Received: 10 March 2017 | Revised: 11 June 2019 | Accepted: 6 October 2019

DOI: 10.1002/jgt.22513

ARTICLE

WILEY

Isometric copies of directed trees in orientations of graphs

Taras Banakh[1,2] | **Adam Idzik**[1,3] | **Oleg Pikhurko**[4] | **Igor Protasov**[5] | **Krzysztof Pszczoła**[6]

[1]Institute of Mathematics, Jan Kochanowski University, Kielce, Poland

[2]Faculty of Mechanics and Mathematics, Ivan Franko University of Lviv, Lviv, Ukraine

[3]Institute of Computer Science, Polish Academy of Sciences, Warsaw, Poland

[4]Mathematics Institute and DIMAP, University of Warwick, Coventry, UK

[5]Faculy of Cybernetics, Taras Shevchenko National University, Kyiv, Ukraine

[6]Institute of Mathematics and Cryptology, Military University of Technology, Warsaw, Poland

Correspondence
Taras Banakh, Institute of Mathematics, Jan Kochanowski University, 25-406, ul. Swietokrzyska 15, Kielce, Poland.
Email: t.o.banakh@gmail.com

Funding information
EPSRC, Grant/Award Number: EP/K012045/1; ERC, Grant/Award Number: 306493

Abstract

The *isometric Ramsey number* $\mathsf{IR}(\vec{\mathcal{H}})$ of a family $\vec{\mathcal{H}}$ of digraphs is the smallest number of vertices in a graph G such that any orientation of the edges of G contains every member of $\vec{\mathcal{H}}$ in the distance-preserving way. We observe that the isometric Ramsey number of a finite family of finite acyclic digraphs is always finite, and present some bounds in special cases. For example, we show that the isometric Ramsey number of the family of all oriented trees with n vertices is at most $n^{2n+o(n)}$.

KEYWORDS

chromatic number, directed tree, girth, isometric embedding, orientation of a graph

JEL CLASSIFICATION

05C20; 05C55; 05C80

1 | INTRODUCTION

In this paper we consider the "isometric" version of the result of Cochand and Duchet [6] who proved (generalizing a result of Rödl [11]) that for every acyclic digraph $\vec{H}$ there exists a finite graph G such that every orientation of G contains an isomorphic copy of $\vec{H}$.

First we recall the necessary definitions from Graph Theory. A *graph* is a pair $G = (V_G, E_G)$ consisting of a set V_G of *vertices* and a set E_G of two-element subsets of V_G, called the *edges* of G. By a *digraph* we will mean a pair $\vec{G} = (V_{\vec{G}}, E_{\vec{G}})$ consisting of a set $V_{\vec{G}}$ of vertices and a set $E_{\vec{G}} \subset V_{\vec{G}} \times V_{\vec{G}}$ of *directed edges*, where neither loops (x, x), nor pairs of opposite arcs (x, y) and

(y, x) are allowed. An *orientation* of a graph $G = (V_G, E_G)$ is a function $\vec{\cdot}: E_G \to V_G^2$ assigning to each edge $e \in E_G$ an ordered pair $\vec{e} = (a, b) \in V_G^2$ such that $e = \{a, b\}$. In this case the pair $\vec{G} = (V_G, \{\vec{e}\}_{e \in E_G})$ is a digraph called an *orientation* of G.

A sequence $(v_0, ..., v_n)$ of distinct vertices of a graph G is called a *path* in G if for every positive $i \le n$ the unordered pair $\{v_{i-1}, v_i\}$ is an edge of G. The *length* of the path $(v_0, ..., v_n)$ is n, that is, the number of edges. The *distance* $d_G(x, y)$ between two vertices v, u of a graph G is the smallest length of a path in G connecting the vertices v and u. If u and v cannot be connected by a path, then we write $d_G(x, y) = \infty$ and assume that $\infty > n$ for all $n \in \omega$. A graph G is called *connected* if any two vertices u, v can be connected by a path in G. A digraph is called *connected* if its underlying undirected graph is connected. The distance in a digraph is taken with respect to the underlying undirected graph.

A sequence $(v_0, ..., v_n)$ of distinct vertices of a digraph $\vec{G}$ is called a *directed path* in $\vec{G}$ if for every positive $i \le n$ the ordered pair (v_{i-1}, v_i) is an edge of $\vec{G}$. A *directed cycle* is a sequence $(v_0, ..., v_n)$ of distinct vertices such that $\{(v_n, v_0)\} \cup \{(v_i, v_{i+1})\}_{0 \le i < n} \subset E_{\vec{G}}$. A digraph $\vec{G}$ is *acyclic* if it does not contain directed cycles. It is well-known that each graph G admits an acyclic orientation $\vec{G}$: take any linear order $\le$ on the set V_G of vertices and for any edge $\{u, v\} \in E_G$ put $(u, v) \in E_{\vec{G}}$ if and only if $u < v$.

Following Rado's arrow notations, for a graph G and a digraph $\vec{H}$ let us write $G \to \vec{H}$ if for every orientation $\vec{G}$ of G there exists an injective function $f : V_{\vec{H}} \to V_G$ such that an ordered pair (u, v) of vertices of $\vec{H}$ is a directed edge in $\vec{H}$ if and only if $(f(u), f(v))$ is a directed edge in $\vec{G}$. (Thus we require that f induces an isomorphism of undirected graphs and preserves all edge orientations.) If, moreover, $d_{\vec{H}}(u, v) = d_G(f(u), f(v))$ for every pair of vertices $u, v \in V_{\vec{H}}$, then we write $G \Rightarrow \vec{H}$ and say that f is an *isometric embedding* of $\vec{H}$ in $\vec{G}$. Since each graph G admits an acyclic orientation, the arrow $G \to \vec{H}$ implies that the digraph $\vec{H}$ is acyclic.

Given a graph G and a class $\vec{\mathcal{H}}$ of digraphs, we write $G \to \vec{\mathcal{H}}$ (resp. $G \Rightarrow \vec{\mathcal{H}}$) if for every oriented graph $\vec{H} \in \vec{\mathcal{H}}$ we have $G \to \vec{H}$ (resp. $G \Rightarrow \vec{H}$). In this case the family $\vec{\mathcal{H}}$ necessarily consists of acyclic digraphs. For a natural number $n \in \mathbb{N}$ by $\vec{\mathcal{T}}_n$ we denote the class of oriented trees on n vertices. By a *tree* we understand a connected graph without cycles. An *oriented tree* is a digraph whose underlying undirected graph is a tree. For $n \in \mathbb{N}$, the *directed path* $\vec{I}_n$ is the digraph with $V_{\vec{I}_n} = \{0, ..., n-1\}$ and $E_{\vec{I}_n} = \{(i-1, i) : 0 < i < n\}$. So, $\vec{I}_n$ has n vertices and $(n-1)$ edges.

