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Anti-Ramsey properties of random graphs

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ABSTRACT

We call a coloring of the edge set of a graph G a b-bounded coloring if no color is used more than b times. We say that a subset of the edges of G is rainbow if each edge is of a different color. A graph has property $\mathcal{A}(b, H)$ if every b-bounded coloring of its edges has a rainbow copy of H. We estimate the threshold for the random graph $G_{n,p}$ to have property $\mathcal{A}(b, H)$.

1. Introduction

We call a coloring of the edge set of a graph G a *b-bounded coloring* if no color is used more than b times. We say that a subset of the edges of G is *rainbow* (or *polychromatic*) if each edge is of a different color. We consider the following question: What relationship between b, G and H implies that every b-bounded coloring of the graph G contains a rainbow copy of the graph H (i.e. a copy of H in which $E(H)$ is rainbow colored)? Note that this can be viewed as a variation on classical Ramsey theory, but here instead of a homogeneous (i.e. monochromatic) copy of H we are interested in a heterogeneous (i.e. rainbow) copy of H. Questions of this form have been studied in a number of contexts. Erdős, Simonovits and Sós considered the minimum number of colors needed to ensure a rainbow copy of H in every coloring of the edge set of K_n where we require that every color is used at least once [5]. Lefmann, Rödl and Wysocka considered some variations on this question where the restriction that each color is used at least once is replaced by other natural restrictions, including b-bounded coloring [13]. The existence of rainbow Hamilton cycles in edge colored copies of complete graphs was studied in [1,4,8,11]. The existence of rainbow stars was studied in Hahn [9,10] and Fraisse, Hahn and Sotteau [7]. The complexity of finding rainbow sub-graphs was studied by Fenner and Frieze [6]. Cooper and Frieze [3] studied the existence of polychromatic Hamilton cycles

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in random graphs. In this paper we study the existence of rainbow copies of a fixed graph H in b-bounded colorings of the random graph $G_{n,p}$.

Let H be a fixed graph. Let v_H and e_H denote the number of vertices and edges of H, respectively. For a positive integer b let $\mathcal{A}(b,H)$ denote the following graph property: $G \in \mathcal{A}(b,H)$ iff *every* b-bounded coloring of $E(G)$ has a rainbow copy of H. Define

$$m_H = \frac{e_H - 1}{v_H - 2},$$
$$m_H^* = \max\{m_{H'}\colon\ H' \subseteq H,\ v_{H'} \geqslant 3\},$$
$$p^* = \frac{1}{n^{1/m_H^*}}.$$

One can show that, unless maximum degree $\Delta_H = 1$, it is enough to consider only connected subgraphs H'.

Note that, when p is not too small, **whp** the number of copies of H in $G_{n,p}$ is $\Theta(n^{v_H} p^{e_H})$ while the number of edges in $G_{n,p}$ is $\Theta(np^2)$. (**Whp** stands for *with high probability*, that is, with probability $1 - o(1)$ as $n \to \infty$.) Thus if $p \ll p^*$ then the number of copies of H in $G_{n,p}$ is much fewer than the number of edges in $G_{n,p}$ and so it *should be the case* that **whp** it is easy to color the edges so that there is no rainbow copy of H. On the other hand, when $p \gg p^*$ there are so many copies of H relative to the number of edges that **whp** a rainbow copy of H should be unavoidable. So, at first glance, it is natural to expect p^* to be the threshold for the anti-Ramsey property $\mathcal{A}(b,H)$. Of course, this reasoning can also be applied to the classical Ramsey property, and p^* is (with a few exceptions) indeed the threshold for the Ramsey property that every coloring of $G_{n,p}$ with a set of r colors has a monochromatic copy of H as shown by Rödl and Ruciński [15]. See also Ruciński and Truszczyński [16] for a version where there are restrictions on the number of colors used locally.

There is one immediate exception to this general framework for the anti-Ramsey property $\mathcal{A}(b,H)$. Note that if H is a forest then $m_H^* = 1$ (assuming that $\Delta_H \geqslant 2$) but it turns out that there are trees that have the property $\mathcal{A}(b,H)$. Since $p = n^{-(k+1)/k}$ is the threshold probability for having a copy of *every* tree with k edges, it follows that $p = 1/n^{m_H^\star} = 1/n$ is not the threshold for the anti-Ramsey property $\mathcal{A}(b,H)$.

So we begin with a general result for arbitrary graphs that are not acyclic.

Theorem 1. *For all graphs H containing at least one cycle there exists a constant $b_0 = b_0(H)$ such that if $b \geqslant b_0$ then there exist $c_1 = c_1(b,H)$ and $c_2 = c_2(b,H)$ such that if $p = cn^{-1/m_H^*}$ then*

$$\lim_{n\to\infty} \mathbf{Pr}\bigl(G_{n,p} \in \mathcal{A}(b,H)\bigr) = \begin{cases} 0 & \text{if } c \leqslant c_1, \\ 1 & \text{if } c \geqslant c_2. \end{cases} \tag{1}$$

In truth 1-statement (when $c \geqslant c_2$) holds for all $b \geqslant 2$, as will be seen from an examination of the proof in Section 4.2.

Our proof of the 1-statement (when $c \geqslant c_2$) has been reduced to a few lines by a clever observation from one of the reviewers of the paper.

We study the threshold for $\mathcal{A}(b,K_3)$ in more detail. For $b = 2$ and $H = K_3$, the situation is completely resolved.

Theorem 2. *Let $p = \frac{c_n}{n^{2/3}}$. Then*

$$\lim_{n\to\infty} \mathbf{Pr}\bigl(G_{n,p} \in \mathcal{A}(2,K_3)\bigr) = \begin{cases} 0 & \text{if } c_n \to 0, \\ 1 - e^{-c^6/24} & \text{if } c_n \to c, \\ 1 & \text{if } c_n \to \infty. \end{cases}$$

Note that Theorem 2 shows that some condition on b is necessary in Theorem 1 (since $m_{K_3}^* = 2$). When $b = 3$ and $H = K_3$ there is an intriguing gap in our result.

