Acta Math. Univ. Comenianae
Vol. LXXXVIII, 3 (2019), pp. 463–468

SHARP BOUNDS FOR DECOMPOSING GRAPHS INTO EDGES AND TRIANGLES

A. BLUMENTHAL, B. LIDICKÝ, O. PIKHURKO, Y. PEHOVA, F. PFENDER AND J. VOLEC

Abstract. Let $\pi_3(G)$ be the minimum of twice the number of edges plus three times the number of triangles over all edge-decompositions of G into copies of K_2 and K_3. We are interested in the value of $\pi_3(n)$, the maximum of $\pi_3(G)$ over graphs G with n vertices. This specific extremal function was first studied by Gyori and Tuza [Decompositions of graphs into complete subgraphs of given order, Studia Sci. Math. Hungar. 22 (1987), 315–320], who showed that $\pi_3(n) \le 9n^2/16$. In a recent advance on this problem, Král', Lidický, Martins and Pehova [arXiv:1710:08486] proved via flag algebras that $\pi_3(n) \le (1/2 + o(1))n^2$, which is tight up to the $o(1)$ term. We extend their proof by giving the exact value of $\pi_3(n)$ for large n, and we show that K_n and $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$ are the only extremal examples.

1. Introduction

In recent work of Král', Lidický, Martins and Pehova [**15**], they proved using the flag algebra method (see [**18, 3, 1, 4, 7, 8, 12, 16**] for applications to other problems in extremal combinatorics) that the edges of any n-vertex graph can be decomposed into copies of K_2 and K_3 whose total number of vertices is at most $(1/2 + o(1))n^2$. This was a conjecture of Győri and Tuza [**19**], but the problem itself can be traced back to Erdős, Goodman and Pósa [**6**] who considered the problem of minimising the total number of cliques in an edge-decomposition of an arbitrary n-vertex graph. They showed the following:

Theorem 1 (Erdős, Goodman, Pósa [**6**])**.** *The edges of every n-vertex graph can be decomposed into at most $n^2/4$ complete graphs.*

The only extremal example for this bound is the bipartite Turán graph $T_2(n)$. Moreover, this result still holds if we restrict the sizes of the cliques used in the decomposition to 2 and 3 (that is, triangles and single edges). In a series of papers published independently by Chung [**13**], Győri and Kostochka [**10**], and Kahn [**14**],

Received May 22, 2019.
2010 *Mathematics Subject Classification.* Primary 05C70.
The second author is supported in part by NSF grant DMS-1600390.
The third author is supported by ERC grant 306493 and EPSRC grant EP/K012045/1.
The fourth author is supported by ERC grant 648509. The fifth author is supported in part by NSF grant DMS-1600483.

they proved that in fact something stronger than Theorem 1 is true, confirming a conjecture by Katona and Tarján:

Theorem 2 (Chung [**13**], Győri and Kostochka [**10**], Kahn [**14**]). *Every n-vertex graph can be edge-decomposed into cliques whose total number of vertices is at most $n^2/2$.*

For a given graph G on n vertices, let $\pi_k(G)$ be the minimum over all decompositions of the edges of G into cliques $C_1, ..., C_\ell$ of size at most k of the sum $|C_1| + |C_2| + \cdots + |C_\ell|$. With this notation, the conclusion of the above theorem is that $\min_{k\in\mathbb{N}} \pi_k(G) \leq n^2/2$. In light of Theorem 2, Tuza [**19**] conjectured that $\pi_3(G) \leq n^2/2 + o(n^2)$, and in fact that $\pi_3(G) \leq n^2/2 + O(1)$. In [**9**] Győri and Tuza showed that $\pi_3(G) \leq 9n^2/16$. This was the best known bound until recently, when using the celebrated flag algebra method by Razborov [**18**], Král', Lidický, Martins and Pehova [**15**] proved the asymptotic version of Tuza's conjecture:

Theorem 3 (Král', Lidický, Martins and Pehova [**15**]). *Every n-vertex graph G satisfies $\pi_3(G) \leq (1/2 + o(1))n^2$.*

We show, by building upon the proof in [**15**], that in fact $\pi_3(G) \leq n^2/2+1$, and that the extremal graphs G which maximise $\pi_3(G)$ are the complete graph K_n and the bipartite Turán graph $T_2(n)$. Which of these two graphs is extremal is a matter of divisibility of n by 6. In the case of the Turán graph, $\pi_3(T_2(n)) = 2\lfloor n/2\rfloor\lceil n/2\rceil$, giving $n^2/2$ for even n and $(n^2 - 1)/2$ for odd n. For graphs with minimum degree $n - o(n)$, the following result shows that we can decompose them only into copies of K_3, as long as they are *triangle-divisible*; that is, if each vertex has even degree and the total number of edges is divisible by three.