For a class $\vec{\mathcal{H}}$ of digraphs let $\mathsf{R}(\vec{\mathcal{H}})$ (resp. $\mathsf{IR}(\vec{\mathcal{H}})$) be the smallest number of vertices of a graph G such that $G \to \vec{\mathcal{H}}$ (resp. $G \Rightarrow \vec{\mathcal{H}}$). If no graph G with $G \to \vec{\mathcal{H}}$ (resp. $G \Rightarrow \vec{\mathcal{H}}$) exists, then we put $\mathsf{R}(\vec{\mathcal{H}}) = \infty$ (resp. $\mathsf{IR}(\vec{\mathcal{H}}) = \infty$). The number $\mathsf{R}(\vec{\mathcal{H}})$ (resp. $\mathsf{IR}(\vec{\mathcal{H}})$) is called the (*isometric*) *Ramsey number* of the family $\vec{\mathcal{H}}$. If the family $\vec{\mathcal{H}}$ consists of a unique digraph $\vec{H}$, then we write $\mathsf{R}(\vec{H})$ and $\mathsf{IR}(\vec{H})$ instead of $\mathsf{R}(\{\vec{H}\})$ and $\mathsf{IR}(\{\vec{H}\})$, respectively.

By Theorem B of Cochand and Duchet [6], for every finite acyclic digraph $\vec{H}$ the Ramsey number $\mathsf{R}(\vec{H})$ is finite. This implies that for every finite family $\vec{\mathcal{H}}$ of finite acyclic digraphs the Ramsey number $\mathsf{R}(\vec{\mathcal{H}}) \le \sum_{\vec{H} \in \vec{\mathcal{H}}} \mathsf{R}(\vec{H})$ is finite, too. In Section 2 we shall apply a deep Ramsey result of Dellamonica and Rödl [7] to prove that the isometric Ramsey number $\mathsf{IR}(\vec{\mathcal{H}})$ is finite, too.

For the family $\vec{\mathcal{T}}_n$ of oriented trees on n vertices Kohayakawa, Łuczak and Rödl [9] proved that $\mathsf{R}(\vec{\mathcal{T}}_n) = O(n^4 \log n)$. In this paper for every $n \in \mathbb{N}$ we construct a graph G_n with $< 2^{2^{n-1}}$

vertices such that $G_n \Rightarrow \vec{\mathcal{T}}_n$, witnessing that $\mathsf{IR}(\vec{\mathcal{T}}_n) < 2^{2^{n-1}}$. Using Bollobás' [3] bounds on the order of graphs of large girth and large chromatic number, we shall improve the upper bounds $\mathsf{IR}(\vec{I}_n) \le \mathsf{IR}(\vec{\mathcal{T}}_n) < 2^{2^{n-1}}$ to $\mathsf{IR}(\vec{I}_n) = o(n^{2n})$ and $\mathsf{IR}(\vec{\mathcal{T}}_n) = o(n^{4n})$. In Theorem 4.5 using random graphs we improve the latter upper bound to $\mathsf{IR}(\vec{\mathcal{T}}_n) \le (4e + o(1))^n (n^2 \ln n)^n = n^{2n+o(n)}$. The technique developed for the proof of Theorem 4.5 allows us to improve the upper bound $\mathsf{R}(\vec{\mathcal{T}}_n) \le (2500e^8 + o(1))\, n^4 \ln n$ obtained by Kohayakawa et al [9] to the upper bound $(K + o(1))\, n^4 \ln n$, where $K = \min_{x>1} 16x^2/(1 - x + x\ln x) \approx 98.8249\ldots$. In Section 5 we search for long directed paths in arbitrary orientations of graphs. In the final Section 6 we prove that every infinite graph G admits an orientation containing no directed path of infinite diameter in G. Some other results and problems related to coloring and orientations in graphs can be found in [10].

2 | THE ISOMETRIC RAMSEY NUMBER FOR A FINITE ACYCLIC DIGRAPH

In this section we prove that each finite acyclic digraph $\vec{H}$ has finite isometric Ramsey number $\mathsf{IR}(\vec{H})$. The idea of the proof of this result was suggested to the authors by Yoshiharu Kohayakawa.

Theorem 2.1. *For any finite acyclic digraph $\vec{H} = (V, \vec{E})$, the isometric Ramsey number $\mathsf{IR}(\vec{H})$ is finite.*

Proof. Clearly, it is enough to the prove the theorem when the digraph $\vec{H}$ is connected. Fix any vertex h of $\vec{H}$ and consider the digraph $\vec{\Gamma}$ with

$$V_{\vec{\Gamma}} := V_{\vec{H}} \times \{0,1\} \quad \text{and} \quad \vec{E}_{\vec{\Gamma}} := \{((h,0),(h,1))\} \cup \{((u,0),(v,0)),((v,1),(u,1)) : (u,v) \in E_{\vec{H}}\}.$$

Observe that the digraph $\vec{\Gamma}$ is acyclic, connected, and contains isometric copies of $\vec{H}$ and the graph $\vec{H}$ with the opposite orientation. Being acyclic, the graph $\vec{\Gamma}$ admits a linear ordering $<$ of vertices such that $u < v$ for any directed edge $(u,v) \in \vec{E}_{\vec{\Gamma}}$.

By Theorem 1.8 of [7], there exists a finite graph G with a linear ordering of vertices such that for any 2-coloring of its edges there exists a map $f : V_{\vec{\Gamma}} \to V_G$ such that

- f is *monotone* in the sense that for any vertices $u < v$ of $\vec{\Gamma}$ we have $f(u) < f(v)$ in G;
- $d_G(f(u), f(v)) = d_{\vec{\Gamma}}(u,v)$ for any vertices u, v of $\vec{\Gamma}$;
- the set $\{\{f(u), f(v)\} : (u,v) \in E_{\vec{\Gamma}}\}$ is monochromatic.

