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ARTICLE

Sharp bounds for decomposing graphs into edges and triangles

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Abstract

For a real constant α, let $\pi_3^\alpha(G)$ be the minimum of twice the number of K_2's plus α times the number of K_3's over all edge decompositions of G into copies of K_2 and K_3, where K_r denotes the complete graph on r vertices. Let $\pi_3^\alpha(n)$ be the maximum of $\pi_3^\alpha(G)$ over all graphs G with n vertices.

The extremal function $\pi_3^3(n)$ was first studied by Győri and Tuza (*Studia Sci. Math. Hungar.* **22** (1987) 315–320). In recent progress on this problem, Král', Lidický, Martins and Pehova (*Combin. Probab. Comput.* **28** (2019) 465–472) proved via flag algebras that $\pi_3^3(n) \leqslant (1/2 + o(1))n^2$. We extend their result by determining the exact value of $\pi_3^\alpha(n)$ and the set of extremal graphs for all α and sufficiently large n. In particular, we show for $\alpha = 3$ that K_n and the complete bipartite graph $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$ are the only possible extremal examples for large n.

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1. Introduction

In recent progress on a problem of Győri and Tuza [27], Král', Lidický, Martins and Pehova [19] proved via flag algebras that the edges of any n-vertex graph can be decomposed into copies of K_2 and K_3 whose total number of vertices is at most $(1/2 + o(1))n^2$, where K_r denotes the clique on

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r vertices. The origins of this problem can be traced back to Erdős, Goodman and Pósa [10], who considered the problem of minimizing the total number of cliques in an edge decomposition of an arbitrary n-vertex graph. They showed the following.

Theorem 1.1 (Erdős, Goodman and Pósa [10]). *The edges of every n-vertex graph can be decomposed into at most $\lfloor n^2/4 \rfloor$ complete graphs.*

The only extremal example for this bound is the *(bipartite) Turán graph* $T_2(n) := K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$, where $K_{a,b}$ denotes the complete bipartite graph with part sizes a and b. Moreover, this result still holds if we restrict the sizes of the cliques used in the decomposition to 2 and 3 (*i.e.* single edges and triangles). In a series of papers published independently by Chung [4], Győri and Kostochka [15] and Kahn [18], they proved that in fact something stronger than Theorem 1.1 is true, confirming a conjecture by Katona and Tarján.

Theorem 1.2 (Chung [4], Győri and Kostochka [15], Kahn [18]). *Every n-vertex graph can be edge-decomposed into cliques whose total number of vertices is at most $\lfloor n^2/2 \rfloor$.*

For a given graph G on n vertices, let $\pi_k(G)$ be the minimum over all decompositions of the edges of G into cliques $C_1, \ldots, C_\ell$ of size at most k of the sum $|C_1| + |C_2| + \cdots + |C_\ell|$, where $|G| := |V(G)|$ denotes the *order* of a graph G. Let $\pi_k(n)$ be the maximum of $\pi_k(G)$ over all graphs G with n vertices. With this notation, the conclusion of the above theorem is that $\min_{k\in\mathbb{N}} \pi_k(n) \leqslant \lfloor n^2/2 \rfloor$. In light of Theorem 1.2, Tuza [27] conjectured that $\pi_3(n) \leqslant n^2/2 + o(n^2)$, and in fact that $\pi_3(n) \leqslant n^2/2 + O(1)$. Győri and Tuza [16] showed that $\pi_3(n) \leqslant 9n^2/16$. This was the best known bound until recently, when using the celebrated flag algebra method of Razborov [24], Král', Lidický, Martins and Pehova [19] proved the asymptotic version of Tuza's conjecture.

Theorem 1.3 (Král' *et al.* [19]). *We have $\pi_3(n) \leqslant (1/2 + o(1))n^2$ as $n \to \infty$.*

In this paper we show, by building upon the proof in [19], that for all large n it holds in fact that $\pi_3(n) \leqslant n^2/2 + 1$. Moreover, if a graph G of order n attains $\pi_3(n)$, then G is the complete graph K_n or the Turán graph $T_2(n)$.

Which of these two graphs is extremal is a matter of divisibility of n by 6. In the case of the Turán graph, we trivially have $\pi_3(T_2(n)) = 2\lfloor n/2 \rfloor \lceil n/2 \rceil$, giving $n^2/2$ for even n and $(n^2-1)/2$ for odd n. In order to determine $\pi_3(K_n)$, we have to determine the maximum number of edge-disjoint triangles in K_n. Clearly, the graph made of their edges is *triangle-divisible*, that is, each vertex has even degree and the total number of edges is divisible by three. It is routine to see that the minimum size of a graph H on n vertices whose complement $\overline{H}$ is triangle-divisible is attained by taking at most one copy of the claw $K_{1,3}$ and a perfect matching on the remaining vertices for even n, and isolated vertices plus at most one copy of the 4-cycle $K_{2,2}$ for odd n. (Note that $\binom{n}{2}$ is never equal to 2 modulo 3.) In fact this gives the value of $\pi_3(K_n)$ for all large n by the following general result (which we will also use in our proof).

Theorem 1.4 (Barber, Kuhn, Lo and Osthus [2]). *For every $\varepsilon > 0$, if G is a triangle-divisible graph of large order n and minimum degree at least $(0.9 + \varepsilon)n$, then G has a perfect triangle decomposition.*

The constant 0.9 in the minimum degree condition in Theorem 1.4 comes from the result of Dross [6] on fractional triangle decompositions, and Nash-Williams [21] conjectured that it can be replaced by 3/4. Very recently, Dukes and Horsley [7] and Delcourt and Postle [5] improved the constant to 0.852 and $(7 + \sqrt{21})/14 = 0.8273\ldots$, respectively.

Table 1. Values of $\pi_3(K_n)$ and $\pi_3(T_2(n))$ for large n

n mod 6	K_2's in an optimal decomposition of K_n	$\pi_3(K_n)$	$\pi_3(T_2(n))$
0	perfect matching	$\frac{n^2}{2}$	$\frac{n^2}{2}$
1	none	$\binom{n}{2}$	$\frac{n^2-1}{2}$
2	perfect matching	$\frac{n^2}{2}$	$\frac{n^2}{2}$
3	none	$\binom{n}{2}$	$\frac{n^2-1}{2}$
4	$K_{1,3}$ + perfect matching	$\frac{n^2}{2}+1$	$\frac{n^2}{2}$
5	C_4	$\binom{n}{2}+4$	$\frac{n^2-1}{2}$

In Table 1 we list the values of π_3 for the graphs K_n and $T_2(n)$ for large n. Let us define

$$\mathcal{E}_n := \begin{cases} \{T_2(n), K_n\} & \text{if } n \equiv 0, 2 \pmod 6, \\ \{T_2(n)\} & \text{if } n \equiv 1, 3, 5 \pmod 6, \\ \{K_n\} & \text{if } n \equiv 4 \pmod 6, \end{cases}$$

and

$$\ell(n) := \begin{cases} n^2/2 & \text{for } n \equiv 0, 2 \pmod 6, \\ (n^2-1)/2 & \text{for } n \equiv 1, 3, 5 \pmod 6, \\ n^2/2+1 & \text{for } n \equiv 4 \pmod 6. \end{cases}$$

Thus, by the calculations of Table 1, we have for all large n that $\mathcal{E}_n$ consists of those graphs in $\{T_2(n), K_n\}$ which maximize π_3 while $\ell(n)$ is this maximum value.

Clearly, $\ell(n)$ is a lower bound on $\pi_3(n)$ for large n. Our main result is that this is equality.