Theorem 3. *Let* $p = \frac{c}{n^{1/2}}$. *Then,*

$$\lim_{n\to\infty} \mathbf{Pr}\big(G_{n,p} \in \mathcal{A}(3, K_3)\big) = \begin{cases} 1 - e^{-c^{10}/120} & \text{if } c < 1/\sqrt{2}, \\ 1 & \text{if } c > \sqrt{2}. \end{cases}$$

Theorem 3 leaves open the possibility of a 'one-sided-sharp' phase transition; to be precise, there could be a critical value $c \in [1/\sqrt{2}, \sqrt{2}]$ at which the probability that $G_{n,c/\sqrt{n}}$ has property $\mathcal{A}(3, K_3)$ quickly jumps from $1 - e^{-c^{10}/120}$ to 1. Finally, we note that (1) holds for $H = K_3$ and $b \geqslant 4$, see Remark 2 after the proof of Theorem 3.

We now turn to the anti-Ramsey thresholds for forests. For a tree T, let $s(b, T)$ be the minimum value s such that there exists a tree with s edges having property $\mathcal{A}(b, T)$. For a fixed forest F, the threshold for $\mathcal{A}(b, F)$ will then be $p = n^{-(s+1)/s}$ where s is the maximum of $s(b, T)$ over all connected components T of F. So the study of thresholds for $\mathcal{A}(b, F)$ amounts to the study of $s(b, T)$. We begin with the following general statement about the growth rate of $s(b, T)$ as b grows.

Theorem 4. *Let T be a fixed tree with diameter l, and set $m = \lfloor l/2 \rfloor$. Then (letting $b \to \infty$) we have*

$$s(b, T) = \Theta\big(b^m\big).$$

The upper bound in Theorem 4 is given by a certain class of trees which we conjecture always determines $s(b, T)$. Let T be a tree, e be an edge in T and b be a positive integer. In Section 5 we define the tree $B_{T,e,b}$ (which we dub the *b-blow up* of T centered at e) and show that $B_{T,e,b} \in \mathcal{A}(b, T)$.

Conjecture 1. *For any $b \geqslant 2$ and tree T,*

$$s(b, T) = \min_{e\in T}\big\{\big|E(B_{T,e,b})\big|\big\}.$$

In support of this conjecture, we verify it for paths and rooted trees with a constant branching factor. Using similar proof techniques we have verified the conjecture for a few other special classes of trees (e.g. the *m-fork* which consists of m leaves added to an endpoint of a path of length 3). The details for these other classes of trees are omitted for the sake of brevity. We note that Picollelli [14] has verified the conjecture for all trees of diameter at most four.

Theorem 5.

(a) *Let P_l be the path with l edges. We have*

$$s(b, P_l) = \begin{cases} (b+1)\sum_{i=0}^{k-1} b^i & \text{if } l = 2k, \\ 1 + 2\sum_{i=1}^{k} b^i & \text{if } l = 2k+1. \end{cases}$$

(b) *Let $T_{d,l}$ be a rooted tree, with all leaves at distance l from the root such that every non-leaf has the same degree d. Then*

$$s(b, T_{d,l}) = 1 + 2\sum_{i=1}^{l-1}\big(b(d-1)\big)^i + \big(b(d-1)\big)^l.$$

We prove our theorems in the following order. Theorem 2 is proved first in Section 2. Theorem 3 is proved in Section 3. The general theorem, Theorem 1, is proved in Section 4, and we discuss trees in Section 5.

A few words on our notation. We will use '$\subseteq$' to denote inclusion. The expression $a_n \sim b_n$ means that $\lim_{n\to\infty} a_n/b_n = 1$. The $O()$-notation is standard.

2. Proof of Theorem 2

We begin by noting that K_4 has the anti-Ramsey property $\mathcal{A}(2, K_3)$ (by proving the following, more general statement).

Lemma 6.

$$K_{r+2} \in \mathcal{A}(r, K_3) \quad \text{for } r \geqslant 1.$$

Proof. Assume for the sake of contradiction that a given r-bounded coloring of K_{r+2} does not have a rainbow triangle. Let C be a largest connected component, in terms of number of vertices, induced by edges of the same color, red say. The number of vertices in C is at most $r+1$ and so there is a vertex $v \notin C$. Consider the edges from v to C. They cannot be colored red and as there are no rainbow triangles they must all be the same color, blue say. But then the connected component induced by the blue edges that contains v has more vertices than C, contradiction. □

Now assume that $p = \frac{c}{n^{2/3}}$ and let Z_4 denote the number of copies of K_4 in $G_{n,p}$. Thus

$$\mathbf{E}(Z_4) = \binom{n}{4} p^6 \to \frac{c^6}{24}.$$

It is well known [2,12,17] that in this case Z_4 is asymptotically Poisson and so

$$\mathbf{Pr}(Z_4 = 0) \to e^{-c^6/24}.$$

Since $K_4 \in \mathcal{A}(2, K_3)$ and the property $\mathcal{A}(b, H)$ is monotone, we can prove Theorem 2 by showing that if $p = \frac{c}{n^{2/3}}$, c constant, then

$$\lim_{n\to\infty} \mathbf{Pr}\big(G_{n,p} \in \mathcal{A}(2, K_3) \mid G_{n,p} \text{ is } K_4\text{-free}\big) = 0. \tag{2}$$

We now define a *triangle graph* $\Gamma = (W, X)$ where W is the set of triangles of $G_{n,p}$ and $(T_1, T_2) \in X$ iff the triangles T_1, T_2 share an edge. If $C = \{T_1, T_2, \ldots, T_\ell\}$ is a connected component of Γ we define the *base graph* of C to be the sub-graph G_C of $G_{n,p}$ with vertex set $V_C = \bigcup_{i=1}^{\ell} V(T_i)$ and edge set $E_C = \bigcup_{i=1}^{\ell} E(T_i)$.

We say that a graph K is *d-degenerate* if there is an ordering $v_1, v_2, \ldots, v_k$ of the vertices of K such that each vertex v has at most d neighbors that appear before v in this ordering; to be precise,

$$\big|\big\{j\colon\ j < i \text{ and } \{v_i, v_j\} \in E(K)\big\}\big| \leqslant d$$

for every $i = 1, \ldots, k$. Note that for any component of Γ we have

$$|E_C| \geqslant 2|V_C| - 3$$

with equality iff G_C is 2-degenerate.

