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HYPERGRAPHS WITH INDEPENDENT NEIGHBORHOODS

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For each $k \geq 2$, let $\rho_k \in (0,1)$ be the largest number such that there exist k-uniform hypergraphs on n vertices with independent neighborhoods and $(\rho_k + o(1))\binom{n}{k}$ edges as $n \to \infty$. We prove that $\rho_k = 1 - 2\log k/k + \Theta(\log\log k/k)$ as $k \to \infty$. This disproves a conjecture of Füredi and the last two authors.

1. Introduction

The *neighborhood* $N(S)$ of a $(k-1)$-set S in a k-uniform hypergraph (henceforth a *k-graph*) is the set of vertices v such that $S \cup \{v\}$ is an edge. For $n \geq k \geq 2$, let $f(n,k)$ be the maximum number of edges in a k-graph on n vertices such that all its neighborhoods are *independent* sets (that is, span no edge). Mantel proved in 1907 that $f(n,2) = \lfloor n^2/4 \rfloor$, and this was the first result in extremal graph theory. Thus the problem of computing $f(n,k)$ is a natural generalization of Mantel's result.

A k-graph is *odd* if it has a vertex partition $X \cup Y$ such that all edges have an odd number of points less than k in Y. It is easy to see that all neighborhoods in an odd k-graph are independent sets. Let $b(n,k)$ be the maximum number of edges in an odd k-graph. Then the previous observation

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implies that $f(n,k) \geq b(n,k)$. It was conjectured in [8] that there exists some function $n_0(k)$ such that $n > n_0(k)$ implies

$$f(n,k) = b(n,k). \tag{1}$$

There was some evidence for this, as it reduces to Mantel's theorem for $k=2$, and it was proved for $k=3$ by Füredi, Pikhurko, and Simonovits [9, 10], thereby settling a conjecture of Mubayi and Rödl [18]. Recently, (1) has also been proved for $k=4$ [8]. As we will show here, (1) is not that far from the truth for $k=5$.

Since exact results are rare in extremal hypergraph theory, one often studies asymptotics. In this case, we can define $\rho_k = \lim_{n\to\infty} f(n,k)/\binom{n}{k}$ which is easily shown to exist [12]. Now conjecture (1) implies that $\rho_k = 1/2$ for all even k and $\rho_k \uparrow 1/2$ as $k \to \infty$ for odd k. Thus a weaker statement than (1) would be that $\rho_k = \lim_{n\to\infty} b(n,k)/\binom{n}{k}$, and an even weaker statement is that $\rho_k \to 1/2$ as $k \to \infty$.

In this paper we show that conjecture (1) is false for all $k \geq 7$, and in fact that $\rho_k \to 1$. This follows from an old construction of Kim and Roush [16] which gives lower bounds for the Turán problem for complete k-graphs. Thus the small cases shed little light on the behavior of ρ_k.

We are able to obtain rather sharp estimates on the rate at which ρ_k converges to 1:

Theorem 1. *As $k \to \infty$, we have*

$$1 - \frac{2\log k}{k} + (1+o(1))\,\frac{\log\log k}{k} \leq \rho_k \leq 1 - \frac{2\log k}{k} + (5+o(1))\,\frac{\log\log k}{k},$$

where log *denotes the natural logarithm. Furthermore, for $k \geq 7$, we have $\rho_k > 1/2$, hence (1) is false for $k \geq 7$.*

This leaves open the cases $k = 5$ and 6, where we believe that (1) still holds.

Conjecture 1. $f(n,k) = b(n,k)$ for $k \in \{5,6\}$ and n sufficiently large.

We will present the lower bounds in Theorem 1 via constructions in the next section. Sections 3 and 4 are devoted to the proof of the upper bound. In Section 5 we prove that $40/81 = 0.493\ldots \leq \rho_5 < 0.534$. We close with some concluding remarks and related open problems.

We associate a k-graph with its edge set. For a vertex subset S of size k–1, let $d(S) = |N(S)|$. Let $\binom{V}{k} = \{X \subset V : |X| = k\}$. We denote $[n] = \{1,\ldots,n\}$. Let $\mathrm{Bin}(k,p)$ denote the binomial distribution with parameters k and p. In Sections 2–4, the asymptotic notation ($O(1)$, $o(1)$, etc.) will refer to the case when k is fixed and $n \to \infty$.

2. Construction

In this section we prove the lower bound in Theorem 1 by means of a construction due to Kim and Roush. As we will mention in Section 6, this is not the only construction that can be used for this result, but it appears to be the simplest one.