Theorem 1.5. *There exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, we have $\pi_3(n) = \ell(n)$, and the set of $\pi_3(n)$-extremal graphs up to isomorphism is exactly $\mathcal{E}_n$.*

A simple corollary of Theorem 1.5 is an affirmative answer to a question of Pyber [23] (see also [27, Problem 45]) for sufficiently large n. A *covering* of a graph G is a collection of subgraphs of G such that every edge of G appears in at least one subgraph. (For comparison, a decomposition requires that every edge appears in exactly one subgraph.)

Corollary 1.1. *There exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, the edge set of every n-vertex graph can be covered with triangles and edges so that the sum of their orders is at most $\lfloor n^2/2 \rfloor$.*

Proof. Theorem 1.5 directly implies the corollary unless $n \equiv 4 \pmod 6$ and the graph under consideration is K_n. So assume that $n \equiv 4 \pmod 6$. Denote the vertices of K_n by $v_1, \ldots, v_n$. Recall that an optimal decomposition for K_n is obtained by taking edges v_1v_2, v_1v_3, v_1v_4 and v_iv_{i+1} for all odd i with $5 \leqslant i \leqslant n-1$. The rest of the graph becomes triangle-divisible and Theorem 1.4 can be applied. This gives a decomposition of cost $n^2/2+1$. A covering of cost at most $n^2/2$ can be obtained from this decomposition by replacing edges v_1v_2 and v_1v_3 with a triangle $v_1v_2v_3$. (Note that the pair v_2v_3 is covered by two triangles in the resulting covering.) □

We also study an extension of Theorem 1.5, where we consider decompositions into K_2's and K_3's but we modify the cost of K_3's to be α (with the cost of K_2 still being 2). The minimum over all costs of such decompositions of a graph G is denoted by $\pi_3^\alpha(G)$. The maximum value of $\pi_3^\alpha(G)$ over all n-vertex graphs G is denoted by $\pi_3^\alpha(n)$. Note that $\pi_3^3(G) = \pi_3(G)$ and $\pi_3^3(n) = \pi_3(n)$. Denote K_n without one edge by K_n^- and K_n without a matching of size two by $K_n^=$. Then the following result holds.

Theorem 1.6. *For every real α there exists $n_0 \in \mathbb{N}$ such that every π_3^α-extremal graph G with $n \geqslant n_0$ vertices satisfies the following (up to isomorphism).*

- *If $\alpha < 3$, then $G = T_2(n)$.*
- *If $\alpha = 3$, then Theorem 1.5 applies.*
- *If $3 < \alpha < 4$ and $n \equiv 0, 2, 4, 5 \pmod 6$, then $G = K_n$.*
- *If $3 < \alpha < 4$ and $n \equiv 1, 3 \pmod 6$, then $G = K_n^=$.*
- *If $\alpha = 4$ and $n \equiv 1, 3 \pmod 6$, then $G \in \{K_n, K_n^-, K_n^=\}$ and, moreover, the three listed graphs are all π_3^α-extremal.*
- *If $\alpha = 4$ and $n \equiv 0, 2, 4, 5 \pmod 6$, then $G = K_n$.*
- *If $4 < \alpha$, then $G = K_n$.*

This paper is organized as follows. In Section 2 we give an outline of the proof of Theorem 1.3 from [19] that we build on. Theorem 1.5 is proved in Section 3. An extension for other weights of triangles is in Section 4. Some related results are mentioned in Section 5.

Notation. We follow standard graph theory notation (see *e.g.* [3]).

For a graph G, we denote the set neighbours of $x \in V(G)$ by $\Gamma_G(x)$ (or just $\Gamma(x)$ when G is understood) and the number of edges in a set $B \subseteq E(G)$ incident with x by $d_B(x)$. We let $K[V_1, V_2]$ denote the complete bipartite graph with vertex partition (V_1, V_2). The term $[X, Y]$-edges refers to edges $xy \in E(G)$ such that $x \in X$ and $y \in Y$. We write $[x, Y]$-edges as shorthand for $[\{x\}, Y]$-edges.

Let $t_2(n) := |E(T_2(n))|$ be the number of edges in the Turán graph $T_2(n)$. Recall that $t_2(n) = \lfloor n^2/4 \rfloor$. By a *cherry* we mean a path with two edges.

We consider graphs up to isomorphism; in particular, we write $G = H$ to denote that G and H are isomorphic graphs.

2. Outline of the proof of Theorem 1.3 from [19]

In this section we give a short outline of the proof of [19, Lemma 5], which was a key step in proving $\pi_3(n) \leqslant n^2/2 + o(n^2)$ and is a starting point of our argument towards Theorem 1.5. For an n-vertex graph G and each $i \in \mathbb{N}$, let $K_i(G)$ be the set of all i-cliques in G. Let $\pi_{3,f}(G)$ be the minimum of

$$2 \sum_{xy \in K_2(G)} c(xy) + 3 \sum_{xyz \in K_3(G)} c(xyz)$$

over *fractional* $\{K_2, K_3\}$*-decompositions* c of $E(G)$, that is, over maps $c \colon K_2(G) \cup K_3(G) \to [0, 1]$ such that for every edge $xy \in E(G)$ we have $c(xy) + \sum_{z\colon xyz \in K_3(G)} c(xyz) \geqslant 1$. Of course, $\pi_{3,f}(G) \leqslant \pi_3(G)$. By a result of Haxell and Rödl [17] or a more general version by Yuster [28], it also holds that $\pi_3(G) \leqslant \pi_{3,f}(G) + o(n^2)$. So, to show that $\pi_3(G) \leqslant n^2/2 + o(n^2)$, it suffices to consider the fractional equivalent $\pi_{3,f}(G)$.

Lemma 2.1. *Let G be an n-vertex graph. Then*

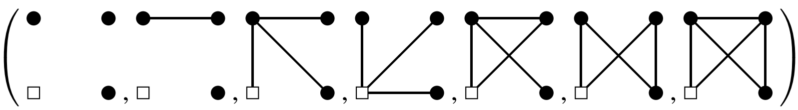
$$\binom{n}{7}^{-1} \sum_{W \in \binom{V(G)}{7}} \pi_{3,f}(G[W]) \leqslant 21 + o(1),$$

where the sum is taken over 7-vertex subsets W of V(G).

Outline of proof. Let M be the positive semidefinite matrix

$$M := \frac{1}{12\cdot 10^9}\begin{pmatrix} 1800000000 & 2444365956 & 640188285 & -1524146769 & 1386815580 & -732139362 & -129387078 \\ 2444365956 & 4759879134 & 1177441152 & -1783771230 & 2546923788 & -1397639394 & -143552208 \\ 640188285 & 1177441152 & 484273772 & -317303211 & 1038156300 & -591902130 & -6783162 \\ -1524146769 & -1783771230 & -317303211 & 1558870290 & -651906630 & 305728704 & 154602378 \\ 1386815580 & 2546923788 & 1038156300 & -651906630 & 2285399634 & -1283125950 & -10755036 \\ -732139362 & -1397639394 & -591902130 & 305728704 & -1283125950 & 734039016 & -1621938 \\ -129387078 & -143552208 & -6783162 & 154602378 & -10755036 & -1621938 & 23860164 \end{pmatrix}$$
$$\succeq 0,$$

and let $\overrightarrow{F} := (F_1, \ldots, F_7)$ be the following vector of rooted graphs, each having four vertices with the root denoted by the white square:

$\vec{F} =$ 

Take any graph G of order $n \to \infty$. For $w \in V(G)$, let $\boldsymbol{v}_{G,w} \in \mathbb{R}^7$ denote the column vector whose ith component is $p(F_i, (G, w))$, the density of the 1-flag F_i in the rooted graph (G, w), which is G with the vertex w designated as the root.