In this case we shall say that f is a monochromatic monotone isometric embedding *of $\vec{\Gamma}$ into G.*

We claim that $G \Rightarrow \vec{H}$. Given any orientation $\vec{G}$ of the graph G, color an edge $\{u,v\} \in E_G$ with $u < v$ in green if $(u,v) \in E_{\vec{G}}$ and in red if $(v,u) \in E_{\vec{G}}$. By the Ramsey property of G, there exists a monochromatic monotone isometric embedding

$f: V_{\vec{\Gamma}} \to V_G$. If the color of the monochromatic set $C = \{\{f(u), f(v)\} : (u, v) \in E_{\vec{\Gamma}}\}$ is green, then the map $g_0 : V_{\vec{H}} \to V_G, g_0 : v \mapsto f(v, 0)$, is a required isometric isomorphic embedding of $\vec{H}$ into $\vec{G}$. If the color of C is red, then the map $g_1 : V_{\vec{H}} \to V_G, g_1 : v \mapsto f(v, 1)$, is an isometric isomorphic embedding of $\vec{H}$ into $\vec{G}$. In both cases we get $G \Rightarrow \vec{H}$. □

Corollary 2.2. *Any finite family $\vec{\mathcal{H}}$ of finite acyclic digraphs has finite isometric Ramsey number $\mathsf{IR}(\vec{\mathcal{H}})$.* □

Corollary 2.3. *For every $n \in \mathbb{N}$ the family $\vec{\mathcal{T}}_n$ of directed trees on n vertices has finite isometric Ramsey number $\mathsf{IR}(\vec{\mathcal{T}}_n)$* □

The proof of [[7], Theorem 1.8] proceeds by a more general induction involving amalgamation and hypergraphs, and seems to give very bad bounds on the isometric Ramsey number $\mathsf{IR}(\vec{\mathcal{A}}_n)$ for the family $\vec{\mathcal{A}}_n$ of all acyclic digraphs on n vertices. It would be interesting to get some reasonable upper bound on this function.

3 | SIMPLE BOUNDS FOR THE ISOMETRIC RAMSEY NUMBERS $\mathsf{IR}(\vec{\mathcal{T}}_n)$

In this section we prove some simple upper bounds on the isometric Ramsey numbers $\mathsf{IR}(\vec{\mathcal{T}}_n)$ and $\mathsf{IR}(\vec{I}_n)$. First we present a simple example of a graph witnessing that $\mathsf{IR}(\vec{\mathcal{T}}_n) < 2^{2^{n-1}}$. The construction of this graph exploits rectangular products of graphs. By definition, the *rectangular product* $G \times H$ of two graphs G, H is the graph such that $V_{G\times H} = V_G \times V_H$ and an unordered pair $\{(g, h), (g', h')\} \subset G \times H$ is an edge of $G \times H$ if and only if either $\{g, g'\} \in E_G$ and $h = h'$ or $g = g'$ and $\{h, h'\} \in E_H$. It can be shown that for any vertices $(g, h), (g', h')$ of $G \times H$ we get

$$d_{G\times H}((g, h), (g', h')) = d_G(g, g') + d_H(h, h').$$

For an (oriented) graph G by $|G|$ we denote the cardinality of the set V_G of vertices of G. For a cardinal number m by K_m we denote the complete graph on m vertices.

Lemma 3.1. *Let $\vec{\mathcal{T}}, \vec{\mathcal{T}}'$ be two families of finite oriented trees such that for every oriented tree $\vec{T}' \in \vec{\mathcal{T}}'$ there is an oriented subtree $\vec{T} \in \vec{\mathcal{T}}$ of $\vec{T}'$ such that $|\vec{T}| = |\vec{T}'| - 1$. Then for any graph G with $G \Rightarrow \vec{\mathcal{T}}$ we get $G \times K_{|G|+1} \Rightarrow \vec{\mathcal{T}}'$.*

Proof. Let $G' = G \times K_{|G|+1}$. To prove that $G' \Rightarrow \vec{\mathcal{T}}'$, take any oriented tree $\vec{T}' \in \vec{\mathcal{T}}'$ and any orientation $\vec{G}'$ of the graph G'. By our assumption, for the tree $\vec{T}'$ there exists an oriented subtree $\vec{T} \in \vec{\mathcal{T}}$ of $\vec{T}'$ such that $|\vec{T}| = |\vec{T}'| - 1$. Let t' be the unique element of the set $V_{\vec{T}'} \setminus V_{\vec{T}}$ and $t \in V_{\vec{T}}$ be the unique vertex of $\vec{T}$ such that (t', t) or (t, t') is an edge of $\vec{T}'$.

For every vertex u of the complete graph $K_{|G|+1}$, identify the graph G with the subgraph $G \times \{u\}$ and denote by $\vec{G}_u$ the orientation of the graph G induced by the orientation of the subgraph $G \times \{u\}$, inherited from the orientation $\vec{G}'$ of the graph

$G' = G \times K_{|G|+1}$. Since $G \Rightarrow \vec{\mathcal{T}}$, there is an isometric embedding $f_u : \vec{T} \to \vec{G}_u$. By the Pigeonhole Principle, there are two distinct vertices u, w in $K_{|G|+1}$ such that $f_u(t) = f_w(t)$. Now look at the orientation of the edges $\{t, t'\}$ and $\{(f_u(t), u), (f_w(t), w)\}$ in the digraphs $\vec{T}'$ and $\vec{G}'$.

If either $(t, t') \in E_{\vec{T}'}$ and $((f_u(t), u), (f_w(t), w)) \in E_{\vec{G}'}$ or $(t', t) \in E_{\vec{T}'}$ and $((f_w(t), w), (f_u(t), u)) \in E_{\vec{G}'}$, then we define a map $f : \vec{T}' \to G'$ by $f(t') = (f_u(t), w)$ and $f(\tau) = (f_u(\tau), u)$ for $\tau \in V_{\vec{T}}$, and observe that f is an isometric embedding of $\vec{T}'$ into $\vec{G}'$.

If either $(t, t') \in E_{\vec{T}'}$ and $((f_w(t), w), (f_u(t), u)) \in E_{\vec{G}'}$ or $(t', t) \in E_{\vec{T}'}$ and $((f_u(t), u), (f_w(t), w)) \in E_{\vec{G}'}$, then we define a map $f : \vec{T}' \to G'$ by $f(t') = (f_w(t), u)$ and $f(\tau) = (f_w(\tau), w)$ for $\tau \in V_{\vec{T}}$, and observe that f is an isometric embedding of $\vec{T}'$ into $\vec{G}'$. □

Corollary 3.2. *If for some $n \in \mathbb{N}$ a graph G satisfies the isometric Ramsey relation $G \Rightarrow \vec{\mathcal{T}}_n$, then $G \times K_{|G|+1} \Rightarrow \vec{\mathcal{T}}_{n+1}$.*

Theorem 3.3. *For every $n \in \mathbb{N}$ $\mathsf{IR}(\vec{\mathcal{T}}_{n+1}) \le \mathsf{IR}(\vec{\mathcal{T}}_n)(\mathsf{IR}(\vec{\mathcal{T}}_n) + 1)$ and $\mathsf{IR}(\vec{\mathcal{T}}_n) < 2^{2^{n-1}}$.*