Construction 1 (Kim and Roush [16]). Let $Y_1 \cup \cdots \cup Y_l$ be a partition of $[n]$ into sets, each of size $\lfloor n/l \rfloor$ or $\lceil n/l \rceil$. Let the k-graph $\mathcal{H}$ consist of all k-sets that have at least one point in each Y_i. Partition $\mathcal{H}$ into $\mathcal{H}_1 \cup \cdots \cup \mathcal{H}_l$, where

$$\mathcal{H}_j = \left\{ S \in \mathcal{H} \colon \sum_{i=1}^{l} i|S \cap Y_i| \equiv j \bmod l \right\}.$$

By the Pigeonhole Principle, we may assume that there is an $a \in [l]$ with $|\mathcal{H}_a| \leq |\mathcal{H}|/l$. Now let

$$\mathcal{F} = \mathcal{H} \setminus \mathcal{H}_a.$$

Proposition 1. *For any $\delta > 0$ there is a $k_0 = k_0(\delta)$ such that for all $k \geq k_0$ and all sufficiently large n (i.e. $n > n_0(k, \delta)$), Construction 1 produces a k-graph $\mathcal{F}$ on n vertices with independent neighborhoods such that*

$$|\mathcal{F}| > \left(1 - \frac{2\log k}{k} + (1-\delta)\frac{\log\log k}{k}\right)\binom{n}{k}.$$

Proof. To see that $\mathcal{F}$ has independent neighborhoods, consider a $(k-1)$-set S. Then $N(S)$ cannot have a point in each Y_i for then $\left\{\sum_{i=1}^{l} i|(S \cup \{v\}) \cap Y_i| \colon v \in N(S)\right\}$ covers all congruence classes modulo l. But then $N(S)$ is an independent set, since every edge of $\mathcal{F}$ has a point in each Y_i.

Let $k > k_0(\delta)$ be fixed and $n \to \infty$. If l is a fixed function of k then we have

$$\begin{aligned}
|\mathcal{F}| &\geq \left(1 - \frac{1}{l}\right)\left(\binom{n}{k} - l\binom{n - \lfloor n/l \rfloor}{k}\right) \\
&\geq \left[\left(1 - \frac{1}{l}\right)\left(1 - l(1-1/l)^k\right) + \Theta\left(\frac{1}{n}\right)\right]\binom{n}{k} \\
&= \left(1 - \frac{1}{l} - (l-1)(1-1/l)^k + \Theta\left(\frac{1}{n}\right)\right)\binom{n}{k}.
\end{aligned}$$

Set $l = \lceil k/((2-\epsilon)\log k) \rceil$, where $\epsilon = \log\log k/\log k$. Then using $(1-1/l)^k < e^{-k/l}$ and $k^\epsilon = \log k$, we obtain

$$|\mathcal{F}| \geq \left(1 - \frac{(2-\epsilon)\log k}{k} - \frac{1}{k} + \Theta\left(\frac{1}{n}\right)\right)\binom{n}{k}.$$

This gives the required bound. ∎

Proposition 2. *For any $k \geq 7$, we have $\rho_k > 1/2$.*

Proof. Let us take $l=3$ in Construction 1. The Inclusion-Exclusion Principle shows that $|\mathcal{H}|/\binom{n}{k} = 1 - 3\cdot(2/3)^k + 3\cdot(1/3)^k + o(1)$. The right-hand side assumes value $\frac{602}{729} > \frac{3}{4}$ for $k=7$ and, as it is not hard to show, is an increasing function of $k \geq 7$. Since $\mathcal{F}$ contains at least $2/3$ edges of $\mathcal{H}$, the proposition follows. ∎

3. Lemmas

This section contains some auxiliary results needed in the proof of the upper bound of Theorem 1. It may be possible to extract the following result from [20] (as pointed out to us by a referee). In any case, we give an independent proof below.

Lemma 1. *For every $k \geq 100$ there is an n_0 such that for all n, x, and y with $x+y=n \geq n_0$ and $\frac{4n}{k-1} \leq y \leq \frac{n}{2}$, we have*

$$\max_{0 \leq i \leq k-1} \frac{\binom{x}{i}\binom{y}{k-i-1}}{\binom{n}{k-1}} \leq 5\left(\frac{n}{ky}\right)^{1/2}.$$

Proof. Let $n_0 = n_0(k)$ be sufficiently large. Let $p = x/n$ and $q = y/n = 1-p$. For $0 \leq i \leq k-1$, let $p_i = \binom{x}{i}\binom{y}{k-i-1}\binom{n}{k-1}^{-1}$ and $b_i = \binom{k-1}{i} p^i q^{k-1-i}$. We begin by noting that the hyper-geometric distribution (as given by p_i) can be bounded by the binomial distribution (as given by b_i). Consider an experiment in which we choose $k-1$ elements of $[n]$ uniformly at random with replacement. Let $X \subset [n]$ with $|X| = x$, and let $\mathcal{D}$ be the event that the $k-1$ random choices are distinct. Note that b_i is the probability that exactly i of our randomly chosen element fall in X and p_i is the probability that exactly i of our randomly chosen elements fall in X when we condition on $\mathcal{D}$. Therefore,

$$p_i \leq \frac{b_i}{Pr(\mathcal{D})} \leq \frac{b_i}{1 - \binom{k-1}{2}\frac{1}{n}}. \tag{2}$$

Note that $b_i < b_{i+1}$ if and only if $i + q < (k-1)p$. Therefore, if we set $i_0 = \lfloor (k-1)p \rfloor$ and $i_1 = i_0 + 1$ then $\max_i b_i = \max\{b_{i_0}, b_{i_1}\}$. Since $k \geq 3$ and $y \leq n/2$ we have $x = n - y \geq \frac{n}{2} \geq \frac{n}{k-1}$ and hence $(k-1)p = (k-1)\frac{x}{n} \geq 1$. Also,

$\frac{4n}{k-1} \leq y$ implies that $i_0 < k-2$. Consequently, $1 \leq i_0 < k-2$ and we can apply a standard estimate for binomial coefficients (e.g., Inequality (1.5) in [2]):

$$b_{i_0} \leq \left(\frac{(k-1)}{2\pi i_0(k-1-i_0)}\right)^{1/2} \left(\frac{(k-1)p}{i_0}\right)^{i_0} \left(\frac{(k-1)q}{k-1-i_0}\right)^{k-1-i_0}.$$

Now let us estimate each of these three terms.