It was shown in [19] that

$$\frac{1}{\binom{n}{7}} \sum_{W \in \binom{V(G)}{7}} \pi_{3,f}(G[W]) + \frac{1}{n} \sum_{w \in V(G)} \boldsymbol{v}_{G,w}^T M \boldsymbol{v}_{G,w} \leqslant 21 + o(1). \tag{2.1}$$

Namely, if we rewrite the left-hand side as a linear combination $\sum_H c_H p(H, G)$, where H ranges over all 7-vertex unlabelled graphs and $p(H, G)$ is the density of H in G, then each coefficient c_H is at most 21. Since $\sum_H p(H, G) = 1$, the claimed inequality (2.1) follows.

In particular, since M is positive semidefinite, the quantity

$$\frac{1}{n} \sum_{w \in V(G)} \boldsymbol{v}_{G,w}^T M \boldsymbol{v}_{G,w}$$

is always non-negative, yielding the result. □

The main result of [19], that $\pi_3(n) \leqslant n^2/2 + o(n^2)$, now follows directly from Lemma 2.1.

Proof of Theorem 1.3. Let G be any graph of order $n \to \infty$. As mentioned before, $\pi_3(G) \leqslant \pi_{3,f}(G) + o(n^2)$. Also, we have

$$\binom{n}{2}^{-1} \pi_{3,f}(G) \leqslant \binom{7}{2}^{-1} \binom{n}{7}^{-1} \sum_{W \in \binom{V(G)}{7}} \pi_{3,f}(G[W]),$$

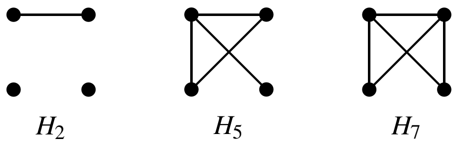


Figure 1. Graphs H_2, H_5 and H_7.

by averaging optimal fractional decompositions of all 7-vertex induced subgraphs. Combining this inequality with Lemma 2.1 immediately gives that $\pi_3(G) \leqslant (1/2 + o(1))n^2$. □

3. Proof of Theorem 1.5

We use the so-called *stability approach*, where the first step is to describe the approximate structure of all almost π_3-extremal graphs of order $n \to \infty$ within $o(n^2)$ adjacencies. Namely, our Corollary 3.2 will show that every such graph is close to K_n or $T_2(n)$.

For this purpose, we start by showing that all almost π_3-extremal graphs contain almost no copies of the three graphs in Figure 1 (which are obtained by taking the unlabelled versions of the corresponding graphs in $\overrightarrow{F}$). This is achieved by the following lemma, which builds on the results from [19].

Lemma 3.1. *For every $c > 0$ there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, if G is a graph of order n with $\pi_3(G) \geqslant (1/2 - \varepsilon)n^2$, then G has at most $c\binom{n}{4}$ copies of each of the graphs*

$$H_2 := (\{a, b, c, d\}, \{ab\}),$$
$$H_5 := (\{a, b, c, d\}, \{ab, bc, ac, ad\}),$$
$$H_7 := (\{a, b, c, d\}, \{ab, bc, ac, bd, ad\})$$

from Figure 1.

Proof. Given $c > 0$, let $\varepsilon \gg 1/n_0 > 0$ be sufficiently small. Let G be a graph as in the lemma. Let M and $\overrightarrow{F}$ be as in the proof of Lemma 2.1.

First, the rank of the matrix M is 6 with $\boldsymbol{v} = (1, 0, 3, 1, 0, 3, 0)$ being the only zero eigenvector. (Thus all other eigenvalues of M are strictly positive by $M \succeq 0$.)

Second, by the almost optimality of G and the fact that each term on the left-hand side of (2.1) is non-negative, we have

$$\sum_{w \in V(G)} \boldsymbol{v}_{G,w}^T M \boldsymbol{v}_{G,w} = o_\varepsilon(n). \tag{3.1}$$

We now show that G must contain few copies of the graphs H_2, H_5 and H_7. Suppose, for contradiction, that G contains at least $c\binom{n}{4}$ copies of H_2. Then, by a simple double-counting argument, we have that at least $cn/4$ vertices in G contain at least $c\binom{n}{3}/4$ copies of the rooted flag F_2. In particular, the second coordinate of at least $cn/4$ of the vectors $\boldsymbol{v}_{G,w}$ is at least $c/4$. For each such vector $\boldsymbol{u}$, let $\boldsymbol{u}' := \boldsymbol{u}/\|\boldsymbol{u}\|_2$ be the scalar multiple of $\boldsymbol{u}$ of ℓ^2-norm 1. Since $\|\boldsymbol{u}\|_2 \leqslant \sqrt{7}$, we have that its second coordinate $\boldsymbol{u}'_2$ is at least $c/4\sqrt{7}$. The scalar product of $\boldsymbol{u}'$ and the ℓ^2-normalized zero eigenvector $\boldsymbol{v}/\sqrt{20}$ (whose second coordinate is 0) is at most

$$\sqrt{1 - (c/4\sqrt{7})^2}.$$

Thus the projection of $\boldsymbol{u}$ on the orthogonal complement $L = \boldsymbol{v}^\perp$ of the zero eigenspace of M has ℓ^2-norm at least $c/4\sqrt{7}$. Thus $\boldsymbol{u}^T M \boldsymbol{u} \geqslant \lambda_2(c/4\sqrt{7})^2$, where $\lambda_2 > 0$ is the smallest positive eigenvalue of M (in fact one can check with the computer that $\lambda_2 = 0.0005228\ldots$). Thus the left-hand side of (3.1), in which each term is non-negative by $M \succeq 0$, is at least $(cn/4) \times \lambda_2(c/4\sqrt{7})^2 = \Omega(n)$, a contradiction.

The analogous argument shows that the densities of H_5 and H_7 in G are also at most c. □

Let us say that two graphs G_1 and G_2 of the same order are *k-close in the edit distance* (or simply *k-close*) if there is a relabelling of the vertices of G_2 so that $|E(G_1) \triangle E(G_2)| \leqslant k$. In other words we can make G_1 and G_2 isomorphic by changing at most k adjacencies.

Corollary 3.2. *For every $\delta > 0$ there exists $n_1 \in \mathbb{N}$ such that if G is a graph of order $n \geqslant n_1$ with $\pi_3(G) \geqslant \ell(n) - n^2/n_1$, then G is δn^2-close in edit distance to K_n or to $T_2(n)$.*

Proof. Given any $\delta > 0$, choose sufficiently small constants $\delta \gg c \gg 1/n_1 > 0$. Take any graph G on $n \geqslant n_1$ vertices such that $\pi_3(G) \geqslant \ell(n) - n^2/n_1$.

By Lemma 3.1 and the Induced Removal Lemma [1], G can be made $\{H_2, H_5, H_7\}$-free by changing at most cn^2 adjacencies. Denote this new graph by G' and note that $\pi_3(G') \geqslant \pi_3(G) - 2cn^2$. By $c \ll \delta$, it is enough to show that G' is $\delta n^2/2$-close to K_n or $T_2(n)$.