Proof. The inequality $\mathsf{IR}(\vec{\mathcal{T}}_{n+1}) \le \mathsf{IR}(\vec{\mathcal{T}}^n)(\mathsf{IR}(\vec{\mathcal{T}}^n) + 1)$ follows from Corollary 3.2. Indeed, for every $n \in \mathbb{N}$ we can choose a graph G with $|G| = \mathsf{IR}(\vec{\mathcal{T}}^n)$ vertices and $G \Rightarrow \vec{\mathcal{T}}_n$. By Corollary 3.2, the graph $G' = G \times K_{|G|+1}$ satisfies the relation $G' \Rightarrow \vec{\mathcal{T}}_{n+1}$ and hence

$$\mathsf{IR}(\vec{\mathcal{T}}_{n+1}) \le |G'| = |G|(|G| + 1) = \mathsf{IR}(\vec{\mathcal{T}}_n)(\mathsf{IR}(\vec{\mathcal{T}}_n) + 1).$$

It remains to prove that $\mathsf{IR}(\vec{\mathcal{T}}_n) + 1 \le 2^{2^{n-1}}$ for $n \in \mathbb{N}$. For $n = 1$ we have the equality $\mathsf{IR}(\vec{\mathcal{T}}_1) + 1 = 1 + 1 = 2^{2^0}$. Assume that for some $n \in \mathbb{N}$ we have proved that $\mathsf{IR}(\vec{\mathcal{T}}_n) + 1 \le 2^{2^{n-1}}$. Then

$$\mathsf{IR}(\vec{\mathcal{T}}_{n+1}) + 1 \le \mathsf{IR}(\vec{\mathcal{T}}_n)(\mathsf{IR}(\vec{\mathcal{T}}_n) + 1) + 1 \le (2^{2^{n-1}} - 1)2^{2^{n-1}} + 1 = 2^{2^n} - 2^{2^{n-1}} + 1 \le 2^{2^n}.$$

□

The upper bound $\mathsf{IR}(\vec{\mathcal{T}}_n) < 2^{2^{n-1}}$ can be greatly improved using known upper bounds on the Erdős function $\mathsf{Erdös}(k, g)$, which assigns to any positive integer numbers k, g the smallest cardinality $|G|$ of a graph G with chromatic number $\chi(G) \ge k$ and girth $g(G) \ge g$. We recall that the *girth* $g(G)$ of a graph is the smallest cardinality of a cycle in G. If G contains no cycles, then we put $g(G) = \infty$. The *chromatic number* $\chi(G)$ of a graph G is the smallest number $k \in \mathbb{N}$ for which there exists a map $\chi : V_G \to \{1, ..., k\}$ such that $\chi(x) \ne \chi(y)$ for any edge $\{x, y\} \in E_G$. The following bounds for the Erdős function $\mathsf{Erdös}(k, g)$ were proved by Erdős [8], Bollobás [3], and Spencer [12], respectively.

Proposition 3.4.

(1) *For any k, g we get $\mathsf{Erdös}(k, g) \ge k^{(g-1)/2}$.*

(2) *For any* $k, g \geq 4$ *we have* $\mathsf{Erdös}(k,g) \leq \lceil h^g \rceil$ *where* $h = 6(k+1)\ln(k+1)$.
(3) *There exists a constant* C *such that for any numbers* $k, g \geq 3$ *and* $m = \mathsf{Erdös}(k,g)$ *we have the inequality* $\sqrt[g-2]{m} \cdot \ln m < Ck$, *which implies that* $\mathsf{Erdös}(k,g) = o(k^{g-2})$ *as* $\max\{k, g\} \to \infty$.

Write $G \rightharpoonup \vec{\mathcal{H}}$ if for every orientation $\vec{G}$ of G and every $\vec{H} \in \vec{\mathcal{H}}$ there is an injective map $f : V_{\vec{H}} \to V_G$ such that for every directed edge (x,y) of $\vec{H}$ the pair $(f(x), f(y))$ is a directed edge of $\vec{G}$. (Note that we do not require that f induces an isomorphism, that is, G can have extra edges inside the set $f(V_{\vec{H}})$.) Another function related to $\mathsf{IR}(\vec{\mathcal{H}})$ is Burr's function $\mathsf{Burr}(\vec{\mathcal{H}})$ assigning to every family $\vec{\mathcal{H}}$ of oriented trees the smallest number k such that $G \rightharpoonup \vec{\mathcal{H}}$ for every graph G with chromatic number $\chi(G) \geq k$. If such number k does not exist, then we put $\mathsf{Burr}(\vec{\mathcal{H}}) = \infty$. By the Gallai-Hasse-Roy-Vitaver Theorem [[13], Theorem 3.13], the chromatic number $\chi(G)$ of a finite graph G is equal to $\max\{n \in \mathbb{N} : G \rightharpoonup \vec{I_n}\}$. This equality implies that $\mathsf{Burr}(\vec{I_n}) = n$ for every $n \in \mathbb{N}$. In [5] Burr considered the numbers $\mathsf{Burr}(\vec{\mathcal{T}_n})$ and proved that $\mathsf{Burr}(\vec{\mathcal{T}_n}) \leq (n-1)^2$. This upper bound was improved to the upper bound $\mathsf{Burr}(\vec{\mathcal{T}_n}) \leq (1/2)n^2 - (1/2)n + 1$ in [2]. According to (still unproved) Conjecture of Burr [5], the equality $\mathsf{Burr}(\vec{\mathcal{T}_n}) = 2n - 2$ holds for all $n \geq 2$.

Proposition 3.5. *For any* $n \in \mathbb{N}$ *and a subclass* $\vec{\mathcal{H}} \subset \vec{\mathcal{T}}_n$ *we get the upper bound*

$$\mathsf{IR}(\vec{\mathcal{H}}) \leq \mathsf{Erdös}(\mathsf{Burr}(\vec{\mathcal{H}}), 2n-2).$$

Proof. Fix a graph G of cardinality $|G| = \mathsf{Erdös}\,(\mathsf{Burr}\,(\vec{\mathcal{H}}), 2n-2)$ with chromatic number $\chi(G) \geq \mathsf{Burr}(\vec{\mathcal{H}})$ and girth $g(G) \geq 2n-2$. Let us prove that $G \Rightarrow \vec{\mathcal{H}}$. Take any orientation $\vec{G}$ of G and $\vec{H} \in \vec{\mathcal{H}}$. Since $G \rightharpoonup \vec{\mathcal{H}}$, there is an orientation-preserving injection $f : \vec{H} \to \vec{G}$. Since $\vec{H}$ is a connected graph with at most n vertices and $g(G) \geq 2n-2$, the map f is an isometric embedding. So, $G \Rightarrow \vec{\mathcal{H}}$. □

Combining Proposition 3.5 with known upper bounds $\mathsf{Burr}\,(\vec{I_n}) = n$ and $\mathsf{Burr}\,(\vec{\mathcal{T}_n}) \leq (1/2)n^2 - (1/2)n + 1$ we get the following upper bounds for the isometric Ramsey numbers $\mathsf{IR}(\vec{I_n})$ and $\mathsf{IR}(\vec{\mathcal{T}_n})$.