- Since $k \geq 100$ and $p \geq 1/2$ we have $\frac{p}{49} \geq \frac{1}{98} \geq \frac{1}{k-1}$. Therefore $\frac{i_0}{k-1} \geq p - \frac{1}{k-1} \geq \frac{48}{49}p$. Also

$$x(k-1-i_0) \geq x(k-1-(k-1)p) = x(k-1)(1-p) = \frac{xy(k-1)}{n} \geq \frac{y(k-1)}{2} \geq \frac{99}{200}yk.$$

This gives

$$\left(\frac{k-1}{2\pi i_0(k-1-i_0)}\right)^{1/2} \leq \left(\frac{49}{96\pi p\,(k-1-i_0)}\right)^{1/2} = \left(\frac{49n}{96\pi x(k-1-i_0)}\right)^{1/2} \leq \alpha\left(\frac{n}{yk}\right)^{1/2}$$

where

$$\alpha = \left(\frac{200 \times 49}{96 \times 99 \times \pi}\right)^{1/2}.$$

- $(k-1)p \leq i_0 + 1$, so

$$\left(\frac{(k-1)p}{i_0}\right)^{i_0} \leq \left(\frac{i_0+1}{i_0}\right)^{i_0} < \mathrm{e}.$$

- Since $q+p=1$, we have $(k-1)(q+p) < k$ and so $(k-1)q < k-(k-1)p \leq k-i_0$. Therefore

$$\left(\frac{(k-1)q}{k-1-i_0}\right)^{k-1-i_0} \leq \left(\frac{k-i_0}{k-1-i_0}\right)^{k-1-i_0} < \mathrm{e}.$$

Altogether we obtain

$$b_{i_0} \leq \alpha\left(\frac{n}{yk}\right)^{1/2} \times \mathrm{e}^2.$$

Now let us do the same for i_1.

- We have $\frac{i_1}{k-1} \geq p$. Also

$$x(k-1-i_1) \geq x(k-1-(k-1)p-1) = x(k-1)(1-p) - x = \frac{xy(k-1)}{n} - x \geq \frac{3xy(k-1)}{4n} \geq \frac{3y(k-1)}{8} \geq \frac{297}{800} yk.$$

This gives

$$\left(\frac{k-1}{2\pi i_1(k-1-i_1)}\right)^{1/2} \leq \left(\frac{1}{2\pi p\,(k-1-i_1)}\right)^{1/2} = \left(\frac{n}{2\pi x(k-1-i_1)}\right)^{1/2} \leq \beta\left(\frac{n}{yk}\right)^{1/2}$$

where

$$\beta = \left(\frac{800}{594\pi}\right)^{1/2}.$$

- $(k-1)p \leq i_1$.
- We have $(k-1)q = k-(k-1)p-1 \leq k-i_1$. Therefore

$$\left(\frac{(k-1)q}{k-1-i_1}\right)^{k-1-i_1} \leq \left(\frac{k-i_1}{k-1-i_1}\right)^{k-1-i_1} < \mathrm{e}.$$

Altogether we obtain

$$b_{i_1} \leq \beta\left(\frac{n}{yk}\right)^{1/2} \times \mathrm{e}.$$

Now the lemma follows from (2) since $\alpha \mathrm{e}^2, \beta\mathrm{e} < 5$. ∎

Lemma 2. *For every $k \geq 100$ there is an n_0 such that for all $n \geq n_0$ the following holds. Suppose that we have two families $\mathcal{F}$ and $\mathcal{G}$ of k-subsets and $(k-1)$-subsets of $[n]$, respectively, such that $|\mathcal{F}| \geq (1-f)\binom{n}{k}$ and $|\mathcal{G}| \geq g\binom{n}{k-1}$. Let $[n] = X \cup Y$ with $x = |X|$ and $y = |Y|$ satisfying $\frac{4n}{k-1} \leq y \leq \frac{n}{2}$. Suppose that reals $0 < f', g' < 1$ satisfy*

$$g'f + f'g > f + f'g' + 5f'\sqrt{n/ky}. \tag{3}$$

Then there is an i, $0 \leq i \leq k-1$, with

$$|\mathcal{F}_i| = |\{K \in \mathcal{F} \colon |K \cap X| = i\}| \geq (1-f')\binom{x}{i}\binom{y}{k-i} \tag{4}$$

and

$$|\mathcal{G}_i| = |\{L \in \mathcal{G} \colon |L \cap X| = i\}| \geq g'\binom{x}{i}\binom{y}{k-i-1}. \tag{5}$$