Let us show that G' is either triangle-free or the disjoint union of at most two cliques. Indeed, if some vertices a, b, c span a triangle in G' then, by the $\{H_5, H_7\}$-freeness of G, all the remaining vertices of G' have either no or three neighbours among $\{a, b, c\}$. Let A_0 be the set of vertices in $G' \setminus \{a, b, c\}$ which see none of $\{a, b, c\}$, and let A_3 be the set of vertices which see all of $\{a, b, c\}$. Then A_3 is a clique because G' is H_7-free. The set A_0 is also a clique because G' is H_2-free. Also, no pair xy in $A_3 \times A_0$ can be an edge, as otherwise, for example, the 4-set $\{a, b, x, y\}$ spans a copy of H_5 in G. It follows that G is the disjoint union of the cliques on A_0 and $A_3 \cup \{a, b, c\}$, as required.

Now, if G' is triangle-free, then

$$e(G') = \pi_3(G')/2 \geqslant \ell(n)/2 - n^2/n_1 - 2cn^2 \geqslant t_2(n) - 3cn^2.$$

Thus, by the stability result for Mantel's theorem by Erdős [8] and Simonovits [26], the graph G' must indeed be $\delta n^2/2$-close in edit distance to $T_2(n)$.

Otherwise G' is the disjoint union of two cliques. Let us show that one of them has size at most $\delta n/2$. Indeed, otherwise G' has a triangle packing covering all but at most $n/2 + 2$ edges by Theorem 1.4, meaning that $\pi_3(G') \leqslant e(G') + n/2 + 2$. Also, $e(G')$ is maximum when clique sizes are as far apart as possible. Thus, by the lower bound on $\pi_3(G) \leqslant \pi_3(G') + 2cn^2$, we conclude that, for example,

$$\ell(n) - 3cn^2 \leqslant \binom{\delta n/2}{2} + \binom{(1-\delta/2)n}{2},$$

leading to a contradiction to our choice of constants. Therefore G' is at most $n \cdot \delta n/2$ adjacency edits away from K_n, as desired. □

The key steps in proving Theorem 1.5 are Lemmas 3.3–3.5.

Lemma 3.3. *There exist constants $\delta > 0$ and $n_1 \in \mathbb{N}$ such that, among all graphs on $n \geqslant n_1$ vertices which are δn^2-close to $T_2(n)$, the maximizer of π_3 is $T_2(n)$.*

Proof. Choose sufficiently small $\varepsilon \gg \delta \gg 1/n_1 > 0$. Let G be an arbitrary graph with $n \geqslant n_1$ vertices which is δn^2-close to $T_2(n)$. We will show that $\pi_3(G) \leqslant \pi_3(T_2(n))$ with equality if and only if $G = T_2(n)$. In fact this claim can be directly derived from the result of Győri [11, Theorem 1] that a graph with n vertices and $t_2(n) + k$ edges, where $n \to \infty$ and $k = o(n^2)$, has at least $k - O(k^2/n^2)$ edge-disjoint triangles. More specifically, for each $\varepsilon > 0$ there exists $\delta > 0$ and $n_0 \in \mathbb{N}$ such that every graph with $n \geqslant n_0$ vertices and $t_2(n) + k$ edges, where $k \leqslant \delta n^2$, has at least $k - \varepsilon k^2/n^2$ edge-disjoint triangles. (See also [12, Theorem 1] for a generalization of this to r-cliques for any fixed $r \geqslant 3$.) Since G is δn^2-close to $T_2(n)$, it must have at most $t_2(n) + \delta n^2$ edges. From this and $1/n \ll \delta \ll \varepsilon \ll 1$, we have that, for $k := e(G) - t_2(n)$,

$$\pi_3(G) \leqslant 2(t_2(n)) + k) - 3(k - \varepsilon k^2/n^2) = 2t_2(n) - k(1 - 3\varepsilon k/n^2) \leqslant 2t_2(n).$$

Clearly, if equality is achieved then $k = 0$, *i.e.* $e(G) = t_2(n)$; furthermore, G must be triangle-free and thus $G = T_2(n)$, as required. □

Next we need to analyse graphs that are close to K_n. If $n \equiv 1, 3 \pmod 6$, then let $\mathcal{E}'_n$ consist of those graphs which are obtained from K_n by removing a matching of size $m \equiv 2 \pmod 3$; otherwise let $\mathcal{E}'_n := \{K_n\}$. Also, define

$$w(n) := \begin{cases} n/2 & n \equiv 0, 2 \pmod 6, \\ 2 & n \equiv 1, 3 \pmod 6, \\ n/2 + 1 & n \equiv 4 \pmod 6, \\ 4 & n \equiv 5 \pmod 6). \end{cases}$$

Using Theorem 1.4 and the calculation for K_n described in Table 1, one can show that $\pi_3(G) = \binom{n}{2} + w(n)$ for all large n and every $G \in \mathcal{E}'_n$. We are going to show that these are exactly the extremal graphs among those close to K_n. It is more convenient to do first the case when we have some bound on the minimum degree of a graph and then derive the general case (in a separate Lemma 3.5).

Lemma 3.4. *There exist constants $\delta > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Let G be a graph on $n \geqslant n_0$ vertices with minimum degree at least $n/8$ such that G is δn^2-close to K_n and $\pi_3(G) \geqslant \binom{n}{2} + w(n)$. Then $G \in \mathcal{E}'_n$.*

Proof. Choose small constants in the following order: $c \gg \delta \gg 1/n_0 > 0$. Suppose that G is a graph of order $n \geqslant n_0$ as in the statement of the lemma. Let $w := w(n)$.

Let

$$U := \{v \in V(G) \colon d_G(v) \leqslant (1 - c)n\}.$$

Then

$$\frac{|U|cn}{2} \leqslant e(\overline{G}) \leqslant \delta n^2,$$

and so $|U| \leqslant (2\delta/c)n$. Denote $W := V(G) \setminus U$, and let $S := \{v \in W \colon d_G(v) \text{ is odd}\}$. Let M be a set of edges forming a maximum matching in $G[S]$, and denote $X := S \setminus V(M)$. Then X is an independent set and thus $\binom{|X|}{2} \leqslant \delta n^2$, which implies that rather roughly

$$|X| < cn. \tag{3.2}$$

Moreover, for every edge $yz \in M$ and any two distinct vertices $y', z' \in X$, at most one of yy' and zz' can be an edge of G (otherwise $y'yzz'$ is an augmenting path contradicting the maximality of M). It follows that if $|X| \neq 1$, then for every edge $yz \in M$ there are at least $|X|$ edges missing between yz and X. Let Y_W denote the set of missing edges in $G[W]$. Thus

$$|Y_W| \geqslant \binom{|X|}{2} + |M|(|X| - \mathbb{1}_{|X|=1}), \tag{3.3}$$

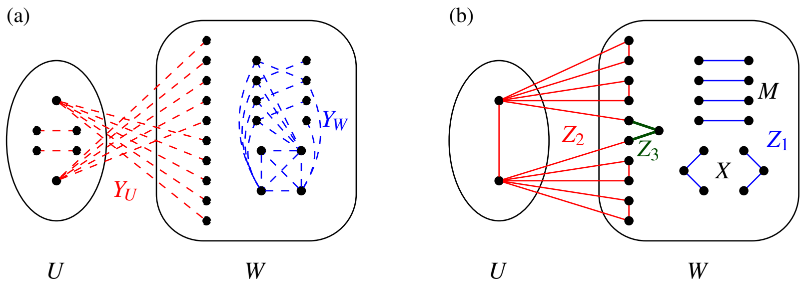


Figure 2. (a) Missing edges in Y_W are coloured blue and edges in Y_U are red. (b) Edges in Z_1 are coloured blue, edges in Z_2 are red and in Z_3 green. The same vertices are in (a), where some of the missing edges are dashed. Note that this is a sketch and vertices in W can incident to both blue and red (dashed) edges.

where the indicator function $\mathbb{1}_{|X|=1}$ is 1 if $|X| = 1$ and is 0 otherwise. Moreover, the set Y_U of missing edges in G with at least one endpoint in U satisfies

$$|Y_U| \geqslant cn|U| - \binom{|U|}{2} \tag{3.4}$$

by the definition of U. Note that $e(G) = \binom{n}{2} - |Y_W| - |Y_U|$. See Figure 2 for a sketch of Y_W and Y_U.