Corollary 3.6. *For every* $n \in \mathbb{N}$ *we get the upper bounds*

$$\mathsf{IR}(\vec{I_n}) \leq \mathsf{Erdös}(n, 2n-2) = o(n^{2n-4}) = o(n^{2n}) \quad \text{and}$$
$$\mathsf{IR}(\vec{\mathcal{T}_n}) \leq \mathsf{Erdös}\left(\frac{1}{2}n^2 - \frac{1}{2}n + 1, 2n-2\right) = o\left(\left(\frac{1}{2}n^2 - \frac{1}{2}n + 1\right)^{2n-4}\right) = o(n^{4n}).$$

In Theorem 4.5 we shall improve the upper bound $o(n^{4n})$ for $\mathsf{IR}(\vec{\mathcal{T}_n})$ to the upper bound $n^{2n+o(n)}$.

Remark 3.7. By Theorem 3 in [9], $\mathsf{R}(\overrightarrow{I_n}) \geq n^2/2$ for all $n \in \mathbb{N}$. This yields the lower bound

$$\frac{1}{2}n^2 \leq \mathsf{R}(\overrightarrow{I_n}) \leq \mathsf{IR}(\overrightarrow{I_n}) \leq \mathsf{IR}(\overrightarrow{\mathcal{T}_n})$$

for the isometric Ramsey numbers $\mathsf{IR}(\overrightarrow{I_n})$ and $\mathsf{IR}(\overrightarrow{\mathcal{T}_n})$.

Remark 3.8. It can be shown that

$$\begin{aligned}
&\mathsf{IR}(\overrightarrow{I_1}) = \mathsf{IR}(\overrightarrow{\mathcal{T}_1}) = 1 = |K_1|,\\
&\mathsf{IR}(\overrightarrow{I_2}) = \mathsf{IR}(\overrightarrow{\mathcal{T}})_2 = 2 = |K_2|,\\
&\mathsf{IR}(\overrightarrow{I_3}) = 5 = |C_5|,\ \mathsf{IR}(\overrightarrow{\mathcal{T}_3}) = 6 = |K_2 \times K_3|,\\
&\mathsf{IR}(\overrightarrow{I_4}) \leq 30 = |C_5 \times K_6|,\ \mathsf{IR}(\overrightarrow{\mathcal{T}_4}) \leq 42 = |K_2 \times K_3 \times K_7|.
\end{aligned}$$

Question 3.9. What is the exact value of the isometric Ramsey numbers $\mathsf{IR}(\overrightarrow{I_4})$ and $\mathsf{IR}(\overrightarrow{\mathcal{T}_4})$? Are they distinct?

4 | ISOMETRIC COPIES OF DIRECTED TREES IN ORIENTATIONS OF RANDOM GRAPHS

In this section we shall apply the technique of random graphs and shall improve the upper bound $\mathsf{IR}(\overrightarrow{\mathcal{T}_n}) = o(n^{4n})$ established in Corollary 3.6 to the upper bound $\mathsf{IR}(\overrightarrow{\mathcal{T}_n}) \leq (4e + o(1))^n\,(n^2 \ln n)^n = n^{2n+o(n)}$.

First we prove some technical lemmas. The first of them uses the idea of the proof of Theorem 1 in [9].

Lemma 4.1. *A graph $G = (V_G, E_G)$ satisfies $G \Rightarrow \overrightarrow{\mathcal{T}_n}$ for some $n \in \mathbb{N}$ if there exist sequences $(w_k)_{k=1}^{n-1}$ and $(d_k)_{k=1}^{n-1}$ of positive real numbers such that for every $2 \leq k < n$ the following conditions hold:*

(1) For every set $S = \{s_1, ..., s_{k-1}\} \subset V_G$ of cardinality $k-1$ and every $v \in V_G \setminus S$, the set $Y := \{y \in V_G \setminus S : \mathrm{dist}_G(y, v) = 1$ and $\mathrm{dist}_{G-v}(y, s_i) \leq i$ for some $i < k\}$ has cardinality $|Y| \leq d_k$.

(2) Every set $W \subset V_G$ of cardinality $|W| = \bar{w}_k := \min\{m \in \mathbb{N} : w_k < m\}$ spans more than $(d_k + k - 1)\bar{w}_k$ edges in G.

(3) $\sum_{k=1}^{n-1} w_k < |V_G|$.

Proof. For a subset $U \subset V_G$ denote by $G[U] := (U, E[U])$ the induced subgraph of G with the set of edges $E[U] := \{\{u, v\} \in E_G : \{u, v\} \subset U\}$. Also, let us write $(G, U) \Rightarrow \overrightarrow{\mathcal{T}_k}$, meaning that, for every $\overrightarrow{T} \in \overrightarrow{\mathcal{T}_k}$, every orientation $\overrightarrow{G}$ of G contains an isometric copy of $\overrightarrow{T}$ which lies inside U.

We shall inductively prove that for every $1 \le k \le n$ and every set $U \subset V_G$ of size $|U| > \sum_{i=1}^{k-1} w_i$, we have $(G, U) \Rightarrow \vec{\mathcal{T}}_k$. The base case $k = 1$ is trivial. Suppose that this holds for some positive $k < n$. Take any $U \subset V_G$ of cardinality $|U| > \sum_{i=1}^{k} w_i$. Take any orientation $\vec{G}$ of the graph G and any directed tree $\vec{T} \in \vec{\mathcal{T}}_{k+1}$. Fix any vertex u of degree 1 in the tree $\vec{T}$. By symmetry, assume that (v, u) is an arc in $\vec{T}$, that is, the arc in $\vec{T}$ goes from the unique neighbor v of u to u.

Let W be the set of vertices in U whose out-degree in $G[U]$ is at most $d_k + k - 1$. We claim that $|W| \le w_k$. Suppose not. Then $|W| > w_k$ and we can choose a subset $W_k \subset W$ of cardinality $|W_k| = \bar{w}_k := \min\{m \in \mathbb{N} : w_k < m\}$. Item 2 guarantees that W_k spans more than $(d_k + k - 1)\bar{w}_k$ edges in G, each edge contributing to out-degree of some vertex in W_k. Thus $(d_k + k - 1)\bar{w}_k = (d_k + k - 1)|W_k| \ge |E[W_k]| > (d_k + k - 1)\bar{w}_k$, which is a desired contradiction showing that $|W| \le w_k$.

