal*.

Let t be optimal. To prove the theorem, we will show that $F(\xi \cdot (I + t)) < F(A)$ (and so $s_k(\xi \cdot (I + t)) < s_k(A)$) for any a-subset $A \subseteq \mathbb{Z}_p$ which is not a dilation of $I + t$.

We will first show that $F(I+t) < F(A)$ for any a-subset A which is not affine equivalent to an interval. By Lemma 13(i), we have that $|\widehat{\mathbb{1}_{I+t}}(\pm 1)| = m_1(a)$ and $\rho(A) \leqslant m_2(a)$. Let $m_2'(a)$ be the maximum of $\widehat{\mathbb{1}_J}(\gamma)$ over all length-a intervals J and $\gamma \in [2, p-2]$. Lemma 13(i) implies that $m_2'(a) < m_1(a)$. Thus

$$\left|F(I+t) - 2(m_1(a))^{k+1}\cos(\theta_t) - F(A)\right| \leqslant (p-1)(m_2(a))^{k+1} + (p-3)\left(m_2'(a)\right)^{k+1}. \quad (20)$$

Now $\cos(\theta_t) \leqslant \cos(\pi - \pi/p) < -0.9$ since $p \geqslant 7$. This together with the fact that $k \geqslant k_0(a,p)$ and Lemma 13 imply that the absolute value of $2(m_1(a))^{k+1}\cos(\theta_t) < 0$ is greater than the right-hand size of (20). Thus $F(I + t) < F(A)$, as required.

The remaining case is when $A = \zeta \cdot (I + v)$ for some non-optimal $v \in \mathbb{Z}_p$ and non-zero $\zeta \in \mathbb{Z}_p$. Since $s_k(A) = s_k(I + v)$, we may assume that $\zeta = 1$. Note that $\cos(\theta_t) \leqslant \cos(\pi - \pi/p) < \cos(\pi - 2\pi/p) \leqslant \cos(\theta_v)$. Thus

$$\begin{aligned} F(I+t) - F(I+v) &\leqslant 2(m_1(a))^{k+1}(\cos(\theta_t) - \cos(\theta_v)) + (2p-4)(m_2'(a))^{k+1} \\ &\leqslant 2(m_1(a))^{k+1}(\cos(\pi - \pi/p) - \cos(\pi - 2\pi/p)) + (2p-4)(m_2'(a))^{k+1} \\ &< 0 \end{aligned}$$

where the last inequality uses the fact that k is sufficiently large. Thus $F(I+t) < F(I+v)$, as required. □

Finally, using similar techniques, we prove Theorem 5.

Proof of Theorem 5. Recall that $p \geqslant 7$, $a \in [3, p-3]$ and $k > k_0(a,p)$ is sufficiently large with $k \equiv 1 \pmod p$. Let $I := [a]$ and $I' = [a-1] \cup \{a\}$.

Suppose first that a and k are both even. Let $A \subseteq \mathbb{Z}_p$ be an arbitrary a-set not affine equivalent to the interval I. By Lemma 13, I attains $m_1(a)$ (exactly at $x = \pm 1$), while $\rho(A) < m_1(a)$. Also, $m_2'(a) < m_1(a)$, where $m_2'(a) := \max_{\gamma \in [2,p-2]} |\widehat{\mathbb{1}_I}(\gamma)|$. Thus

$$\begin{aligned} F(I) - F(A) &\overset{(14),(15)}{\leqslant} 2\sum_{\gamma=1}^{\frac{p-1}{2}} (-1)^\gamma \left|\widehat{\mathbb{1}_I}(\gamma)\right|^{k+1} + \sum_{\gamma=1}^{p-1} \left|\widehat{\mathbb{1}_A}(\gamma)\right|^{k+1} \\ &\leqslant -2(m_1(a))^{k+1} + (2p-4)(\max\{m_2(a), m_2'(a)\})^{k+1} < 0, \end{aligned}$$

where the last inequality uses the fact that k is sufficiently large. So $s_k(a) = s_k(I)$. Using Lemma 13, the same argument shows that, for all $B \in \binom{\mathbb{Z}_p}{a}$, we have $s_k(B) = s_k(a)$ if and only if B is an affine image of I. This completes the proof of Part 1 of the theorem.

Suppose now that at least one of a, k is odd. Let A be an a-set not equivalent to I. Again by Lemma 13, we have

$$\begin{aligned} F(I)-F(A) &\geqslant \sum_{\gamma=1}^{p-1}\left|\widehat{\mathbb{1}_I}(\gamma)\right|^{k+1}-\sum_{\gamma=1}^{p-1}\left|\widehat{\mathbb{1}_A}(\gamma)\right|^{k+1} \\ &\geqslant 2(m_1(a))^{k+1}-(p-1)(m_2(a))^{k+1} \;>\; 0. \end{aligned}$$

So the interval I and its affine images have in fact the largest number of additive $(k+1)$-tuples among all a-subsets of $\mathbb{Z}_p$. In particular, $s_k(a) < s_k(I)$.

Suppose that there is some $A \in \binom{\mathbb{Z}_p}{a}$ which is not affine equivalent to I or I'. (If there is no such A, then the unique extremal sets are affine images of I' for all $k > k_0(a,p)$, giving the required.) Write $\rho := re^{\theta i} = \widehat{\mathbb{1}_{I'}}(1)$. Then by Lemma 13(ii), we have $r = m_2(a)$, and $\rho(A) \leqslant m_3(a)$. Given $k \geqslant 2$, let $s \in \mathbb{N}$ be such that $k = sp+1$. Then

$$\left|F(I')-2m_2(a)^{k+1}\cos(sp\theta)-F(A)\right| \leqslant (p-1)m_3(a)^{k+1}+(p-3)\left(m_2'(a)\right)^{k+1}. \tag{21}$$

Proposition 11 implies that there is an even integer $\ell \in \mathbb{N}$ for which $c := p\theta - \ell\pi \in (-\pi,\pi)\setminus\{0\}$. Let $\varepsilon := \frac{1}{3}\min\{|c|, \pi-|c|\} > 0$. Given an integer t, say that $s \in \mathbb{N}$ is *t-good* if $sc \in ((t-\frac{1}{2})\pi+\varepsilon, (t+\frac{1}{2})\pi-\varepsilon)$. This real interval has length $\pi - 2\varepsilon > |c| > 0$, so must contain at least one integer multiple of c. In other words, for all $t \in \mathbb{Z}\setminus\{0\}$ with the same sign as c, there exists a t-good integer $s > 0$. As $sp\theta \equiv sc \pmod{2\pi}$, the sign of $\cos(sp\theta)$ is $(-1)^t$. Moreover, Lemma 13 implies that $m_2(a) > m_3(a), m_2'(a)$. Thus, when $k = sp+1 > k_0(a,p)$, the absolute value of $2m_2(a)^{k+1}\cos(sp\theta)$ is greater than the right-hand side of (21). Thus, for large $|t|$, we have $F(A) > F(I')$ if t is even and $F(A) < F(I')$ if t is odd, implying the theorem by (14). □

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