We now build a decomposition $\mathcal{D}$ of G into edges and triangles, starting with $\mathcal{D} = \emptyset$. If we add edges/triangles to $\mathcal{D}$, we regard them as removed from $E(G)$. It is convenient to split our argument into the following two cases.

Case 1. $U \neq \emptyset$ or $S = \emptyset$.

In this case, our procedure for constructing $\mathcal{D}$ is as follows.

Step 1. Add the following to $\mathcal{D}$ as K_2's: the edges of the matching M and the edges of some $\lfloor |X|/2 \rfloor$ cherries with distinct endpoints in X such that their middle points are pairwise distinct.

Step 2. For each $u \in U$, one at a time, add to $\mathcal{D}$ a maximum set of edge-disjoint K_3's containing u and two vertices from W. Add all remaining edges incident to vertices in U as K_2's to $\mathcal{D}$.

Step 3. (a) Let $S' \subseteq V(G)$ be the set of vertices with odd degree after Step 2. Add to $\mathcal{D}$ the edges of some $|S'|/2$ cherries with distinct endpoints in S' such that their middle points are pairwise distinct.

(b) If the number of remaining edges is not divisible by 3, then fix this by adding to $\mathcal{D}$ (as single edges) the edge set of some cycle of length 4 or 5.

Step 4. Add a perfect triangle decomposition of the remaining edges to $\mathcal{D}$.

For $i \in \{1, 2, 3\}$, let Z_i be the set of edges that are added to $\mathcal{D}$ in Step i as copies of K_2. See Figure 2 for some illustrations of the above steps.

Claim. *The above steps can be carried out as stated. Moreover, the obtained decomposition $\mathcal{D}$ of G has at most $|M| + |X| + \binom{|U|}{2} + 2|U| + 6$ copies of K_2.*

Proof of Claim. In order to carry out Step 1 as stated, we can iteratively pick any two new vertices $x, y \in X$ and then an arbitrary vertex z which is suitable as the middle point for a cherry on xy. Note that the number of choices for z is at least $n - 2 - 2cn$, the number of common neighbours of $x, y \in X \subseteq W$, minus $|X| - 1$, the number of vertices previously used as middle points. This is positive by (3.2) and $c \ll 1$, so we can always proceed. Note for future reference that every vertex is incident to at most three edges removed in Step 1. Also, Step 1 adds $|Z_1| = |M| + 2(\lfloor |X|/2 \rfloor) \leqslant |M| + |X|$ copies of K_2 to $\mathcal{D}$.

Clearly, Step 2 can always be processed. Consider the moment when we apply Step 2 to some $u \in U$. In the current graph, the induced subgraph $G[\Gamma(u) \cap W]$ has minimum degree at least $|\Gamma(u) \cap W| - cn - 3$, which is at least $|\Gamma(u) \cap W|/2$ since $|\Gamma(u)| \geqslant n/8 - 3$. So by Dirac's theorem, this subgraph has a matching covering all but at most one vertex, that is, all edges between u and W except at most one are decomposed as triangles in Step 2. Let U' be the set of those $u \in U$ for which an exceptional edge occurs. Thus we have $|U'| \leqslant |U|$ copies of K_2 connecting U to W that are added to $\mathcal{D}$ in Step 2. There are trivially at most $\binom{|U|}{2}$ edges with both endpoints in U. So Step 2 adds $|Z_2| \leqslant \binom{|U|}{2} + |U|$ copies of K_2 to $\mathcal{D}$. Note that all edges incident to U are decomposed after Step 2.

Since all vertices of W but at most one had even degrees before Step 2, we have that S' has at most $|U'| + 1 \leqslant |U| + 1$ vertices. As in Step 1, a simple greedy algorithm finds all cherries as stated in Step 3(a). (Note that S', as the set of all odd-degree vertices, has even size.)

The minimum degree of $G[W]$ after Step 3(a) is at least $0.99n$, since each $w \in W$ has at most $2|U| + 6$ incident edges removed (at most $2|U|$ from Step 2 and at most 3 in each of Steps 1 and 3(a)). Thus we can find the required 4- or 5-cycle in Step 3(b).

Clearly, we add $|Z_3| \leqslant |S'| + 5 \leqslant |U| + 6$ copies of K_2 to $\mathcal{D}$ in Step 3.

Note that, at the end of Step 3, the graph $G[W]$ has minimum degree at least, say, $0.98n$ while all its degrees are even. By Theorem 1.4, all remaining edges can be decomposed using only triangles, so Step 4 indeed removes all remaining edges.

Step 4 adds no additional K_2's, so the total number of K_2's in $\mathcal{D}$ is

$$|Z_1| + |Z_2| + |Z_3| \leqslant |M| + |X| + \binom{|U|}{2} + 2|U| + 6,$$

finishing the proof of the claim. □

Now we compute the cost of $\mathcal{D}$. Using the notation from above, we have

$$\begin{aligned} w &\leqslant \pi_3(G) - \binom{n}{2} \\ &\leqslant -|Y_U| - |Y_W| + |Z_1| + |Z_2| + |Z_3| \\ &\leqslant -|Y_U| - |Y_W| + |M| + |X| + \binom{|U|}{2} + 2|U| + 6. \end{aligned} \tag{3.5}$$

Substituting the bounds from (3.3) and (3.4) and rearranging the terms, we get

$$w \leqslant \left(2\binom{|U|}{2} + 2|U| - cn|U| + 6\right) + (3 - |X|)\left(\frac{|X|}{2} + |M|\right) + (\mathbb{1}_{|X|=1} - 2)|M|. \tag{3.6}$$

First, suppose that $|U| > 0$. Then the estimate $|U| \leqslant 2\delta n/c$ yields that

$$2\binom{|U|}{2} + 2|U| - cn|U| + 6 \leqslant -cn|U|/2 \leqslant -cn/2.$$

Since $w \geqslant 2$, we must have that $|X| \leqslant 1$. Observe that n is odd as otherwise $w \geqslant n/2$ and, by $|M| \leqslant n/2$, the cases $|X| \in \{0, 1\}$ also contradict (3.6). So every vertex of degree $n - 1$ has even degree, meaning that every vertex of S is in some pair from Y_W or Y_U. Hence $2|M| \leqslant 2|Y_W| + |Y_U|$. Substituting this into the right-hand side of (3.5) and using our bound on $|Y_U|$ from (3.4), we obtain

$$w \leqslant -\frac{|Y_U|}{2} + |X| + \binom{|U|}{2} + 2|U| + 6 \leqslant \frac{3}{2}\binom{|U|}{2} + 2|U| - \frac{cn|U|}{2} + 7,$$

which again is negative for $|U| > 0$ and large n, contradicting $w \geqslant 2$.

Thus U is empty and, by the assumption of Case 1, S is also empty (and so are X and M). This gives that the initial graph G has minimum degree at least $(1-c)n$, $|Z_1|=|Z_2|=0$, $S'=\emptyset$, and no K_2's are added to $\mathcal{D}$ in Step 3(a).