Thus $U' = U \backslash W$ has cardinality $|U'| = |U| - |W| > (\sum_{i=1}^{k} w_i) - w_k = \sum_{i=1}^{k-1} w_i$. By the inductive assumption, $(G, U') \Rightarrow \vec{\mathcal{T}}_k$, which implies that the digraph $\vec{G}$ contains an isometric copy $\vec{T}'$ of the oriented tree $\vec{T} - u$ such that $V_{\vec{T}'} \subset U'$. Let us identify the tree $\vec{T} - u$ with its isometric copy $\vec{T}'$ in $\vec{G}[U']$. Let $\{s_1, \ldots, s_{k-1}\}$ be an enumeration of the set $S := V_{\vec{T}'} \backslash \{v\} \subset U'$ such that $\mathrm{dist}(s_i, v) \le i$ for every $i < k$. Let Y be defined as in Item 1 with respect to v and $\{s_1, \ldots, s_{k-1}\}$. By Item 1, $|Y| \le d_k$. On the other hand, the neighbor $v \in V_{\vec{T}'} \subset U'$ of u must have out-degree in $U \backslash S$ greater than $d_k + k - 1 - |S| = d_k$. Thus there is an out-neighbor of v which is in $U \backslash (S \cup Y)$. Let u be mapped to this vertex. Then $(v, u) \in \vec{E}(G[U])$ is oriented from v to u, as desired. Since $d_{G-v}(u, s_i) > i$ for each $i < k$, the addition of u cannot violate the G-isometry property (since all vertices of $\vec{T} - u$ are embedded into $S \cup \{v\}$). This gives the required isometric embedding of $\vec{T}$ and finishes the proof. □

Our next elementary lemma yields an upper bound on the sum of a geometric progression.

Lemma 4.2. *For positive real numbers a, c with $c(a-1) > 1$ we get $\sum_{i=0}^{n-1} a^i < (1+c)a^{n-1}$ for every $n \in \mathbb{N}$.*

Proof. Since $\sum_{i=0}^{n-1} a^i = (a^n - 1)/(a - 1)$, the inequality is equivalent to $a^n - 1 < (1+c)a^{n-1}(a-1) = a^n - a^{n-1} + ca^{n-1}(a-1)$ and to $a^{n-1} - 1 < ca^{n-1}(a-1)$. The latter inequality follows from $a^{n-1} < ca^{n-1}(a-1)$, which is equivalent to $1 < c(a-1)$. □

In the proof of Lemma 4.4 we shall use the following Chernoff-type bounds; for a proof see, for example [[1], §A.1].

Lemma 4.3 (Chernoff bounds). *Let $X_1, \ldots, X_n$ be independent random variables taking values in $\{0, 1\}$ and let $\mathbb{E}X$ be the expected value of their sum $X = \sum_{i=1}^{n} X_i$. Then*

$$\mathbb{P}\{X \ge C \cdot \mathbb{E}X\} \le \left(\frac{e^{C-1}}{C^C}\right)^{\mathbb{E}X}, \quad \mathbb{P}\{X \ge (1+c)\mathbb{E}X\} \le e^{-(c^2/3)\mathbb{E}X} \quad \text{and} \quad \mathbb{P}\{X \le (1-c)\mathbb{E}X\} \le e^{-(c^2/2)\mathbb{E}X}$$

for every $C > 1$ and $0 < c < 1$.

Lemma 4.4. *For positive integers n, N the inequality $\mathsf{IR}(\overrightarrow{\mathcal{T}_n}) \le N$ holds if there exist real numbers $c, p \in (0,1)$, $C \in (1,\infty)$ satisfying the following inequalities:*

(1) $c^2 p(N-1) > 3\ln(3N)$;
(2) $(1 - C + C\ln C)p(1+c)^n(pN)^{n-2} > (n-1)\ln N + \ln(1+c) + \ln(3)$;
(3) $c^2 C^2 (1+c)^{2n}(pN)^{2n-4} > N\ln 2 + \ln(3n)$;
(4) $(n-1) + (n-1)(n-2)/(1-c)p + (2C/(1-c))(n-1)(1+c)^n(pN)^{n-2} < N$.

Proof. Assume that the numbers n, N, p, c, C satisfy the assumptions of the lemma. Let $G = G(N,p)$ be a random graph on N vertices in which an edge $\{u,v\} \subset V_G$ appears with probability p. We shall prove that with nonzero probability the random graph G satisfies $G \Rightarrow \overrightarrow{\mathcal{T}_n}$.

Let

$$\hbar := (1+c)^n(pN)^{n-2}.$$

For every positive integer $k < n$ let

$$d_k = Cp\hbar \quad \text{and} \quad w_k = \frac{2(d_k + k - 1)}{(1-c)p} + 1.$$

Chernoff bound implies that any fixed vertex of G has degree $\ge (1+c)p(N-1)$ with probability $< e^{-(c^2/3)p(N-1)}$. Consequently, with probability $P_1 > 1 - Ne^{-(c^2/3)p(N-1)}$ all vertices of G have degree $< (1+c)pN$. The condition (1) implies that $-(c^2/3)p(N-1) < -\ln(3N)$ and hence

$$P_1 > 1 - Ne^{-(c^2/3)p(N-1)} > 1 - Ne^{-\ln(3N)} = \frac{2}{3}.$$

For every $k < n$, take any pairwise distinct points $v, s_1, \ldots, s_{k-1} \in V_G$. If all vertices of G have degree at most $(1+c)pN$, then for every $i < k$ the ball $B(s_i, i) = \{x \in V_G : \mathrm{dist}_G(x, s_i) \le i\}$ has cardinality

$$|B(s_i, i)| \le \sum_{j=0}^{i} ((1+c)pN)^j < (1+c)((1+c)pN)^i.$$

The latter strict inequality can be derived from Lemma 4.2 and the inequality $cpN \ge c^2pN > 3\ln(3N) \ge 3$.