Theorem 4 (Barber, Kuhn, Lo, Osthus [**2**] and Dross [**5**]). *For every $\varepsilon > 0$, if G is a triangle-divisible graph of large order n and minimum degree at least $(0.9 + \varepsilon)n$, then G has a triangle decomposition.*

In particular, for each residue class of n mod 6, the optimal triangle-edge decompositions of K_n are in Table 1.

Table 1. Values of $\pi_3(K_n)$ and $\pi_3(T_2(n))$ for large n.

n mod 6	optimal decomposition of K_n	$\pi_3(K_n)$	$\pi_3(T_2(n))$
0	triangle-divisible + perfect matching	$\frac{n^2}{2}$	$\frac{n^2}{2}$
1	triangle-divisible	$\binom{n}{2}$	$\frac{n^2-1}{2}$
2	triangle-divisible + perfect matching	$\frac{n^2}{2}$	$\frac{n^2}{2}$
3	triangle-divisible	$\binom{n}{2}$	$\frac{n^2-1}{2}$
4	triangle-divisible + perfect matching + $K_{1,3}$	$\frac{n^2}{2}+1$	$\frac{n^2}{2}$
5	triangle-divisible + C_4	$\binom{n}{2}+4$	$\frac{n^2-1}{2}$

Let us define

$$\mathcal{E}_n = \begin{cases} \{T_2(n)\} & \text{if } n \equiv 1,3,5 \pmod 6, \\ \{K_n\} & \text{if } n \equiv 4 \pmod 6, \\ \{T_2(n), K_n\} & \text{if } n \equiv 0,2 \pmod 6, \end{cases}$$

the graphs in $\{T_2(n), K_n\}$ which maximise π_3 in each residue class mod 6. For $n \in \mathbb{N}$, let L_n be any member of $\mathcal{E}_n$ and define $\ell(n) := \pi_3(L_n)$. Clearly, $\ell(n)$ is a lower bound on $\pi_3(n)$, the maximum over all n-vertex graphs G of $\pi_3(G)$.

Then, our main result is the following:

Theorem 5. *There exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $\pi_3(n) = \ell(n)$ and the set of $\pi_3(n)$-extremal graphs is exactly $\mathcal{E}_n$. This gives*

$$\pi_3(n) = \begin{cases} n^2/2 & \text{for } n \equiv 0,2 \bmod 6 & \text{attained only by } T_2(n) \text{ and } K_n, \\ (n^2-1)/2 & \text{for } n \equiv 1,3,5 \bmod 6 & \text{attained only by } T_2(n), \\ n^2/2+1 & \text{for } n \equiv 4 \bmod 6 & \text{attained only by } K_n. \end{cases}$$

A simple corollary of Theorem 5 is an affirmative answer to a question of Pyber [**17**], see also Problem 45 [**19**], for sufficiently large n. In a decomposition of a graph, every edge is used exactly once. In a covering, every edge is used at least once.

Corollary 6. *For sufficiently large n, an edge set of every n-vertex graph be covered with triangles of weight 3 and edges of weight 2 such that their total weight is at most $\lfloor n^2/2 \rfloor$.*

2. Proof of Theorem 5

By analysing the dual solution to the optimisation problem considered in [**15**], we may obtain the following:

Proposition 7. *For every $\delta > 0$ there exists $n_1 \in \mathbb{N}$ such that if G is a graph of order $n \geq n_1$ with $\pi_3(G) \geq \ell(n) - n^2/n_1$, then G is δn^2-close[1] in edit distance to K_n or to $T_2(n)$.*