Proof. Suppose on the contrary that no such i exists. Consider

$$s = \frac{(1-g')}{\binom{n}{k}}|\mathcal{F}| + \frac{f'}{\binom{n}{k-1}}|\mathcal{G}| \geq (1-g')(1-f) + f'g. \tag{6}$$

Observe that we always have $|\mathcal{F}_i| \leq \binom{x}{i}\binom{y}{k-i}$ and $|\mathcal{G}_i| \leq \binom{x}{i}\binom{y}{k-1-i}$. Since for each i, either $\mathcal{F}_i$ or $\mathcal{G}_i$ is small (as defined by (4), (5)), we have $s \leq \sum_{i=0}^{k} \max(a_i, b_i)$, where

$$a_i = (1-g')(1-f')\frac{\binom{x}{i}\binom{y}{k-i}}{\binom{n}{k}} + f'\frac{\binom{x}{i}\binom{y}{k-i-1}}{\binom{n}{k-1}}$$
$$b_i = (1-g')\frac{\binom{x}{i}\binom{y}{k-i}}{\binom{n}{k}} + f'g'\frac{\binom{x}{i}\binom{y}{k-i-1}}{\binom{n}{k-1}}.$$

Since

$$a_i - b_i = \frac{\binom{x}{i}\binom{y}{k-i}}{\binom{n}{k}} \times (1-g')f' \times \left(-1 + \frac{(n-k+1)(k-i)}{k(y-k+i+1)}\right),$$

there is an i_0 such that $a_i \geq b_i$ for $0 \leq i < i_0$ and $a_i \leq b_i$ for $i_0 \leq i \leq k$. Hence,

$$s \leq \sum_{i=0}^{i_0-1} a_i + \sum_{i=i_0}^{k} b_i.$$

Let $P = \binom{n}{k}^{-1}\sum_{i=0}^{i_0-1}\binom{x}{i}\binom{y}{k-i}$ and $P' = \binom{n}{k-1}^{-1}\sum_{i=0}^{i_0-1}\binom{x}{i}\binom{y}{k-i-1}$. Let us choose a random $(k-1)$-subset L of $[n]$ and then let K be obtained from L by adding a random vertex $x \notin L$. Then K is also uniformly distributed. Note that P (resp. P') is the probability that K (resp L) has less than i_0 vertices in X. Since $L \subset K$, $P \leq P'$. On the other hand $P' - P$ is exactly the probability that $x \in X$ and $|L \cap X| = i_0 - 1$. It follows from Lemma 1 that $\Pr(|L \cap X| = i_0 - 1) \leq 5\sqrt{n/ky}$ and so $P' - P \leq 5\sqrt{n/ky}$. Hence,

$$\begin{aligned} s &\leq P(1-g')(1-f') + P'f' + (1-P)(1-g') + (1-P')f'g' \\ &\leq P(1-g')(1-f') + Pf' + (1-P)(1-g') + (1-P)f'g' + 5f'\sqrt{n/ky} \\ &= 1 - g' + f'g' + 5f'\sqrt{n/ky}. \end{aligned}$$

From (6), we obtain that

$$(1-g')(1-f) + f'g \leq s \leq 1 - g' + f'g' + 5f'\sqrt{n/ky},$$

and this contradicts (3). ∎

4. The Upper Bound on ρ_k

Before embarking on the formal proof, let us briefly describe the main idea. Suppose that $\mathcal{F}$ is an n vertex k-graph with $\rho\binom{n}{k}$ edges and independent neighborhoods. We may assume that k is large but fixed and $n \to \infty$. By simple averaging, there is a $(k-1)$-set S with $d(S) = |N(S)| \geq \rho(n-k+1)$. No k-set within $N(S)$ can be in $\mathcal{F}$, since $\mathcal{F}$ has independent neighborhoods. Consequently, we obtain

$$(1-\rho)\binom{n}{k} = \binom{n}{k} - |\mathcal{F}| \geq \binom{\rho(n-k+1)}{k}.$$

This yields

$$1-\rho \geq (1-o(1))\rho^k$$

and solving for ρ gives the bound $\rho \leq 1-(1+o(1))\frac{\log k}{k}$. This is where the main term $\frac{\log k}{k}$ comes from.

Now suppose we could find not just one neighborhood of size $(1-o(1))\rho n$ but we could in fact find $k^{1-o(1)}$ such neighborhoods. No k-set in any of these neighborhoods lies in $\mathcal{F}$ so we would (roughly) obtain

$$(1-\rho)\binom{n}{k} = \binom{n}{k} - |\mathcal{F}| \geq k^{1-o(1)}\binom{\rho(n-k+1)}{k}.$$

This yields

$$1-\rho \geq k^{1-o(1)}\rho^k$$

and solving for ρ now yields $\rho \leq 1-(1+o(1))\frac{2\log k}{k}$. However, the above calculation is not precise since we have over counted some k-sets, namely those that lie in two distinct neighborhoods. Thus the main technical details of the proof are concerned with controlling the total amount of over counting in this inclusion/exclusion calculation. We now begin the formal proof.