By above, the set X of vertices of $G - v$ at distance at most $i < k$ in $G - v$ from at least one s_i has size at most

$$\sum_{i=1}^{k-1} |B(s_i, i)| < (1+c)\sum_{i=1}^{k-1} ((1+c)pN)^i < (1+c)^{k+1}(pN)^{k-1} \le \hbar.$$

Consider the set Y of neighbors of v that fall into the set X. The definition of X does not depend on the edges incident to v, so conditioned on X (of size at most $\hbar$) the size of Y is dominated by $Y' \sim Bin(\hbar, p)$. Chernoff bound shows that the probability that Y' is at least $Cp\hbar = C\mathbb{E}Y'$ is at most $(e^{C-1}/C^C)^{p\hbar}$. Since the number of possible choices of $v, s_1, \ldots, s_{k-1}$ is equal to $N!/(N-k)! \leq N^k$, with probability

$$P_2 \geq 1 - \sum_{k=1}^{n-1} N^k \left(\frac{e^{C-1}}{C^C}\right)^{p\hbar} > 1 - (1+c)N^{n-1}\left(\frac{e^{C-1}}{C^C}\right)^{p\hbar},$$

the condition (1) of Lemma 4.1 is satisfied or we have a vertex of degree $\geq (1+c)pN$. We claim that $P_2 > 2/3$. It suffices to prove that

$$\ln(1+c) + (n-1)\ln N + p\hbar(C - 1 - C\ln C) < -\ln(3).$$

But this follows from condition (2).

Next, we prove that with probability $> 2/3$ the condition (2) of Lemma 4.1 holds. Take any positive $k < n$ and put $\bar{w}_k = \min\{m \in \mathbb{N} : w_k < m\}$. For any fixed set $W \subset V_G$ of cardinality $|W| = \bar{w}_k$, the number of edges it spans is $Bin\left(\binom{\bar{w}_k}{2}, p\right)$. By Chernoff bound, the probability that it is less than $(1-c)p\binom{\bar{w}_k}{2}$ is less than $e^{-(1/2)c^2p\binom{\bar{w}_k}{2}}$. The probability $P_{3,k}$ that some set $W \subset V_G$ of cardinality $|W| = \bar{w}_k$ spans less than $(1-c)p\binom{\bar{w}_k}{2}$ edges is $P_{3,k} < \binom{N}{\bar{w}_k}e^{-(1/2)c^2p\binom{\bar{w}_k}{2}} < 2^N e^{-(1/4)c^2p\bar{w}_k(\bar{w}_k-1)}$. We claim that $P_{3,k} < 1/3n$ which will follow as soon as we show that $N\ \ln 2 - (1/4)c^2p\bar{w}_k(\bar{w}_k - 1) < -\ln\ (3n)$. For this it suffices to check that $(1/4)c^2p\bar{w}_k(\bar{w}_k - 1) > N\ln 2 + \ln(3n)$.

This follows from the chain of the inequalities

$$\begin{aligned}\frac{1}{4}c^2\bar{w}_k(\bar{w}_k - 1) &> \frac{1}{4}c^2(w_k - 1)^2 > c^2C^2\hbar^2 = c^2C^2(1+c)^{2n}(pN)^{2n-4} \\ &> N\ln 2 + \ln(3n),\end{aligned}$$

the last inequality postulated in (3). Therefore, $P_{3,k} < 1/3n$ and the probability P_3 that for every $k < n$ every set $W \subset V_G$ of cardinality $|W| = \bar{w}_k$ spans at least

$$(1-c)p\binom{\bar{w}_k}{2} = \frac{1}{2}(1-c)p(\bar{w}_k - 1)\bar{w}_k > \frac{1}{2}(1-c)p(w_k - 1)\bar{w}_k = (d_k + k - 1)\bar{w}_k,$$

edges is $> 1 - \sum_{k=1}^{n-1} P_{3,k} > 1 - ((n-1)/3n) > 2/3$. So, with probability $> 2/3$ the condition (2) of Lemma 4.1 holds.

Since $(1 - P_1) + (1 - P_2) + (1 - P_3) < 1$, there is a non-zero probability that the random graph $G = G(N, p)$ satisfies the conditions (1) and (2) of Lemma 4.1.

It remains to show that the condition (3) of Lemma 4.1 holds, too. For this observe that

$$\sum_{k=1}^{n-1} w_k = (n-1) + \sum_{k=1}^{n-1} \frac{2(Cp\hbar + k - 1)}{(1-c)p} = (n-1) + \frac{2}{(1-c)p}\sum_{k=1}^{n-1}(k-1) + \frac{2C}{1-c}(n-1)\hbar$$
$$= (n-1) + \frac{(n-1)(n-2)}{(1-c)p} + \frac{2C}{1-c}(n-1)(1+c)^n(pN)^{n-2} < N.$$

The last inequality follows from the condition (4) of the Lemma.

Now it is legal to apply Lemma 4.1 and conclude that $G \Rightarrow \overrightarrow{\mathcal{T}}_n$ and hence $\mathsf{IR}(\overrightarrow{\mathcal{T}}_n) \le |G| = N$. □

Now we are able to prove the promised upper bound $\mathsf{IR}(\overrightarrow{\mathcal{T}}_n) \le (4e + o(1))^n (n^2 \ln n)^n = n^{2n+o(n)}$.

Theorem 4.5. *For every $\varepsilon \in (0,1)$ there is $n_\varepsilon \in \mathbb{N}$ such that $\mathsf{IR}(\overrightarrow{\mathcal{T}}_n) \le (4e(1+\varepsilon)\, n^2 \ln n)^n$ for all $n \ge n_\varepsilon$.*

Proof. Choose any positive $\delta, c \in (0,1)$ such that

$$(1+\delta)(1+c) < 1+\varepsilon \quad \text{and} \quad 4(1+\delta)\,\frac{1-c}{2+c} > 2+\delta.$$

For every $n \in \mathbb{N}$ let N be the smallest integer number, which is greater than

$$\frac{(2+c)e^n}{1-c}(n-1)(1+c)^n(4(1+\delta)\, n^2 \ln n)^{n-2}$$

and let

$$p := \frac{4(1+\delta)\, n^2 \ln n}{N}.$$

So, $N > ((2+c)e^n/(1-c))(n-1)(1+c)^n(pN)^{n-2} \ge N-1$. It is easy to see that

$$N = o((4e(1+\varepsilon)\, n^2 \ln n)^n)$$

and for $C = e^n$ the conditions (1), (3) and (4) of Lemma 4.4 hold for all sufficiently large n. To verify the condition (2), observe that

$$(1 - C + C\ln C)p(1+c)^n(pN)^{n-2} \ge (1 - e^n + e^n n)p\frac{(N-1)(1-c)}{(2+c)e^n(n-1)}$$
$$= \frac{1 + e^n(n-1)}{e^n(n-1)}\frac{1-c}{2+c}\frac{N-1}{N}pN$$
$$= \left(1 + \frac{1}{e^n(n-1)}\right)\frac{N-1}{N}\frac{1-c}{2+c}\; 4(1+\delta)n^2 \ln n$$
$$> \left(1 + \frac{1}{e^n(n-1)}\right)\frac{N-1}{N}(2+\delta)\, n^2 \ln n = (2 + \delta + o(1))\, n^2 \ln n.$$

On the other hand, $(n-1)\ln N + \ln(1+c) + \ln 3 = (2+o(1))\, n^2 \ln n$. So, the condition (2) holds for large n. Applying Lemma 4.4, we conclude that

$$\mathsf{IR}(\overrightarrow{\mathcal{T}_n}) \le N \le (4e(1+\varepsilon)\, n^2 \ln n)^n$$

for all sufficiently large n. □

By Corollary 3.6 and Theorem 4.5, $\mathsf{IR}(\overrightarrow{I_n}) = o(n^{2n})$ and $\mathsf{IR}(\overrightarrow{\mathcal{T}_n}) \le n^{2n+o(n)}$.