Lemma 7. *Let Γ be the triangle graph of $G_{n,p}$ with $p = c/n^{2/3}$ where c is constant.* **Whp** *every component C of Γ satisfies one of the following two conditions*

(a) *G_C is isomorphic to K_4, or*
(b) *G_C is 2-degenerate.*

Proof. We first show that **whp** $|V_C| \leqslant 6$ for all components C of Γ. Indeed, if there exists a component C of Γ such that $|V_C| \geqslant 7$ then there is a set of 7 vertices in $G_{n,p}$ that spans at least 11 edges. A simple first moment calculation shows **whp** that no such sub-graph of $G_{n,p}$ exists.

It remains to show that **whp** there are no components C of Γ such that G_C is not 2-degenerate and $V_C = 5$ or 6. However, these correspond to sub-graphs of $G_{n,p}$ with 5 vertices and 8 edges and sub-graphs of $G_{n,p}$ with 6 vertices and 10 edges, respectively. By the first moment method no such sub-graphs of $G_{n,p}$ exist. □

We are now ready to prove (2). Suppose $G_{n,p}$ is K_4-free and that every component C of Γ has G_C 2-degenerate. We color the edge set of $G_{n,p}$ by considering each component of Γ in turn. Consider a 2-degenerate ordering $v_1, \dots v_k$ of the vertices of G_C. We introduce one color for each vertex and color the edge $\{v_i, v_j\}$ with the color corresponding to the maximum of i and j. If $\{v_a, v_b, v_c\}$ is a triangle in C then the color corresponding to the maximum of a, b and c appears on 2 of the edges in triangle. Thus, this gives a 2-bounded coloring of the edges of $G_{n,p}$ with no rainbow K_3.

3. Proof of Theorem 3

Suppose first that $p = \frac{c}{n^{1/2}}$ and $c < 1/\sqrt{2}$.

Let Z_5 denote the number of copies of K_5 in $G_{n,p}$. We have

$$\mathbf{E}(Z_5) = \binom{n}{5} p^{10} \to \frac{c^{10}}{120} \quad \text{and} \quad \mathbf{Pr}(Z_5 = 0) \to e^{-c^{10}/120}.$$

Since $K_5 \in \mathcal{A}(3, K_3)$ by Lemma 6, we can prove the first part of Theorem 3 by showing that if $p = \frac{c}{n^{1/2}}$ and $c < 1/\sqrt{2}$ then

$$\lim_{n\to\infty} \mathbf{Pr}\big(G_{n,p} \in \mathcal{A}(3, K_3) \bigm| G_{n,p} \text{ is } K_5\text{-free}\big) = 0. \tag{3}$$

Let the triangle graph Γ be as defined in Section 2. A component C of Γ is *safe* if

$$|E_C| \leqslant 2|V_C|.$$

Lemma 8. Whp *every connected component C of Γ is safe.*

Proof. Consider the following process that generates all connected components of Γ. Choose 3 vertices u, v, w and let $V_0 = \{u, v, w\}$ and let $E_0 = \{\{u, v\}, \{u, w\}, \{v, w\}\}$. If u, v, w generate a triangle in G continue as follows: Suppose that we have generated a disjoint sequence of vertex sets $V_0, V_1, \dots, V_k$ and edge sets $E_1, E_2, \dots, E_k$. Initialize $V_{k+1} = E_{k+1} = \emptyset$ and then perform the following steps:

A. For each $z \notin V^{(k)} = \bigcup_{i=0}^{k} V_i$ and $e = \{x, y\} \in E_k$ see if both edges $\{x, z\}, \{y, z\}$ exist in $G_{n,p}$. If so, add these edges to E_{k+1} and z to V_{k+1}. This is done one vertex at a time and for each vertex it is done one edge at a time. We place z in V_{k+1} on the first success and then move on to the next vertex.
B. For each pair of vertices consisting of a vertex z in V_{k+1} and a vertex a in $V^{(k+1)}$ see if the edge $\{z, a\}$ is in $G_{n,p}$. If so add this edge to E_{k+1}.

Of course, we terminate when $V_{k+1} = \emptyset$ after step A. Let V_{final} and E_{final} be the vertex and edge sets, respectively, that are formed at the end of this process and let $C = C_{u,v,w}$ be the triangle component containing the triangle u, v, w (if this triangle appears). Note that E_{final} is not necessarily equal to E_C as we add edges in step B that are not necessarily involved in triangles. However, we do have $E_C \subseteq E_{\text{final}}$. Also,

$$|E_{\text{final}} \setminus E_C| \geqslant 2|V_{\text{final}} \setminus V_C|.$$

Thus, if $|E_{\text{final}}|$ is at most $2|V_{\text{final}}|$ then $C_{u,v,w}$ is safe. Since edges and vertices join at a ratio of 2 edges to each vertex during step A, we have $|E_{\text{final}}| \leqslant 2|V_{\text{final}}|$ iff the number of edges that join during a step B is at most 3.

Note that throughout our process the conditioning we impose on $G_{n,p}$ is of a very special form. At any given point we have fully queried certain edges (i.e. we are conditioning on the event that some set of edges appears and some other set of edges does not appear). Furthermore, since we have checked to see if certain pairs of edges appear in $G_{n,p}$ in step A we also condition on the event that a certain collection of pairs of edges do not appear. Since the latter is a downwardly closed event, it follows from the FKG inequality that when we condition on this event the probability that any set of k edges (that have not been fully queried) lie in $G_{n,p}$ is at most p^k.

We view our process as a sort of branching process in which the edges are the individuals and each edge that joins has a 'parent' edge that it attaches to. Let A_i be the number of step A children of the ith edge to join. Let B_i be the number of step B children of the ith edge to join. Note that there is ambiguity in the parent of a type B edge. We assign paternity to an arbitrarily chosen incident – and previously appearing – edge. Note also that each of these B_i edges shares a common vertex with its parent.

Define the constant $c' = 6/(\delta c)^2$ where $\delta > 0$ is defined by $c + \delta = 1/\sqrt{2}$. Let $K = c' \log n$. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. $Bi(n, p^2)$ random variables. Here Bi is used to denote the binomial random variable. Let $Y_1, Y_2, \ldots$ be a sequence of i.i.d. $Bi(2K, p)$ random variables. Note that $2X_i$ dominates A_i for all i while Y_i dominates B_i for $i = 1, \ldots, K$.