In the case when G is δn^2 edges away from $T_2(n)$, a result by Györi [**11**] that a graph with n vertices and $e(T_2(n)) + k$ edges where $k = o(n^2)$ has at least $k - O(k^2/n^2)$ edge-disjoint triangles almost immediately implies the desired result. More specifically, for each $\varepsilon > 0$ there exists $\delta > 0$ such that for large n every n-vertex graph with $t_2(n) + k$ edges where $k \leq \delta n^2$ has at least $k - \varepsilon k^2/n^2$ edge-disjoint triangles. Since G is δn^2-close to $T_2(n)$, then it must have at most $t_2(n) + \delta n^2$ edges. From this we have that $\pi_3(G) \leq 2(t_2(n)) + k) - 3(k - \varepsilon k^2/n^2) = 2t_2(n) - k(1 - 3\varepsilon k/n^2) \leq 2t_2(n)$ for $\delta \ll \varepsilon \ll 1$. Equality is achieved only if $k = 0$, that is, if $G \cong T_2(n)$.

[1]Two graphs G_1 and G_2 on the same vertex set are said to be *k-close in edit distance* (or simply *k-close*) if $|E(G_1) \triangle E(G_2)| \leq k$.

In the case when G is δn^2 edges away from K_n, by iteratively removing vertices of small degree, we may assume that $\delta(G) \geq (\ell(n) - \ell(n-1))/2$.

We now proceed to decompose the edges of G into edges and triangles in 3 stages:

Stage 1: Denote by U the set of all vertices which have degree less than $(1-c)n$ for some $c \ll 1$, and let $W = V(G)\backslash U$. By a double-counting argument $|U| = u \leq (2\delta/c)n^2$. For each vertex $u \in U$ in turn, remove a maximum family of edge-disjoint triangles, each containing u and two vertices from W. Denote the resulting graph induced on W by G'. Through a simple neighbourhood-chasing argument, we can show that $|\Gamma_{G'}(u) \cap W| \leq 1$ for all $u \in U$, that is, up to parity, the triangles removed during Stage 1 cover all $[U, W]$-edges in G.

Stage 2: Remove a maximum collection of edge-disjoint triangles from G'. Using Theorem 4 we may consider this as a problem of setting aside a set X of edges (which induce a bounded-degree graph) such that $G' - X$ is triangle-divisible. By considering the even- and odd-degree vertices in G' separately, we can construct a set X of size $|X| = p \leq (n-u)/2 + 2$ and maximum degree 2.

Stage 3: Decompose X trivially into copies of K_2.

Let t_1 and t_2 denote the number of triangles removed respectively in Stages 1 and 2. By counting pairs of vertices inside W, we conclude that $t_1 + 3t_2 + p \leq \binom{n-u}{2}$. Moreover, since each vertex of U has degree at most $(1-c)n$ in G, we also have that $t_1 \leq u(1-c)n/2$.

Thus we obtain

$$\begin{aligned}
\pi_3(G) &\leq 3t_1 + 3t_2 + 2\binom{u}{2} + 2p + 2u \\
&\leq 3t_1 + \left(\binom{n-u}{2} - p - t_1\right) + 2\binom{u}{2} + 2p + 2u \\
&\leq u(1-c)n + \binom{n-u}{2} + 2\binom{u}{2} + p + 2u \\
&= \binom{n}{2} + \frac{3u^2}{2} + \frac{3u}{2} - cnu + p.
\end{aligned}$$

We now compare this bound with the conjectured maxima presented in Table 1.

First, suppose that n is even. Here the larger value is achieved by K_n and it is at least $\pi_3(K_n) \geq n^2/2 = \binom{n}{2} + \frac{n}{2}$. Since $u \leq (2\delta/c)n$, we have that $3u^2/2 + \frac{3u}{2} \leq cnu/2$ and so

$$\pi_3(G) - \pi_3(K_n) \leq -cnu/2 + (n-u)/2 + 2 - (n+u)/2,$$

which is non-negative only if $u = 0$, and since G is extremal, all inequalities we used in upper-bounding $\pi_3(G)$ are tight. In particular, we get that $t_1 = u(1-c)n/2 = 0$, and hence $e(G) = 3t_2 + p = \binom{n-u}{2} = \binom{n}{2}$, meaning that $G \cong K_n$.