Take small $\delta > 0$. Let $k \geq k_0(\delta) \geq 100$ be sufficiently large. Choose large $n_0 = n_0(k,\delta)$. With foresight, we define

$$c_0 = 4+\delta \qquad c_1 = 5+2\delta \qquad c_2 = 5+3\delta \qquad c_3 = 5+6\delta.$$

For brevity of notation, let $\epsilon = \log\log k/\log k$. We will show that for all $k > k_0$ we have

$$\rho_k < 1 - \frac{(2-(5+7\delta)\epsilon)\log k}{k} = 1 - \frac{2\log k}{k} + (5+7\delta)\frac{\log\log k}{k}.$$

Suppose that this is false for some $k > k_0$. Then for infinitely many n, in particular for some $n > n_0(k,\delta)$, we can find a k-graph $\mathcal{F}$ with vertex set $[n]$ and independent neighborhoods such that

$$|\mathcal{F}| > \left(1 - \frac{(2 - c_3\epsilon)\log k}{k}\right)\binom{n}{k}.$$

Define

$$l = \left\lceil \frac{k}{(\log k)^{c_0}} \right\rceil.$$

Our goal is to find sets $A_1, \dots, A_l, B_1, \dots, B_l \subset [n]$ such that the following conditions hold.

Condition 1. For every $i \in [l]$, the set A_i is independent (with respect to $\mathcal{F}$), is disjoint from B_i, and has size

$$a = \left\lceil \left(1 - \frac{(2 - c_1\epsilon)\log k}{k}\right) n \right\rceil. \tag{7}$$

Condition 2. The sets $B_1, \dots, B_l$ are pairwise disjoint, each of size

$$b = \left\lceil \frac{(2 - c_2\epsilon)\log k}{k} n \right\rceil. \tag{8}$$

Indeed, if we have such sets then, for any $1 \le i < j \le l$, the set $A_i \cap A_j$ has at most $n - 2b$ elements because its complement contains $B_i \cup B_j$ as a subset. Since every k-set in $\bigcup_{i=1}^{l} \binom{A_i}{k}$ is missing from $\mathcal{F}$, we have by a simple version of the Inclusion-Exclusion Principle that

$$l\binom{a}{k} - \binom{l}{2}\binom{n - 2b}{k} \le \binom{n}{k} - |\mathcal{F}| < \frac{2\log k}{k}\binom{n}{k}.$$

Dividing by $\binom{n}{k}$ and using $k > k_0(\delta)$ and $n > n_0(k,\delta)$, we get

$$(1 - \delta)\left(\frac{l}{k^{2 - c_1\epsilon}} - \frac{l^2}{2k^{4 - 2c_2\epsilon}}\right) \le \frac{2\log k}{k},$$

which is a contradiction (for $\delta < 1$ and $k \ge k_0(\delta)$).

Before proceeding with an argument that gives the sets $A_1, \dots, A_l, B_1, \dots, B_l$, we need two observations regarding $(k-1)$-sets of large degree. First, observe that for every $(k-1)$-set S, we have

$$d(S) < \left(1 - \frac{\log k - 2\log\log k}{k}\right) n, \tag{9}$$

for otherwise $\binom{n}{k}-|\mathcal{F}|\geq\binom{d(S)}{k}>\frac{1}{2}\frac{\log^2 k}{k}\binom{n}{k}$ which is a contradiction.

We will obtain the sets A_i as neighborhoods of $(k-1)$-sets. Our strategy is to use the global lower bound on the number of edges to show that there are many $(k-1)$-sets S with large neighborhoods $d(S)$. We would therefore like to restrict our attention to those $(k-1)$-sets with large neighborhoods. Let $\mathcal{G}$ be the collection of $(k-1)$-sets $S\in\binom{[n]}{k-1}$ such that $d(S)\geq n-b$.

Claim 1. $|\mathcal{G}|\geq 2\delta\epsilon\binom{n}{k-1}$.

Proof of Claim. Let $|\mathcal{G}|=g\binom{n}{k-1}$. We have

$$k\left(1-\frac{(2-c_3\epsilon)\log k}{k}\right)\binom{n}{k}\leq k|\mathcal{F}|=\sum_{S\in\binom{[n]}{k-1}}d(S)$$
$$\leq\binom{n}{k-1}(1-g)\left(1-\frac{(2-c_2\epsilon)\log k}{k}\right)n$$
$$+g\binom{n}{k-1}\left(1-\frac{\log k-2\log\log k}{k}\right)n,$$

where the last expression comes from (9). Solving for g yields

$$g\geq\frac{(c_3-c_2)\epsilon-2k^2/n}{1-c_2\epsilon+2\log\log k/\log k}>2\delta\epsilon.$$

(We used the facts that $c_3-c_2=3\delta$ and $c_2>2$ in the last inequality.) This completes the proof of Claim 1. ∎

Now we describe how to inductively construct the sets A_i and B_i. Suppose that we have constructed $A_1,\dots,A_p,B_1,\dots,B_p$ with $0\leq p<l$ satisfying Conditions 1 and 2. Let

(10) $$y=\left\lfloor\frac{2n}{(\log k)^{c_0-1}}\right\rfloor$$

and $x=n-y$. Take an arbitrary partition $[n]=X\cup Y$ with $Y\supset\bigcup_{j=1}^{p}([n]\setminus A_j)$ and $|Y|=y$, which is possible since each set $[n]\setminus A_i$ has $n-a\leq 2n\log k/k$ elements and $p<l$. Our task now is to construct A_{p+1} and B_{p+1}.