Let $\mathcal{S}$ be the event that the edges $\{u,v\}, \{u,w\}, \{v,w\}$ appear in $G_{n,p}$. If $\mathcal{S}$ occurs and $|E_{\text{final}}| > K$ then $\sum_{i=1}^K A_i > K - \sum_{i=1}^K B_i - 3$. If $\mathcal{S}$ occurs and $K \geqslant |E_{\text{final}}| > 2|V_{\text{final}}|$ then $\sum_{i=1}^K B_i \geqslant 4$.

Thus we have

$$\begin{aligned} Pr\big(|E_{\text{final}}| > 2|V_{\text{final}}| \,\big|\, \mathcal{S}\big) &\leqslant Pr\big(|E_{\text{final}}| > K \,\big|\, \mathcal{S}\big) + Pr\big(|E_{\text{final}}| > 2|V_{\text{final}}| \,\big|\, \mathcal{S} \wedge |E_{\text{final}}| \leqslant K\big) \\ &\leqslant \left[Pr\left(\sum_{i=1}^K 2X_i > K - 6 \right) + Pr\left(\sum_{i=1}^K Y_i \geqslant 4 \right) \right] + Pr\left(\sum_{i=1}^K Y_i \geqslant 4 \right). \end{aligned}$$

Now we apply the Chernoff bounds. Since the sum $\sum_{i=1}^K X_i$ is distributed as $Bi(Kn, p^2)$ we have

$$Pr\left(\sum_{i=1}^K X_i \geqslant \frac{K-6}{2} \right) \leqslant Pr\left(\sum_{i=1}^K X_i \geqslant Knp^2(1+\delta) \right) \leqslant \exp\left\{ -\delta^2 Knp^2/3 \right\} = \frac{1}{n^2}.$$

(Note that we use the fact $(1/\sqrt{2} - x)^2(1+x) < 1/2$ for x in the interval $(0, 1/\sqrt{2})$ and that we assume that n is sufficiently large.) For the sum of the Y_i's we simply have

$$Pr\left(\sum_{i=1}^K Y_i \geqslant 4 \right) \leqslant \binom{2K^2}{4} p^4 = O\left(\frac{(\log n)^8}{n^2} \right).$$

Therefore, by the union bound, the probability that there is a triangle component C that is not safe is

$$O\left(n^3 \left(\frac{1}{\sqrt{n}} \right)^3 \frac{(\log n)^8}{n^2} \right) = o(1). \qquad \square$$

Assume that all triangle components C are safe. We give an algorithm for coloring each triangle component in such a way that no triangle is rainbow. Consider a fixed component C of Γ. We define the graph D to be K_6 minus a perfect matching. Let $v_1, v_2, v_3, \ldots, v_\ell$ be the vertices of G_C listed so that

(i) If G_C contains a copy of D then this graph comes at the beginning of the sequence. If there is no copy of D, but there is a copy of $K_5 - e$ then this graph comes at the beginning of the sequence. Finally, if there is no copy of D or $K_5 - e$, but there is a copy of K_4 then this graph comes at the beginning of the sequence.
If G_C does not contain any of these graphs then the first three vertices in the sequence form a triangle.

(ii) Let v_k be the last vertex in our initial graph as defined in (i). Each subsequent vertex v_i, $i > k$ has at least 2 neighbors (called *back-neighbors* below) among $v_1, \ldots, v_{i-1}$ and the set of neighbors of v_i among $v_1, \ldots, v_{i-1}$ span at least one edge.

Property (ii) follows from the facts that the ordering of vertices by their addition to G_C satisfies it and that we can start growing G_C from any triangle, in particular, from one belonging to the targeted initial graph.

For $i = k+1, \dots, v$ let d_i be the number of neighbors v_i has among $v_1, \dots, v_{i-1}$. By assumption $d_i \geqslant 2$ for all $i > k$. Let $I_t = \{i > k\colon\ d_i = t\}$, $t \geqslant 3$ and $I = I_3 \cup I_4 \cup I_5$. Note that our assumption that C is safe implies that $I_t = \emptyset$ for $t \geqslant 6$ and $|I_3| + 2|I_4| + 3|I_5| \leqslant 3$. Furthermore,

$$G_C \text{ contains } D \implies I = \emptyset,$$
$$G_C \text{ contains } K_5 - e \implies I_4 \cup I_5 = \emptyset,$$
$$G_C \text{ contains } K_4 \implies I_5 = \emptyset \text{ and } |I_3| + 2|I_4| \leqslant 2.$$

We first check that $K_5 - e$ and D can be colored without creating a rainbow triangle.

$K_5 - e$: Suppose that $e = \{4, 5\}$. The following table shows a coloring without a rainbow triangle:

{1,2}	{1,3}	{1,4}	{1,5}	{2,3}	{2,4}	{2,5}	{3,4}	{3,5}
1	1	2	3	1	2	3	2	3

D: Suppose that the deleted matching is $\{1,4\}, \{2,5\}, \{3,6\}$. The following table shows a coloring without a rainbow triangle:

{1,2}	{1,3}	{1,5}	{1,6}	{2,3}	{2,4}	{2,6}	{3,4}	{3,5}	{4,5}	{4,6}	{5,6}
1	2	2	1	1	3	4	3	3	4	4	2

We then use the following basic coloring algorithm to color the remainder of E_C: color the edges between v_i and $v_1, \dots, v_{i-1}$ with the same color i. This always gives a coloring with no rainbow K_3 (the color of the last vertex in each triangle appears on 2 of the edges in the triangle). However, the coloring is 3-bounded only if $d_i \leqslant 3$ for all i. For example, the algorithm succeeds if G_C contains a copy of D or $K_5 - e$, because here $I_4 = I_5 = \emptyset$. We henceforth assume that G_C does not contain either of these graphs. We now describe how to modify this algorithm for the remaining cases. The availability of *free colors* (that is, colors used less than three times in this basic coloring) will help us in this task. For the sake of brevity, we will mention only the changes needed to fix this coloring.

Case 1. There exists an i such that $d_i = 5$.

In this case we have $d_j = 2$ for $j > k$, $j \neq i$. If the back-neighbors of v_i are $v_{i_1}, \dots, v_{i_5}$ then we recolor each $\{v_i, v_{i_s}\}$ with color i_s. Any triangle formed by i and 2 of its back-neighbors can be expressed as v_i, v_{i_s}, v_{i_t} where $i_s > i_t$, say. This triangle will then have two edges of color i_s.