If n is even, then every vertex of G has at least one missing edge,

$$e(G) \leqslant \binom{n}{2} - \frac{n}{2},$$

and

$$\pi_3(G) \leqslant \binom{n}{2} - \frac{n}{2} + |Z_3| \leqslant \binom{n}{2} - \frac{n}{2} + 5,$$

which is strictly less than $\pi_3(K_n)$, a contradiction.

Let n be odd and let $r := \binom{n}{2} - e(G)$ be the number of missing edges in G. Suppose that $r > 0$, as otherwise $G = K_n$ and we are done. The upper bound on $\pi_3(G)$ given by $\mathcal{D}$ is $\rho_r + \binom{n}{2} - r$, where we define ρ_r as the unique element of $\{0, 4, 5\}$ with $\binom{n}{2} - \rho_r - r \equiv 0 \pmod 3$. Therefore $r \leqslant 3$ as otherwise $\pi_3(G) \leqslant \binom{n}{2} + 1$, contradicting $w \geqslant 2$. On the other hand all the degrees of $\overline{G}$ are even so $r = 3$ and the only non-empty component of $\overline{G}$ is a triangle. However, this contradicts $w \geqslant 2$ because

$$\pi_3(G) = \begin{cases} \binom{n}{2} - 1 & n \equiv 1, 3 \pmod 6, \\ \binom{n}{2} + 1 & n \equiv 5 \pmod 6. \end{cases}$$

Case 2. $U = \emptyset$ and $S \neq \emptyset$.

Some things simplify in this case (as we do not need to deal with U). On the other hand we have to be a bit more careful with calculations, as the new extremal graphs (K_n minus a matching) fall into this case. In particular, removing a 4- or 5-cycle may be too wasteful here. So we construct a decomposition $\mathcal{D}$ of G as follows. Recall that M is a maximum matching in $G[S]$ and X is the set of vertices of S not matched by M.

Step 1. Make the graph triangle-divisible by removing the following as K_2's. If $X = \emptyset$, then remove all but one edge $xy \in M$ and a path of length $\rho + 1 \in \{1, 2, 3\}$ whose endpoints are x and y (thus, for $\rho = 0$, we remove just the matching M). If X is non-empty, then remove M and the edge sets of some $|X|/2 - 1$ paths of length 2 and one path of length $\rho + 2 \in \{2, 3, 4\}$ so that their degree-1 vertices partition X and their degree-2 vertices are pairwise distinct.

Step 2. Decompose the rest perfectly into triangles.

Note that S, the set of all odd-degree vertices of G, has even size (and also $|X| = |S| - 2|M|$ is even). Since the minimal degree of G is at least $(1-c)n$, a simple greedy algorithm achieves Step 1 (and Theorem 1.4 takes care of Step 2).

The decomposition $\mathcal{D}$ has exactly $|M| + |X| + \rho$ copies of K_2. Also, $e(G) = \binom{n}{2} - |Y_W|$. Thus

$$w \leqslant \pi_3(G) - \binom{n}{2} \leqslant -|Y_W| + |M| + |X| + \rho. \tag{3.7}$$

Using (3.3) and that $|X| \neq 1$ (since $|X|$ is even), we obtain

$$w \leqslant (3 - |X|)\left(\frac{|X|}{2} + |M|\right) - 2|M| + \rho. \tag{3.8}$$

Moreover, $|X| \leqslant 2$ as otherwise $2 \leqslant w \leqslant \rho - 2 - 3|M|$, contradicting $\rho \leqslant 2$. Thus X has either 0 or 2 elements.

Suppose that $X = \emptyset$. First, let n be even. Then every vertex not in S is incident to at least one non-edge of G, $|Y_W| \geqslant (n-2|M|)/2$, and by (3.7),

$$n/2 \leqslant w \leqslant 2|M| + \rho - n/2.$$

If $2|M| \leqslant n-2$, then all inequalities here become equalities and thus $|M| = (n-2)/2$, $|Y_W| = 1$, $\rho = 2$, $w = n/2$, and $n \equiv 0, 2 \pmod 6$. However, then the graph after Step 1 has exactly

$$\binom{n}{2} - 1 - \frac{n-2}{2} - 2$$

edges, which is not divisible by 3, a contradiction. Thus $2|M| = n$, the copies of K_2 in the decomposition contain a perfect matching of G, and $\pi_3(G) \leqslant \pi_3(K_n)$ with equality only if $G = K_n$, as desired. So suppose that n is odd. Since every vertex of S has to be incident to a missing edge of G, we have $|Y_W| \geqslant |S|/2 = |M|$ and the bound in (3.7) becomes $w \leqslant \rho$. It follows that we have equality throughout, $|Y_W| = |M|$, $w = \rho = 2$, $n \equiv 1, 3 \pmod 6$, and $\binom{n}{2} - |M| - \rho \equiv 0 \pmod 3$; the last gives that $|M| \equiv 2 \pmod 3$. Thus G is as required.

Finally, it remains to consider the case when $|X| = 2$. This time, (3.8) yields that

$$2 \leqslant w \leqslant \rho - |M| + 1 \leqslant 3.$$

Therefore $|M| \leqslant 1$, and $n \equiv 1, 3 \pmod 6$ as otherwise $w \geqslant 4$. If $|M| = 1$, then we have equality everywhere, giving $w = \rho = 2$, $|S| = 4$ and $|Y_W| = 3$. However, then the graph after Step 1 has

$$\binom{n}{2} - |Y_W| - |M| - |X| - \rho = \binom{n}{2} - 8$$

edges, which is not divisible by 3, a contradiction. Thus M is empty, $\rho \in \{1, 2\}$ and $S = X$. By (3.7), $|Y_W| \leqslant 2$ and hence $|Y_W| = 1$. In other words, $G = K_n^-$. However, then the graph after Step 1 has

$$\binom{n}{2} - 1 - (2 + \rho)$$

edges, which is not divisible by 3. (Alternatively, Theorem 1.4 gives that $\pi_3(K_n^-) - \binom{n}{2} < 2 = w$.) This contradiction finishes Case 2 and the proof of the lemma. □

Lemma 3.5. *There exist constants $\delta > 0$ and $n_1 \in \mathbb{N}$ such that the following holds. Let G be a graph on $n \geqslant n_1$ vertices maximizing $\pi_3(G)$ among all graphs that are δn^2-close to K_n. Then $G \in \mathcal{E}'_n$.*

Proof. Let n_0 and δ be the constants from Lemma 3.4. We claim that, for example, $n_1 := 2n_0$ is enough for the conclusion of Lemma 3.5 to hold. Indeed, take any extremal graph G of order $n \geqslant n_1$. If G satisfies the assumption on minimum degree of Lemma 3.4, then we are done. Hence assume that the minimum degree of G is less than $n/8$. Let $G_n := G$, and iteratively define a sequence of graphs $G_{n-1}, G_{n-2}, \ldots$ as follows. Given a graph G_i of order i, if it has a vertex x of degree less than $i/8$, let $G_{i-1} := G_i - x$ be obtained from G_i by removing the vertex x; otherwise stop. Note that the process does not reach $i < n/2$ for otherwise G has roughly at least $(n/2) \times (n/4)$ non-edges, which is a contradiction to G being δn^2-close to K_n.