Now, suppose that n is odd. In this case we have $\pi_3(T_2(n)) - n/2 \geq \pi_3(K_n) - O(1) = \binom{n}{2} - O(1)$. Similarly to the previous case, $\pi_3(G) - \pi_3(T_2(n)) \leq -cnu/2+$

$O(1)$, which again is non-negative only if $u = 0$ and $e(G) = \binom{n}{2}$. But since n is odd, this means that G has no odd-degree vertices and in fact $\pi_3(G) \le \binom{n}{2} + 2 < \pi_3(T_2(n))$, a contradiction to the extremality of G.

References

1. Balogh J., Hu P., Lidický B. and Pfender F., *Maximum density of induced 5-cycle is achieved by an iterated blow-up of 5-cycle*, European J. Combin. **52** (2016), 47–58.

2. Barber B., Kühn D., Lo A. and Osthus D., *Edge-decompositions of graphs with high minimum degree*, Adv. Math. **288** (2016), 337–385.

3. Baber R. and Talbot J., *Hypergraphs do jump*, Combin. Probab. Comput. **20** (2011), 161–171.

4. Das S., Huang H., Ma J., Naves H. and Sudakov B., *A problem of Erdős on the minimum number of k-cliques*, J. Combin. Theory Ser. B **103** (2013), 344–373.

5. Dross F., *Fractional triangle decompositions in graphs with large minimum degree*, SIAM J. Discrete Math. **30** (2016), 36–42.

6. Erdős P., Goodman A. W. and Pósa L., *The representation of a graph by set intersections*, Canad. J. Math. **18** (1966), 106-112.

7. Even-Zohar Ch. and Linial N., *A note on the inducibility of 4-vertex graphs*, Graphs Combin. **31** (2015) 1367–1380.

8. Glebov R., Král' D. and Volec, J., *A problem of Erdős and Sós on 3-graphs*, Israel J. Math. **211** (2016), 349–366.

9. Győri E. and Tuza Zs., *Decompositions of graphs into complete subgraphs of given order*, Studia Sci. Math. Hungar. **22** (1987), 315–320.

10. Győri E. and Kostochka A. V., *On a problem of G. O. H. Katona and T. Tarján*, Acta Math. Acad. Sci. Hungar. **34** (1979), 321–327.

11. Győri E., *On the number of edge-disjoint triangles in graphs of given size*, Colloq. Math. Soc. János Bolyai **52** (1988), 267–276.

12. Hladký J., Král' D. and Norin S., *Counting flags in triangle-free digraphs*, Combinatorica **37** (2017), 49–76.

13. Chung F. R. K., *On the decomposition of graphs*, SIAM J. Algebraic Discrete Methods **2** (1981), 1–12.

14. Kahn J., *Proof of a conjecture of Katona and Tarján*, Period. Math. Hungar. **12** (1981), 81–82.

15. Král' D., Lidický B., Martins T. L. and Pehova Y., *Decomposing graphs into edges and triangles*, Combin. Probab. Comput. **28** (2019), 465–472.

16. Král' D., Mach L. and Sereni, J.-S., *A new lower bound based on Gromov's method of selecting heavily covered points*, Discrete Comput. Geom. **48** (2012), 487–498.

17. Pyber L., *Covering the edges of a graph by ...*, Colloq. Math. Soc. János Bolyai **60** (1992), 583–610.

18. Razborov A., *Flag algebras*, J. Symb. Log. **72** (2007), 1239–1282.

19. Tuza Zs., *Unsolved Combinatorial Problems, Part I*, BRICS Lecture Series LS-01-1, 2001.

A. Blumenthal, Department of Mathematics, Iowa State University, Ames, IA, USA,
e-mail: `ablument@iastate.edu`

B. Lidický, Department of Mathematics, Iowa State University, Ames, IA, USA,
e-mail: `lidicky@iastate.edu`

O. Pikhurko, Mathematics Institute, University of Warwick, Coventry, UK,
e-mail: `o.pikhurko@warwick.ac.uk`

Y. Pehova, Mathematics Institute, University of Warwick, Coventry, UK,
e-mail: `y.pehova@warwick.ac.uk`

F. Pfender, Department of Mathematical and Statistical Sciences, University of Colorado Denver, USA,
e-mail: `florian.pfender@ucdenver.edu`

J. Volec, Mathematics and Science Center, Atlanta, USA,
e-mail: `jan@ucw.cz`