Case 2. There exists an index i such that $d_i = 4$.

Let the back-neighbors of v_i be $v_{i_1}, \dots, v_{i_4}$ where $i_1 < i_2 < i_3 < i_4$.

Case 2a. $d_{i_4} = 2$.

Here, we use the color i_4 for the edge $\{v_{i_4}, v_i\}$.

Note that if C contains a copy of K_4 then, assuming that $d_i = 4$, we are in Case 2a (otherwise $i_4 \leqslant k = 4$ and we have a copy of K_5, a contradiction). Assume for the remaining sub-cases that C does not contain a copy of K_4.

Case 2b. $d_{i_4} = 3$.

We have

$$d_j = 2 \quad \text{for } j < i,\ j \neq i_4.$$

Now we consider the graph X induced by $\{v_{i_1}, \dots, v_{i_4}\}$. By assumption X has at most 4 edges (otherwise we have a K_5 or $K_5 - e$). Since C does not contain a copy of K_4, X does not contain a triangle. We may assume that v_{i_4} is adjacent to v_{i_3}: Otherwise we can just recolor the edge $\{v_{i_3}, v_i\}$ with color i_3. Also, we may assume that X has no isolated vertex v_p: Otherwise $p = i_1$ or i_2, and we can recolor the edge $\{v_p, v_i\}$ with color p.

Therefore, we can now restrict our attention to one of the following cases listed below:

Case 2bi. X is 2 disjoint edges.

One of the edges in X is $\{v_{i_1}, v_{i_2}\}$. The color i_2 is a free color, so we can use it to recolor the edge $\{v_i, v_{i_2}\}$.

Case 2bii. X is a path of length 3.

If v_{i_2} is an endpoint connected to v_{i_1}, then we are done by recoloring $\{v_{i_2}, v_i\}$ with color i_2. Thus we can assume that our path is the union of two sub-paths going monotonely up and ending in v_{i_4}. (One of the sub-paths can be empty.) Take the longer sub-path, let it begin with edge $\{v_{i_b}, v_{i_a}\}$, $b < a \leqslant 3$. We recolor the edge $\{v_{i_b}, v_i\}$ with color i_a (thus color i_a forms a path of length 3 after the recoloring).

Case 2biii. X is a 3-star.

Note that the center of the star cannot be v_{i_4} (since the back-neighbors of v_{i_4} must span an edge). Therefore, one of the edges in the star has a free color. Use this color on the edge from the corresponding leaf to i.

Case 2biv. X is 4-cycle.

Let v_p be the vertex not in $\{v_{i_1}, v_{i_2}, v_{i_3}\}$ that is a back-neighbor of v_{i_4}. Let v_q be the neighbor of v_{i_4} in $\{v_{i_1}, v_{i_2}\}$. Let v_s be the other vertex in $\{v_{i_1}, v_{i_2}\}$. We have the following sub-cases.

- v_p is not adjacent to v_q.
  The edge $\{v_{i_4}, v_q\}$ has no conditions on its color (relative to v_{i_4}) i.e. v_{i_4}, v_p, v_q and v_{i_4}, v_{i_3}, v_q do not form triangles. Using this observation we can proceed as follows. We re-color edge $\{v_{i_4}, v_q\}$ with color i. We then color edge $\{v_i, v_{i_4}\}$ with color i_4. We color edge $\{v_i, v_{i_3}\}$ with color i_3 (and the edges $\{v_i, v_{i_1}\}$, $\{v_i, v_{i_2}\}$ keep color i).
- v_p is adjacent to v_q but not to v_{i_3}.
  We replace the color on $\{v_p, v_{i_4}\}$ with the color on $\{v_p, v_q\}$. Then use color i_4 on the other edges incident to v_{i_4}, including the edge to v_i.
- v_p is adjacent to v_q and to v_{i_3}.
  Note first that $p < i_3$ as otherwise the back neighbors of v_p do not span an edge. Since v_p and v_s are the (only) back-neighbors of v_{i_3} there must be an edge between them. Now we have a copy of D with the deleted matching being $\{v_{i_3}, v_q\}, \{v_{i_4}, v_s\}, \{v_i, v_p\}$, contradiction.

This completes the proof of (3) and the first part of the proof of Theorem 3.

Suppose now that $c > \sqrt{2}$. **Whp** $G_{n,p}$ has $(1+o(1))cn^{3/2}/2$ edges, $(1+o(1))c^3n^{3/2}/6$ triangles and $o(n^{3/2})$ copies of K_4. Suppose that we have a 3-bounded coloring and A_i is the set of colors that are used i times and $a_i = |A_i|$ for $i = 1, 2, 3$. Thus,

$$a_1 + 2a_2 + 3a_3 = \big(1 + o(1)\big)cn^{3/2}/2. \tag{4}$$

Suppose that there are no rainbow triangles. Then each triangle T contains a pair of edges of the same color $c(T)$. For color x let $t(x)$ be the number of triangles T such that $c(T) = x$. So $t(x) = 0$ for

$x \in A_1$, $t(x) \leqslant 1$ for $x \in A_2$ and $t(x) \leqslant 2$ for $x \in A_3$, unless x is used to color three edges of a copy of K_4. These latter colors are relatively rare (since the total number of K_4-sub-graphs is $o(n^{3/2})$) and so we have

$$a_2 + 2a_3 \geqslant \big(1 + o(1)\big)c^3 n^{3/2}/6. \tag{5}$$

It follows from (4) and (5) that

$$\frac{c^3}{4} \leqslant \frac{c}{2} \text{ or } c \leqslant \sqrt{2}.$$

This contradiction completes the proof of Theorem 3.

Remark 1. The bound $c > \sqrt{2}$ in Theorem 3 can be improved. For example we could remove from our accounting those edges that are not in triangles. Or we could note that isolated triangles (which **whp** form a non-negligible proportion of the triangles) must be accounted for by colors x such that $t(x) = 1$. While these arguments improve this upper bound, they do not completely close the gap between the bounds in Theorem 3.

Remark 2. If $b \geqslant 4$ and $c < 1/\sqrt{2}$ then $\lim_{n\to\infty} \mathbf{Pr}(G_{n,p} \in \mathcal{A}(b, K_3)) = 0$. To see this observe that $K_5 \notin \mathcal{A}(b, K_3)$ for $b \geqslant 4$. Second, **whp** no two copies of K_5 in $G_{n,p}$ share an edge. Thus we can color all copies of K_5 without creating a rainbow copy of K_3. The rest of the edges can now be colored as in the proof of Theorem 3.