For an integer i, define

$$\mathcal{F}_i=\{S\in\mathcal{F}\colon |S\cap X|=i\}\quad\text{and}\quad\mathcal{G}_i=\{S\in\mathcal{G}\colon |S\cap X|=i\}.$$

Also, let

$$f=2\log k/k,\quad g=2\delta\epsilon,\quad f'=\log^{2+\delta}k/k,\quad g'=\delta\epsilon.$$

A short calculation shows by (10) that (3) holds:

$$f'(g-g')+(g'f-f-5f'\sqrt{n/ky}) > \frac{\delta\epsilon\log^{2+\delta}k}{k} - C\frac{\log k}{k} > 0,$$

for some absolute constant C. So Lemma 2 implies that there is an i such that $|\mathcal{F}_i| \geq (1-f')\binom{x}{i}\binom{y}{k-i}$ and $|\mathcal{G}_i| \geq \delta\epsilon\binom{x}{i}\binom{y}{k-i-1}$.

Let $\lambda = f'/(\delta\epsilon)$. Let us show that there is a $(k-1)$-set $T_0 \in \mathcal{G}_i$ such that

$$|Y \setminus N(T_0)| \leq \lambda(y-k+i+1). \tag{11}$$

Suppose on the contrary that no such T_0 exists. Let us count the number γ of pairs (K,z) with $K \in \mathcal{F}_i$ and $z \in K \cap Y$ in two different ways. On the one hand, we can first choose K and then z. This gives

$$(k-i)(1-f')\binom{x}{i}\binom{y}{k-i} \leq (k-i)|\mathcal{F}_i| = \gamma.$$

On the other hand, we can first choose $K-\{z\}$ and then z. The set $K-\{z\}$ is either in $\mathcal{G}_i$ or not. Taking both cases into account yields

$$\gamma < |\mathcal{G}_i|(1-\lambda)(y-k+i+1) + \left(\binom{x}{i}\binom{y}{k-1-i} - |\mathcal{G}_i|\right)(y-k+i+1).$$

It follows that

$$\lambda|\mathcal{G}_i|(y-k+i+1) < f'\binom{x}{i}\binom{y}{k-i}(k-i).$$

Since $|\mathcal{G}_i| \geq \delta\epsilon\binom{x}{i}\binom{y}{k-i-1}$, this contradicts the choice of λ.

Choose an arbitrary set $B_{p+1} \subset X$ that contains all of $X \setminus N(T_0)$ and such that $|B_{p+1}| = b$. (This is possible because $|X| \geq n-lb \geq b$ and $T_0 \in \mathcal{G}$, so $|X \setminus N(T_0)| \leq n-d(T_0) \leq b$.) For every $j \in [p]$, the set $B_j \subset Y$ is disjoint from $B_{p+1} \subset X$, so Condition 2 holds. Let $Z = Y \setminus N(T_0)$ and $A' = [n] \setminus (B_{p+1} \cup Z)$. Note that A', as a subset of $N(T_0)$, is an independent set. Moreover, by the definition of T_0 (i.e. by (11)), we have

$$|A'| \geq n-b-\lambda y \geq n - \left\lceil \frac{(2-c_2\epsilon)\log k}{k}n \right\rceil - \frac{\log^{2+\delta}k}{\delta\epsilon k} \times \frac{2n}{(\log k)^{c_0-1}} \geq a.$$

Let us take for A_{p+1} an arbitrary a-subset of A'. Condition 1 clearly holds, finishing the proof. ∎

5. $k = 5$

As we have mentioned in Conjecture 1, the cases $k=5,6$ remain interesting open questions. By suitably modifying the proof that $\rho_4 = 1/2$ from [8], we can obtain fairly good bounds for ρ_5.

Theorem 2. $0.493 \le \rho_5 \le 0.534$.

Proof. (Sketch) Suppose that $\mathcal{G}$ is a 5-graph with independent neighborhoods and $\pi\binom{n}{5}$ edges which is maximum possible with this restriction. Let I be the 5-graph

$$\{12345, 12346, 12347, 12348, 12349, 56789\}.$$

Then a 5-graph with independent neighborhoods is precisely one with no copy of I. Consequently, $\mathcal{G}$ contains no copy of I. Since I has the property that every two of its vertices lie in an edge, we conclude that if we duplicate any vertex of $\mathcal{G}$ then the resulting 5-graph also contains no copy of I. Now if there are vertices $u, v \in \mathcal{G}$ and any small positive $\epsilon > 0$ such that $d(u) > d(v) + \epsilon n^4$, then we could delete v and duplicate u to obtain another 5-graph $\mathcal{G}'$ with n vertices, independent neighborhoods, and more edges than $\mathcal{G}$ (such a process is sometimes called Zykov symmetrization). This contradiction shows that we may assume all vertex degrees of $\mathcal{G}$ are $(\pi + o(1))\binom{n}{4}$.