Let G_s with $|G_s| = s \geqslant n/2 \geqslant n_0$ be the graph for which the above process terminates. By Lemma 3.4, we have that $\pi_3(G_s) \leqslant s^2/2 + 1$. By decomposing all edges in $E(G) \setminus E(G_s)$ as K_2's, we obtain

$$\pi_3(G_n) \leqslant \pi_3(G_s) + 2(n-s) \cdot \frac{n}{8} \leqslant \frac{s^2}{2} + 1 + (n-s) \cdot \frac{n}{4}.$$

This is a convex function in s so it is maximized on the boundary of $n/2 \leqslant s \leqslant n-1$. If $s = n/2$, we get

$$\pi_3(G_n) \leqslant n^2/4 + 2 < \binom{n}{2} \leqslant \pi_3(K_n).$$

If $s = n-1$, we get

$$\pi_3(G_n) \leqslant \pi_3(G_s) + 2(n-s) \cdot \frac{n}{8} \leqslant \frac{(n-1)^2}{2} + 1 + \frac{n}{4} \leqslant \binom{n}{2} - \frac{n}{4} + 2 < \pi_3(K_n).$$

In both cases, we get a contradiction to G_n being extremal. □

Proof of Theorem 1.5. Choose sufficiently small constants in this order $1 \gg \delta \gg 1/n_0 > 0$. In particular, n_0 is sufficiently large to satisfy Corollary 3.2 for this δ as well as Lemmas 3.3 and 3.5. Let G be an arbitrary graph of order $n \geqslant n_0$ with $\pi_3(G) \geqslant \ell(n)$. By Corollary 3.2, G is δn^2-close to either $T_2(n)$ or K_n.

If G is close to $T_2(n)$ then it must be $T_2(n)$ by Lemma 3.3. If G is close to K_n then it must be in $\mathcal{E}'_n$ by Lemma 3.5. By comparing the costs of optimal decompositions, we conclude that $G \in \mathcal{E}_n$. □

4. Extension to an arbitrary cost α

The goal of this section is to prove Theorem 1.6. Everywhere in this section, let n be sufficiently large.

First, note that the case $\alpha \geqslant 6$ is trivial. Indeed, the cost of a triangle is not better than a cost of three edges. Thus, for every graph G, an optimal decomposition is to decompose all edges of G as K_2's. The unique graph maximizing the number of edges is K_n, so it is also the unique maximizer of π_3^α for every $\alpha \geqslant 6$.

Next let us make some easy general observations which apply when $\alpha < 6$. First,

$$\pi_3^\alpha(G) = \alpha\nu(G) + 2(e(G) - 3\nu(G)) = 2e(G) - (6-\alpha)\nu(G),$$

where $\nu(G)$ denotes the maximum number of edge-disjoint triangles contained in G. Also, if $\alpha_1 \leqslant \alpha_2 < 6$, $\nu(G_1) \geqslant \nu(G_2)$ and $\pi_3^{\alpha_1}(G_1) > \pi_3^{\alpha_1}(G_2)$ for some graphs G_1 and G_2, then

$$\pi_3^{\alpha_2}(G_1) - \pi_3^{\alpha_2}(G_2) = \pi_3^{\alpha_1}(G_1) - \pi_3^{\alpha_1}(G_2) + (\alpha_2 - \alpha_1)(\nu(G_1) - \nu(G_2)) > 0. \quad (4.1)$$

In particular, if K_n is the maximizer of $\pi_3^{\alpha_1}$, it is also a maximizer for $\pi_3^{\alpha_2}$.

4.1 The case $\alpha < 3$

Next we discuss the case $\alpha < 3$. Let n be large and let G be a $\pi_3^\alpha(n)$-extremal graphs. Since

$$\pi_3^3(G) \geqslant \pi_3^\alpha(G) \geqslant \pi_3^\alpha(T_2(n)) = \pi_3^3(T_2(n)) = (1/2 + o(1))n^2,$$

Corollary 3.2 gives that G is $o(n^2)$-close to K_n or $T_2(n)$. Since $\alpha < 3$, we have that $\pi_3^\alpha(T_2(n)) \geqslant (1 + \Omega(1))\pi_3^\alpha(K_n)$ and thus G is close to $T_2(n)$. Now, Lemma 3.3 implies that $\pi_3^\alpha(G) \leqslant \pi_3^3(G) \leqslant \pi_3^3(T_2(n)) = \pi_3^\alpha(T_2(n))$, with equality if and only if $G = T_2(n)$, as desired.

4.2 The case $3 < \alpha < 4$

This subsection proves Theorem 1.6 for $3 < \alpha < 4$.

First let us show that every π_3^α-maximizer G is in K_n or $K_n^=$. Suppose for a contradiction that G violates this. In particular, we have $\pi_3^\alpha(G) \geqslant \pi_3^\alpha(K_n)$. By (4.1), we have that $\pi_3^3(G) \geqslant \pi_3^3(K_n)$. For

$n \to \infty$, it holds by Table 1 that $\pi_3^\alpha(K_n) \geqslant (1+\Omega(1))\,\pi_3^\alpha(T_2(n))$. Hence G needs to be close to K_n and Lemma 3.5 applies to G. In particular, this means that $n \equiv 1, 3 \pmod 6$. Lemma 3.5 gives that all π_3^3-extremal graphs are obtained from K_n by removing a matching of size congruent to 2 modulo 3. It follows from (4.1) that, among these graphs, π_3^α is strictly maximized by $K_n^=$ since this graph has the largest ν.

Theorem 1.4 gives that $3\nu(K_n^=) = \binom{n}{2} - 6$. Since $\pi_3^\alpha(G) \geqslant \pi_3^\alpha(K_n^=)$ and $\pi_3^3(G) < \pi_3^3(K_n^=)$, this implies by (4.1) that $\nu(G) > \nu(K_n^=)$. Since also $\nu(G) < \nu(K_n)$ (otherwise $\pi_3^\alpha(G) < \pi_3^\alpha(K_n)$), we conclude that $3\nu(G) = \binom{n}{2} - 3$, that is, exactly three pairs of vertices of G are not included in some triangle from an optimal decomposition of G. This implies that G is a complete graph without one edge, or a path on three vertices, or a triangle. Among these three candidates (that have the same ν), K^- has the largest size and thus maximizes π_3^α. So K^- is the only possible candidate for G. However, $\pi_3^\alpha(K_n^=) > \pi_3^\alpha(K_n^-)$ if $\alpha < 4$. This contradiction finishes the proof for $3 < \alpha < 4$.

Thus every π_3^α-maximizer is in $\{K_n, K_n^=\}$. It remains to compare these two graphs. Calculations based on Theorem 1.4 show that

$$\frac{\pi_3^\alpha(K_n^=) - \pi_3^\alpha(K_n) + 4}{6-\alpha} = \nu(K_n) - \nu(K_n^=) = \begin{cases} 0 & n \equiv 0, 2, 4, 5 \pmod 6, \\ 2 & n \equiv 1, 3 \pmod 6. \end{cases}$$

Thus $\pi_3^\alpha(K_n) > \pi_3^\alpha(K_n^=)$ if $n \equiv 0, 2, 4, 5 \pmod 6$ and $\pi_3^\alpha(K_n^=) > \pi_3^\alpha(K_n)$ otherwise, as required.

4.3 The case $4 \leqslant \alpha < 6$

In this case we provide a direct proof, without using flag algebras or fractional decompositions. Let n be large and let G be any graph of order n such that $\pi_3^\alpha(G) = \pi_3^\alpha(n)$. Let $\mathcal{D}$ be a decomposition of G with minimum weight consisting of t triangles and ℓ edges.

If G is a complete graph, then we are done. Hence we assume there exists some pair of vertices $x, y \in G$ such that $xy \notin E(G)$. Let G' be obtained from G by adding the edge xy. Let $\mathcal{D}'$ be an optimal decomposition of G' containing t' triangles and ℓ' edges. Recall that finding an optimal decomposition is equivalent to maximizing a triangle packing, that is, $t' = \nu(G')$. Hence $t' \geqslant t$.