4. Proof of Theorem 1

4.1. Small c

Let $p = cn^{-1/m_H^*}$. We first consider the case where c is sufficiently small. We can assume in fact that

$$m_H > m_{H'} \quad \text{for all } H' \subsetneq H \text{ with } v_{H'} \geqslant 3.$$

For if not, and $m_H^* = m_{H'}$ for a sub-graph H' of H then we can show that it is possible to color $G_{n,p}$ without creating a rainbow copy of H', which of course shows there is no rainbow copy of H. It follows that if $H' \subsetneq H$ and $v_{H'} \geqslant 3$ then

$$\delta_{H'} \stackrel{\text{def}}{=} \frac{e_H - e_{H'}}{m_H} - v_H + v_{H'} = (v_{H'} - 2)\left(1 - \frac{m_{H'}}{m_H}\right) > 0.$$

Define

$$\delta_H = \min\big\{\delta_{H'}\colon\ H' \subsetneq H,\ v_{H'} > 2\big\}.$$

We follow a similar strategy to that in the previous section. In place of the triangle graph Γ we will have the H-graph Γ_H whose vertices are the copies of H in $G_{n,p}$ and in which two vertices H_1, H_2 are joined by an edge in Γ_H if H_1, H_2 share at least one edge in $G_{n,p}$.

A component C of Γ_H is *safe* if G_C is $b(H)$-degenerate where we set

$$b(H) \stackrel{\text{def}}{=} \Delta_H + m_H v_H - e_H + 1$$

and Δ_H is the maximum degree in H. Recall that G_C is $b(H)$-degenerate if we can order $V_C = \{v_1, v_2, \ldots, v_\ell\}$, $\ell = |V_C|$ such that each v_i has at most $b(H)$ neighbors among $v_1, v_2, \ldots, v_{i-1}$.

Lemma 9. Whp *every connected component C of Γ_H is safe.*

Proof. In analogy to the proof of Lemma 8, we consider a process where we choose a set of vertices $V_0 = \{v_1, v_2, \dots, v_{v_H}\}$, let E_0 consist of all edges spanned by V_0, and if E_0 contains a copy of H, we do a search that generates an edge set E_{final} that contains E_C where C is the corresponding component of Γ_H. We generate sets V_i, E_i, $i = 1, 2, \dots, k$ via an iterative application of the following 2 steps until $V_{k+1} = \emptyset$ after step A:

A. For each set of v_H vertices that contains some $z \notin V^{(k)} \stackrel{\text{def}}{=} \bigcup_{i=0}^k V_i$ and some $e \in E_k$ determine if this set of vertices gives a copy of H. When we find such a copy of H we add $V(H) \setminus V^{(k)}$ to V_{k+1}, add $E(H) \setminus E^{(k)}$ to E_{k+1} and move on. It is important to stress that once a vertex z is added to V_{k+1} we do not query any other vertex set that contains z.
B. For each pair of vertices consisting of a vertex z in V_{k+1} and a vertex a in $V^{(k+1)}$ see if the edge $\{z, a\}$ is in $G_{n,p}$. If so add this edge to E_{k+1}.

As in the proof of Lemma 8, the conditioning on $G_{n,p}$ imposed by this search is of a very special form. At any stage, certain edges are fully queried and we further condition on the event that certain sets of edges do not appear. Under any conditioning of this form, the probability that any set of k (not fully queried) edges appears in $G_{n,p}$ is at most p^k. Note further that after step B we have fully queried all edges within $V^{(k+1)}$. Let E_{final} be the edge set generated when this process terminates.

Again, we view this as a branching process where the edges are individuals. Here we have three ways in which an edge $e \in E_k$ can have offspring:

1. copies of H found in step A such that $V(H) \cap V^{(k)} = e$,
2. copies of H found in step A such that $e \in E(H)$ but $|V(H) \cap V^{(k)}| \geqslant 3$, and
3. edges added during step B.

Of course, there is some ambiguity in assigning the paternity of edges of types 2 and 3. This is done arbitrarily. Let A_i, B_i and C_i be the number of type 1, 2 and 3 offspring, respectively, of the ith edge to join E_{final}. For simplicity of our formulas, the edges that are in E_0 but not in the initial copy of H, are accounted for by increasing appropriate C_i's.

Let $K = C \log n$, where $C = C(c, H)$ is a sufficiently large constant. Let $X_1, X_2, \dots$ be i.i.d. $Bi(n^{v_H - 2}, p^{e_H - 1})$ random variables, let $Y_1, Y_2, \dots$ be i.i.d.

$$\sum_{H' \subsetneq H, v_{H'} \geqslant 3} Bi\big(K^{v_{H'}-2} n^{v_H - v_{H'}}, p^{e_H - e_{H'}}\big)$$

random variables, and let $Z_1, Z_2, \dots$ be i.i.d. $Bi(2K, p)$ random variables. We see that A_i, B_i and C_i are dominated by $(e_H - 1)X_i$, $(e_H - 1)Y_i$ and Z_i, respectively, for $i \leqslant K$. We have

$$E\left[\sum_{i=1}^K (e_H - 1)X_i\right] = K(e_H - 1)n^{v_H - 2} p^{e_H - 1} = K(e_H - 1)c^{e_H - 1}.$$

So if c is sufficiently small the Chernoff bound implies

$$\begin{aligned} &Pr\left(\sum_{i=1}^K (e_H - 1)X_i \geqslant K - e_H - (m_H v_H - e_H + 1) - \frac{e_H - 1}{\delta_H}\left[v_H - \frac{e_H}{m_H} + 1\right]\right) \\ &\quad = O\big(n^{e_H/m_H - v_H - 1}\big). \end{aligned} \tag{6}$$

The sum $\sum_{i=1}^K Y_i$ is distributed as

$$\sum_{H' \subsetneq H : v_{H'} \geqslant 3} Bi\big(K \cdot K^{v_{H'}-2} n^{v_H - v_{H'}}, p^{e_H - e_{H'}}\big).$$