Now let A be a neighborhood of maximum size, say $|A| = \alpha n$, and $B = [n] \setminus A$. Let h_i be the number of edges of $\mathcal{G}$ with exactly i points in B; note that $h_0 = 0$ by our hypothesis. Let σ_i be the sum, over all 4-sets S with i points in B and $4 - i$ points in A, of $d(S)$. Then one obtains

$$\binom{\alpha n}{3}(1-\alpha)n \times \alpha n \ge \sigma_1 = 4h_1 + 2h_2 \tag{12}$$

$$\binom{\alpha n}{2}\binom{(1-\alpha)n}{2} \times \alpha n \ge \sigma_2 = 3h_2 + 3h_3 \tag{13}$$

$$\alpha n \binom{(1-\alpha)n}{3} \times \alpha n \ge \sigma_3 = 2h_3 + 4h_4. \tag{14}$$

On the other hand, using the fact that all degrees are almost equal we get

$$(1-\alpha)n \times (\pi + o(1))\binom{n}{4} = \sum_{x \in B} d(x) = h_1 + 2h_2 + 3h_3 + 4h_4 + 5h_5. \tag{15}$$

Now consider $3/4\times(12)+1/6\times(13)+1/4\times(14)+(15)$, observe that $\sum_{i=1}^{5} h_i=|\mathcal{G}|$, and divide by n^5. This gives that, as $n\to\infty$,

$$\pi \le \frac{\alpha}{5\alpha-1}\left(15(1-\alpha)\alpha^3+5(1-\alpha)^2\alpha^2+5(1-\alpha)^3\alpha\right)+o(1).$$

Maximizing this function over all $\alpha\in(0.5,1)$ yields $\pi<0.534$ and hence $\rho_5<0.534$.

For the lower bound, observe that $b(n,5)=\left(\frac{40}{81}+o(1)\right)\binom{n}{5}$ (take $|Y|=\left(\frac{1}{3}+o(1)\right)n$). This shows that $\rho_5\ge\frac{40}{81}>0.493$.

6. Concluding Remarks and Open Problems

• Our results are similar in flavor to the following problem about the Turán numbers of complete hypergraphs. Let t_k denote the maximum proportion of edges in a k-graph on n vertices, as $n\to\infty$, that contains no copy of the complete k-graph on $k+1$ vertices. Thus $t_2=1/2$ by Mantel's theorem. The most famous conjecture in this area, due to Turán [24], is that $t_3=5/9$, which is achieved by (among others) the 3-graph with vertex partition Y_1,Y_2,Y_3 into almost equal parts and all edges with two points in Y_i and one point in Y_{i+1} (indices modulo 3) or one point in each Y_i. Perhaps just as interesting is to determine the growth rate of t_k as $k\to\infty$. Frankl and Rödl [7] proved that $1–t_k=O(\log k/k)$ via a construction that has similarities to Construction 1 in this paper. On the other hand, the known upper bound is $t_k=1-\Omega(1/k)$, where the best results are due to Chung and Lu [3]. It would be very interesting to obtain sharper estimates for t_k. Perhaps the methods of this paper can be used to show that $1-t_k=\omega(1/k)$, an open question for whose solution de Caen [4, Page 190] offered 500 Canadian dollars.

• For $2\le m\le k$ let the *book* $B_{k,m}$ be the k-graph with the following $m+1$ edges: $[k-1]\cup\{k+i-1\}$ for $i\in[m]$, and $\{k,k+1,\dots,2k-1\}$. The problem of computing the Turán function $\mathrm{ex}(n,B_{k,m})$ has been actively studied [1,5,6, 8–10,14,18,21,23]. Clearly, the property not containing $B_{k,k}$ as a subgraph is equivalent to having empty neighborhoods, so $f(n,k)=\mathrm{ex}(n,B_{k,k})$. Our results can be modified to show, for example, that for any function $m=m(k)<c_1\log k$, where c_1 is a constant, we have

$$\pi(B_{k,k-m})=1-\Theta\left(\frac{\log k}{k}\right) \tag{16}$$

as $k \to \infty$, where $\pi(F) = \lim_{n\to\infty} \mathrm{ex}(n,F)/\binom{n}{k}$ denotes the *Turán density* of a k-graph F. Indeed, the upper bound on $\pi(B_{k,k-m})$ follows from Theorem 1 and the trivial observation that $\mathrm{ex}(n,B_{k,k}) \ge \mathrm{ex}(n,B_{k,k-m})$. The lower bound (16) can be obtained by taking the k-graph $\mathcal{F}$ of Construction 1 with $l = k/c_2 \log k$ where $c_2 \gg \max(c_1,1)$ and removing those edges of $\mathcal{F}$ that intersect some part Y_i in at most m vertices. As $n \to \infty$, the proportion of edges that we delete is approximately at most

$$l \times \Pr\big(\mathrm{Bin}(k,1/l) < m\big) \le l\,\mathrm{e}^{-c_2 \log k/4} < \frac{1}{k^2}.$$

(We apply the Chernoff bound here, see e.g. [11, Corollary 2.3].) Therefore, the size of the family $\mathcal{F}$ is at least $(1-1/l)(1-1/k^2)\binom{n}{k}$, and (16) follows.