If xy is used as an edge in $\mathcal{D}'$, then removing xy from $\mathcal{D}'$ gives a decomposition of G with cost $\pi_3^\alpha(G') - 2$, contradicting the maximality of G. Therefore xy must appear in a triangle $xyz \in \mathcal{D}'$. We now construct a decomposition $\mathcal{D}^*$ of G by removing xyz from $\mathcal{D}'$ and adding the edges xz and yz. Since the total cost of $\mathcal{D}^*$ is $\alpha(t'-1) + 2(\ell'+2)$, we have

$$\pi_3^\alpha(G) \leqslant \operatorname{cost}(\mathcal{D}^*) = \alpha(t'-1) + 2(\ell'+2) = \alpha t' + 2\ell' - \alpha + 4 \leqslant \alpha t' + 2\ell' = \pi_3^\alpha(G'),$$

which contradicts the maximality of $\pi_3^\alpha(G)$ if at least one of the inequalities is strict. Hence $\alpha = 4$, xy must be in a triangle in $\mathcal{D}'$, and $\pi_3^\alpha(G') = \pi_3^\alpha(n)$.

This means that it is possible to keep adding edges to G, which results in a sequence of graphs $G, G', \ldots, K_n$ where an optimal decomposition of each of these graphs has cost $\pi_3^\alpha(n)$, *i.e.* they are all π_3^α-extremal graphs. Note that we can add missing edges to G in any order, always obtaining a sequence of extremal graphs.

This allows us to reverse the process and examine a sequence of edge removals from K_n.

Suppose that G is obtained from K_n by removing the edge xy, *i.e.* G' is K_n. Note that if $\ell' > 0$, *i.e.* the optimal decomposition of K_n contains an edge, then there exists an option for $\mathcal{D}'$ that contains the edge xy, which was already ruled out. This means that K_n is triangle-divisible, which is the case if and only if $n \equiv 1, 3 \pmod 6$.

Now assume that G is missing more than one edge. Hence K_n^- must be also extremal. By the above, $n \equiv 1, 3 \pmod 6$, K_n is triangle-divisible, and $\pi_3^4(n) = 4\nu(K_n)$, where $\nu(K_n) = \frac{1}{3}\binom{n}{2}$.

Suppose that G is obtained from K_n by removing two edges uv and xy. First suppose that $u = x$. Let $\mathcal{D}^\star$ be a decomposition of G into triangles and one edge vy. This gives

$$\pi_3^4(G) \leqslant \operatorname{cost}(\mathcal{D}^\star) = 4(\nu(K_n) - 1) + 2 < 4\nu(K_n) = \pi_3^4(n),$$

contradicting the maximality of $\pi_3^4(G)$. Hence xy and uv form a matching. Note that x, y, u and v have odd degrees in G, so $\ell \geqslant 2$, for else we are unable to fix the parity of the vertices x, y, u and v. Now $\binom{n}{2} - \ell - 2$ needs to be divisible by 3, so $\ell \geqslant 4$. There indeed exists a decomposition with $\ell = 4$ by taking edges xu, xv, yu and yv and the rest as triangles. This gives

$$\pi_3^4(G) = 4(\nu(K_n) - 2) + 2 \cdot 4 = \pi_3^4(n).$$

Therefore G is extremal.

Suppose that G is obtained from K_n by removing three edges uv, xy and zw. Since G' must be K_n without a matching, uv, xy and zw also form a matching. Let $\mathcal{D}^\star$ be a decomposition of G into triangles and edges ux, yz and vw. This gives

$$\pi_3^4(G) \leqslant \mathrm{cost}(\mathcal{D}^\star) = 4(\nu(K_n) - 2) + 6 < 4\nu(K_n) = \pi_3^4(n),$$

contradicting the maximality of $\pi_3^4(G)$. This implies that G cannot be obtained from K_n by deleting three or more edges, thus finishing the proof of this case and of Theorem 1.6.

5. Related results

A related question of Erdős (see *e.g.* [9]) asks for the largest $t = t(n, m)$ such that every graph with n vertices and $t_2(n) + m$ edges has at least t edge-disjoint triangles. Of course, $t \leqslant m$. Győri [11] (see [13] for a correction) showed, for large n, that $t \geqslant m - O(m^2/n^2)$ if $m = o(n^2)$, and $t = m$ if n is odd and $m \leqslant 2n - 10$ or n is even and $m \leqslant 3n/2 - 5$. Moreover, the last two bounds on m are sharp.

More recently, Győri and Keszegh [14] proved that every K_4-free graph with $t_2(n) + m$ edges has m edge-disjoint triangles.

Theorem 1.5 shows that the maximum of $\pi_3(G)$ is attained for $G = T_2(n)$ or $G = K_n$. However, if we restrict the set of graphs under consideration to graphs of a particular edge density, the decomposition is perhaps cheaper. Note that if the optimal decomposition of a graph G contains t triangles and ℓ edges, then $\pi_3(G) = 2e(G) - 3t$. That is, we have that $\pi_3(G) = 2e(G) - 3\nu(G)$, where as before $\nu(G)$ denotes the maximum number of edge-disjoint triangles in G. Then Theorem 1.3 implies an inequality between the edge density of G and its *triangle packing density*, which we denote by $\nu_d(G) := 3\nu(G)/\binom{n}{2}$.

Corollary 5.1 (of Theorem 1.3)**.** *Let G be a graph with $d\binom{n}{2}$ edges. Then*

$$\nu_d(G) \geqslant 2d - 1 + o(1).$$

We also have that $\nu_d(G) \leqslant d$, which is tight for all graphs which are the union of edge-disjoint triangles.

A question reminiscent of the seminal result of Razborov on the minimal triangle density in graphs [25] (see also [22] and [20]) would be to determine the exact lower bound on $\nu_d(G)$ in terms of d (answering asymptotically the question of Erdős stated above).

Some flag algebra computations yield numerical asymptotic lower bounds on $\nu_d(G)$ with different edge densities between 0.5 and 1. The result, depicted in Figure 3, suggests that the true asymptotic shape of the region $\{(d, \nu_d(G)) \colon 0 \leqslant d \leqslant 1, G \text{ graph}\}$ may indeed have a richer structure.

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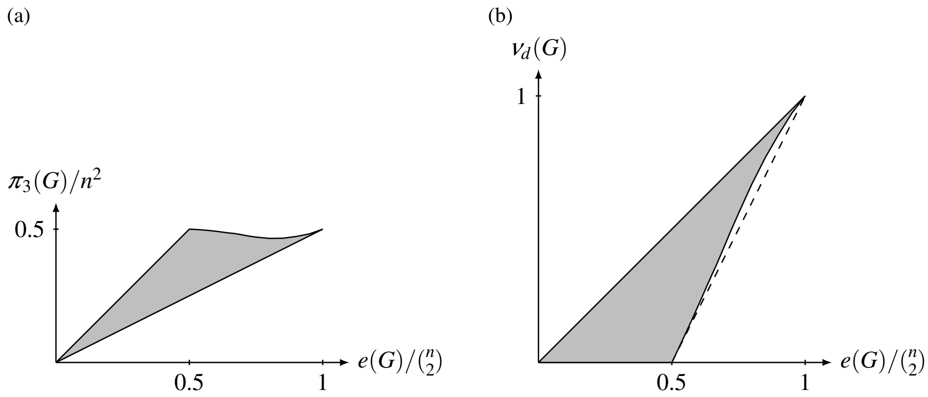

Figure 3. Asymptotic bounds on possible values of $\pi_3(G)$ and $\nu_d(G)$. The dashed line is simply $y = 2x - 1$ for a better display of the shape.

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