Let $\mathcal{I}$ denote the set of sequences of non-negative integers $(i_{H'}\colon\ H' \subsetneq H,\ v_{H'} \geqslant 3)$ such that the sum of the $i_{H'}$'s is $\lceil \frac{1}{\delta_H}[v_H - \frac{e_H}{m_H} + 1]\rceil$. The probability that $\sum_{i=1}^{K}(e_H - 1)Y_i$ is at least $\frac{e_H-1}{\delta_H}[v_H - \frac{e_H}{m_H} + 1]$ is bounded by

$$\begin{aligned}
&\sum_{(i_{H'})\in\mathcal{I}} \prod_{H'\subsetneq H\colon\ v_{H'}\geqslant 3} \binom{K\cdot K^{v_{H'}-2}n^{v_H-v_{H'}}}{i_{H'}} \left(p^{e_H-e_{H'}}\right)^{i_{H'}} \\
&\quad < \sum_{(i_{H'})\in\mathcal{I}} \prod_{H'\subsetneq H\colon\ v_{H'}\geqslant 3} K^{v_H i_{H'}} \left(n^{v_H-v_{H'}} p^{e_H-e_{H'}}\right)^{i_{H'}} \\
&\quad < \sum_{(i_{H'})\in\mathcal{I}} \prod_{H'\subsetneq H\colon\ v_{H'}\geqslant 3} K^{v_H i_{H'}} n^{-\delta_{H'} i_{H'}} \\
&\quad < K^{O(1)} \sum_{(i_{H'})\in\mathcal{I}} \prod_{H'\subsetneq H\colon\ v_{H'}\geqslant 3} n^{-\delta_H i_{H'}} \\
&\quad \leqslant K^{O(1)} \sum_{(i_{H'})\in\mathcal{I}} n^{e_H/m_H - v_H - 1}.
\end{aligned}$$

(Note that $\delta_H \leqslant \delta_{H'}$ for any $H' \subsetneq H$ by definition.) Since there are $|\mathcal{I}| = K^{O(1)}$ sequences we have

$$Pr\left(\sum_{i=1}^{K}(e_H - 1)Y_i \geqslant \frac{e_H - 1}{\delta_H}\left[v_H - \frac{e_H}{m_H} + 1\right]\right) = K^{O(1)} n^{e_H/m_H - v_H - 1}. \tag{7}$$

Finally, we have

$$\begin{aligned}
Pr\left(\sum_{i=1}^{K} Z_i \geqslant m_H v_H - e_H + 1\right) &\leqslant \binom{2K^2}{m_H v_H - e_H + 1} p^{m_H v_H - e_H + 1} \\
&= K^{O(1)} n^{e_H/m_H - v_H - 1/m_H}.
\end{aligned} \tag{8}$$

Since the expected number of the initial graphs H is at most $n^{v_H} p^{e_H} = n^{v_H - e_H/m_H}$, the union bound applied to (6), (7) and (8) shows that **whp** every component of Γ_H has at most K edges. The desired $b(H)$-degenerate ordering then follows from (8). □

Of course, if every component of Γ_H is safe and $b \geqslant b(H)$ then one can color the edges of G so that there are no rainbow copies of H. To color E_C for a component C, we simply use the same new color for every edge from v_i to $\{v_1, v_2, \ldots, v_{i-1}\}$ for $1 \leqslant i \leqslant |V_C|$. Then every copy of H in C has a last vertex in the order and our coloring prevents this copy being rainbow. (Note that we use the fact that H has minimum degree at least 2, which follows from the assumption $m_H > m_{H'}$ for all sub-graphs H' and the inequality $m_H > 1$.)

4.2. Large c

As already mentioned, the following proof is due to a reviewer of the paper. We will show that if every coloring of the edges of graph G with b colors contains a monochromatic copy of H then $G \in \mathcal{A}(b, H)$. Thus the claimed result for large c follows immediately from Rödl and Ruciński [15].

Indeed, given a b-bounded coloring of G, let the edges colored i be denoted $e_{i,1}, e_{i,2}, \ldots, e_{i,b_i}$ where $b_i \leqslant b$ for all i. Now consider the auxilliary coloring in which edge $e_{i,j}$ is colored with j. At most b colors are used and so in the auxilliary coloring there will be a monochromatic copy of H. The definition of the auxilliary coloring implies that this copy of H is rainbow in the original coloring.

5. Trees

We first define the tree $B_{T,e,b}$ and prove that $B_{T,e,b} \in \mathcal{A}(b,T)$.

Let $e = \{x, y\}$ be an edge of the tree T. For each vertex v in T let ℓ_v be the distance from v to e (so $\ell_x = \ell_y = 0$) and let S_v be the set of all strings of the form $(v, i_1, i_2, \ldots, i_{\ell_v})$ where $i_1, i_2, \ldots, i_{\ell_v}$ are integers in the set $\{1, 2, \ldots, b\}$. Note that we have $S_x = \{(x)\}$ and $S_y = \{(y)\}$. The vertex set of the $B_{T,e,b}$ is $\bigcup_{v\in V(T)} S_v$. In addition to the edge $\{(x), (y)\}$, we place an edge between vertex $(v, i_1, \ldots, i_{\ell_v})$ and $(w, j_1, \ldots, j_{\ell_w})$ if and only if

(a) w and v are adjacent in T, and
(b) $i_k = j_k$ for $k = 1, \ldots, \ell_v$ (where we assume $\ell_w = \ell_v + 1$).

We call the set of edges in $B_{T,e,b}$ between a vertex in S_v and a set of vertices in S_w, where $\ell_w = \ell_v + 1$, a *bundle* of edges. We also let the singleton edge $\{(x), (y)\}$ form a bundle. Note that the edge set of $B_{T,e,b}$ is the disjoint union of the set $\mathcal{B}$ of bundles.

Let Ω be the set of colors in an arbitrary b-bounded coloring of $B_{T,e,b}$. For each bundle $B \in \mathcal{B}$ let C_B be the set of colors used on the edges in B. Let $X \subseteq \mathcal{B}$. Since the coloring is b-bounded we have

$$\left|\bigcup_{B\in X} C_B\right| \geqslant \frac{1}{b}\sum_{B\in X}|B| \geqslant \frac{(|X|-1)b+1}{b}.$$

Since the cardinality of this union is an integer, it is at least $|X|$. So, by Hall's Theorem, there is a system of distinct representatives of the sets $\{C_X\colon\ X \in \mathcal{B}\}$.