On the other hand, it is easy to show that $\pi(B_{k,m}) = o(1)$ if $m = o(k)$. Determining the behavior of $\pi(B_{k,m})$ for the intermediate values of m is an interesting open problem.

• A related problem which has been studied a fair amount recently (see, e.g., [13,15,17,22]) is the maximum possible minimum degree (of $(k-1)$-sets) that a k-graph can have without containing some fixed configuration. Let $g(n,k)$ denote the maximum minimum degree of a k-graph on n vertices with independent neighborhoods. Then it was shown in [19] that the limit $\gamma_k = \lim_{n\to\infty} g(n,k)/n$ exists. It is trivial to see that $\gamma_k \le 1/2$ for all k, and odd k-graphs show that if k is even, we have equality. It would be interesting to determine the behavior of γ_k for k odd. As with t_k, the small cases seem difficult. For $k=3$, the construction for t_3 above minus the edges with one point in each Y_i shows that $\gamma_3 \ge 1/3$. In fact, we make the following conjecture.

Conjecture 2. For every $\epsilon > 0$, there exists n_0 such that if $n > n_0$ and $\mathcal{G}$ is an n-vertex 3-graph with every pair lying in at least $(1/3+\epsilon)n$ edges, then $\mathcal{G}$ contains a neighborhood that is not an independent set. In particular, $\gamma_3 = 1/3$.

• Construction 1 has the following generalization. We begin with some definitions that establish the general setting. Let $a, l \ge 2$ be fixed parameters. Consider the digraph D with vertex set $\mathbb{Z}_a^l$ and an arc from $x = (x_1,\dots,x_l)$ to $y = (y_1,\dots,y_l)$ if and only if there exists a coordinate k such that

$$y_i = \begin{cases} x_i & \text{if } i \ne k, \\ x_i - 1 & \text{if } i = k. \end{cases}$$

Note that the out-degree of each vertex is l. We say that a subset X of $\mathbb{Z}_a^l$ is a *perfect cover* of D if the out-neighborhoods of the elements of X form a partition of $\mathbb{Z}_a^l$. In other words, the set X is a perfect cover if for every $y \in \mathbb{Z}_a^l$ there exists a unique $x \in X$ such that the arc (x,y) (i.e. the arc directed from x to y) is in D. Note that a perfect cover contains a^l/l vertices.

Suppose X is a perfect cover of D. Let n be large and fix a partition $Y_1,\dots,Y_l$ of $[n]$. For each k-set S let $y_S \in \mathbb{Z}_a^l$ be the vector $y_S = (y_1,\dots,y_l)$ where $y_i \equiv |S \cap Y_i| \mod a$ for $i=1,\dots,l$. Now we are ready to define our family with independent neighborhoods. Let $\mathcal{F}$ be the collection of k-sets S such that $S \cap Y_i \neq \emptyset$ for $i=1,\dots,l$ and $y_S \notin X$. We claim that the collection $\mathcal{F}$ has independent neighborhoods. To see this, consider a $(k-1)$-set T. Since X is a perfect cover, there exists $x \in X$ such that (x, y_T) is an arc in D. It follows that there exists an index k such that $T \cup \{z\} \notin \mathcal{F}$ for all $z \in Y_k$. In other words, the neighborhood of T (in the hypergraph $\mathcal{F}$) does not intersect Y_k. Since every edge in $\mathcal{F}$ intersects Y_k, it follows that $\mathcal{F}$ has independent neighborhoods.

In order to ensure a lower bound on the cardinality of the collection $\mathcal{F}$, we consider situations where there is a partition of $\mathbb{Z}_a^l$ into perfect covers $X_1,\dots,X_l$. Each X_i corresponds to a collection $\mathcal{F}_i$. Furthermore, each set S that intersects $Y_1,\dots,Y_l$ is excluded from exactly one of the collections $\mathcal{F}_i$. Therefore, there is an index i such that $|\mathcal{F}_i|$ is at least $(1-1/l)$ times the number of k-sets S that intersect $Y_1,\dots,Y_l$.

Note that Construction 1 is given by this general setting by taking $a=l$ and letting

$$X_j = \left\{ x \in \mathbb{Z}_l^l \colon \sum_{i=1}^{l} i x_i = j \right\}.$$

For a second example, set $a=2$ and suppose $l=2^b$ for some integer $b \geq 2$. Fix a Hamming code $H \subseteq \{0,1\}^{l-1}$; that is, fix a set of strings $H \subseteq \{0,1\}^{l-1}$ with the property that every string in $\{0,1\}^{l-1}$ is either in H or adjacent (in the $(l-1)$-cube) to exactly one element of H. Note that

$$X = \left\{ (x_1,\dots,x_l) \in \mathbb{Z}_2^l \colon (x_1,\dots,x_{l-1}) \in H \right\}$$

is a perfect cover of $\mathbb{Z}_2^l$. Furthermore the collection $X, X+e_1, X+e_2, \dots, X+e_{l-1}$ is a partition of $\mathbb{Z}_2^l$ into perfect covers. Thus, the Hamming code gives another construction that achieves the bound given by Construction 1.

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