This system of distinct representatives corresponds to a set Y of edges in $B_{T,e,b}$ such that there is exactly one edge from each bundle in Y and the colors on the edges in Y are all different. This set of edges defines a rainbow copy of T (as well as some extra components) and shows that $B_{T,e,b} \in \mathcal{A}(b,T)$.

5.1. Special cases: proof of Theorem 5

We begin by showing that for the path P_l with l edges we have

$$s(b, P_l) = \begin{cases} (b+1)\sum_{i=0}^{m-1} b^i & \text{if } l = 2m, \\ 1 + 2\sum_{i=1}^{m} b^i & \text{if } l = 2m+1. \end{cases} \tag{9}$$

Observe first that since the b-blow up of P_l centered on the edge e at the middle of the path is in $\mathcal{A}(b, P_l)$, the above expression is an upper bound on $s(b, P_l)$.

For the lower bound we use induction on l with cases $l = 1, 2$ being trivial. Let a tree U give a rainbow P_l for every b-bounded coloring of U. Partition $E(U)$ as $X \cup F_1 \cup \cdots \cup F_k$ so that

(i) for each $1 \leqslant i \leqslant k$, $|F_i| \leqslant b$,
(ii) for each $1 \leqslant i \leqslant k$ and every path (x_1, x_2, x_3, x_4) in H, if the edge $\{x_2, x_3\}$ belongs to F_i, then F_i contains the edge $\{x_1, x_2\}$ or the edge $\{x_3, x_4\}$ (in other words each F_i consists of all edges in U that intersect some set of vertices), and
(iii) $|X|$ is as small as possible (given (i) and (ii)).

Note that every b-bounded coloring of the forest X yields a rainbow P_{l-2}; otherwise, we color the forest X with no rainbow P_{l-2} and each F_i with its own color to give a coloring of U with no rainbow P_l. Thus for some component Y of X we have $|E(Y)| \geqslant s(b, P_{l-2})$. By induction $|E(Y)|$ is bounded below by the expression in (9).

In order to count the edges in U, we assign the other components of X and the parts F_i to vertices of Y according to their vertex of attachment. (I.e. the vertex z and its incident edges are assigned to $y \in Y$ if the path from z to y is edge disjoint from $E(Y)$.)

We claim that for each vertex $y \in Y$ of Y-degree d there are at least $b - d + 1$ edges attached to Y in this way. Indeed, if this is not true, then form a new F_i-set by putting together all edges of Y incident to y, plus all parts attached to y. The new F_i has at most b edges and $|X|$ has strictly decreased. Take any path (x_1, x_2, x_3, x_4) with the edge $\{x_2, x_3\} \in F_i$. If $y \notin \{x_2, x_3\}$, then both edges $\{x_1, x_2\}$ and $\{x_3, x_4\}$ are in F_i. If, say, $y = x_2$ then the edge $\{x_1, x_2\} \in F_i$. The claim has been proved.

For $x \in V(Y)$ let $d_Y(x)$ be the Y-degree and $f_Y(x)$ the aggregate number of edges of $E(U \setminus Y)$ assigned to x. We have

$$\sum_{x \in V(Y)} \big(d_Y(x) + f_Y(x)\big) \geqslant (b+1)\big|V(Y)\big|.$$

But this sum equals $2|E(Y)| + |E(U) \setminus E(Y)| = |E(U)| + |E(Y)|$. So,

$$\big|E(U)\big| \geqslant (b+1)\big|V(Y)\big| - \big|E(Y)\big| = b\big|E(Y)\big| + b + 1 \geqslant b \cdot s(b, P_{l-2}) + b + 1,$$

as required to complete the proof of (9).

Now we turn to part (b) of Theorem 5. We will show that

$$s(b, T_{d,l}) = 1 + 2\sum_{i=1}^{l-1}\big(b(d-1)\big)^i + \big(b(d-1)\big)^l \tag{10}$$

where $T_{d,l}$ is a rooted tree, with all leaves at distance l from the root such that every non-leaf has the same degree d.

Observe first that the tree $B_{T_{d,l},e,b}$ with e being any edge incident with the root shows that our expression is an upper bound for $s(b, T_{d,l})$.

For the lower bound we again proceed by induction on l. The case $l = 1$ is simple: a tree with at most $b(d-1)$ edges can be colored using only $d-1$ colors. Let $l \geqslant 2$ and let U be a tree with the coloring property (i.e. $U \in \mathcal{A}(b, T_{d,l})$). We again grow classes F_i as in the proof of (9) but this time the restriction on their size is $(d-1)b$ (to be precise, we partition U into $X, F_1, \ldots, F_k$ such that we have $|F_i| \leqslant (d-1)b$, (ii) and (iii)). Note that every b-bounded coloring of the forest X yields a rainbow $T_{d,l-1}$; otherwise, we color the forest X with no rainbow $T_{d,l-1}$ and each F_i with its own set of $d-1$ colors to give a coloring of U with no rainbow $T_{d,l}$. Therefore, X has a component Y such that $|E(Y)| \geqslant s(b, T_{d,l-1})$, which is equal to the expression in (10) by induction. We again assign the other components of X and the parts F_i to vertices of Y according to their vertex of attachment and let $d_Y(x)$ be the Y-degree and $f_Y(x)$ the aggregate number of edges of $E(U \setminus Y)$ assigned to x. We have

$$\big|E(U)\big| + \big|E(Y)\big| = \sum_{x \in V(Y)} \big(d_Y(x) + f_Y(x)\big) \geqslant \big((d-1)b + 1\big)\big|V(Y)\big|.$$

But then

$$\big|E(U)\big| \geqslant \big((d-1)b + 1\big)\big|V(Y)\big| - \big|E(Y)\big| = (d-1)b\big|E(Y)\big| + (d-1)b + 1,$$

giving the required lower bound.

5.2. *Proof of Theorem 4*

For the upper bound, consider a b-blow up of T centered on an edge e that is in the middle of a longest path in T. The upper bound follows from the fact that this blow-up is in $\mathcal{A}(b, T)$ and has $O(b^m)$ vertices.

For the lower bound it is enough to note that if a tree H is a sub-graph of T then $s(b, H) \leqslant s(b, T)$. Since T contains the path P_l and $s(b, P_l) = \Omega(b^m)$, we have the desired lower bound.

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