Question 4.6. What is the growth rate of the sequence $\mathsf{IR}(\overrightarrow{\mathcal{T}_n})$? Is $\mathsf{IR}(\overrightarrow{\mathcal{T}_n}) = n^{o(n)}$?

The technique developed for the proof of Theorem 4.5 allows us to improve the upper bound

$$\mathsf{R}(\overrightarrow{\mathcal{T}_n}) \le (4(5e^2)^4 + o(1))\, n^4 \ln n,$$

obtained by Kohayakawa, Łuczak and Rödl in (the proof of) Theorem 1 of [9], and replace the constant $4(5e^2)^4 = 2500e^8 \approx 7452395.96\ldots$ by the much smaller constant $K \approx 98.82\ldots$.

Theorem 4.7. *Let* $K := \min_{x>1} 16x^2/(1 - x + x\ln x) \approx 98.8249\ldots$ *For any positive* $\varepsilon > 0$ *there exists* $n_\varepsilon \in \mathbb{N}$ *such that* $\mathsf{R}(\overrightarrow{\mathcal{T}_n}) < (K+\varepsilon)\, n^4 \ln n$ *for all* $n \ge n_\varepsilon$. *Consequently,* $\mathsf{R}(\overrightarrow{\mathcal{T}_n}) < 99\, n^4 \ln n$ *for all sufficiently large* n.

Proof. We indicate which changes should be made in the proof of Theorem 4.5 to obtain Theorem 4.7.

In the condition (1) of Lemma 4.1 the inequality $d_{G-v}(y, s_i) \le i$ should be replaced by $d_{G-v}(y, s_i) \le 1$.

In the proof of Lemma 4.4 the constant $\hbar$ should be redefined as $\hbar := (1+c)(n-2)pN$ and the conditions (1) to (4) of Lemma 4.4 should be changed to the conditions:

(1′) $c^2 pN > 3\ln(3N)$;

(2′) $(1 - C + C\ln C)(1+c)(n-2)p^2 N > (n-1)\ln N + \ln(1+c) + \ln(3)$;

(3′) $(cC(1+c)(n-2)pN)^2 > N\ln 2 + \ln(3n)$;

(4′) $(n-1) + n(n-1)/(1-c)p + (2C(1+c)/(1-c))(n-1)(n-2)pN < N$.

Now we are able to prove Theorem 4.7. Let $C \approx 4.92155\ldots$ be the unique real number in $(1, \infty)$ such that

$$\frac{16C^2}{1 - C + C\ln C} = K := \min_{x>1} \frac{16x^2}{1 - x + x\ln x} \approx 98.8249\ldots$$

[1]Given any $\varepsilon > 0$, choose real numbers $\delta, c \in (0,1)$ such that $K\delta < \varepsilon$ and

$$4(1+\delta)\frac{(1-c)^2}{(1+c)^3} > 4 + \delta.$$

[1]The approximate values of C and K were found by the online WolframAlpha computational knowledge engine at www.wolframalpha.com

For every $n \in \mathbb{N}$ let

$$p := \frac{1-c}{2C(1+c)^2 n^2}$$

and let N be the smallest integer greater than $K(1+\delta)n^4 \ln n$. It is easy to see that the conditions (1′), (3′), and (4′) are satisfied for all sufficiently large n. To see that (2′) holds, observe that

$$\begin{aligned}(1-C+C\ln C)(1+c)(n-2)p^2N &\geq \frac{(1-C+C\ln C)(1+c)(1-c)^2}{(2C(1+c)^2n^2)^2}(n-2) \\ &\quad K(1+\delta)\, n^4 \ln n \\ &= \frac{1-C+C\ln C}{C^2}\frac{(1-c)^2}{4(1+c)^3}K(1+\delta)(n-2)\ln n \\ &= \frac{16}{K}\frac{(1-c)^2}{4(1+c)^3}K(1+\delta)(n-2)\ln n \\ &> (4+\delta)(n-2)\ln n = (4+\delta+o(1))\, n\ln n.\end{aligned}$$

On the other hand,

$$\begin{aligned}(n-1)\ln N + \ln(1+c) + \ln 3 &\leq (n-1)\ln(1+K(1+\delta)\, n^4 \ln n) + \ln(1+c) + \ln 3 \\ &= (4+o(1))\, n\, \ln n,\end{aligned}$$

so for large n the condition (2′) is satisfied, too.

Applying the modified version of Lemma 4.4, we get

$$\mathsf{R}(\vec{\mathcal{T}}_n) \leq N \leq (K+\varepsilon)\, n^4 \ln n$$

for all sufficiently large numbers n. □

5 | LONG DIRECTED PATHS IN ORIENTATIONS OF A GRAPH

By the Gallai-Hasse-Roy-Vitaver Theorem [[13], Theorem 3.13], each finite graph G has chromatic number

$$\chi(G) = \max\{n \in \mathbb{N} : G \rightharpoonup \vec{I}_n\}.$$

Having this characterization in mind, for every graph G consider the numbers

$$\bar{\bar{\chi}}_I(G) = \sup\{n \in \mathbb{N} : G \Rightarrow \vec{I}_n\}, \quad \bar{\bar{\chi}}_T(G) = \sup\{n \in \mathbb{N} : G \Rightarrow \vec{\mathcal{T}}_n\},$$

and observe that $\bar{\bar{\chi}}_T(G) \le \bar{\bar{\chi}}_I(G) \le \chi(G)$ and

$$\bar{\bar{\chi}}_I(G) \le \sup\{\mathrm{diam}(G') + 1 : G' \quad \text{is a connected component of} \quad G\}.$$

Observe that $\mathsf{IR}(\vec{I_n})$ (resp. $\mathsf{IR}(\vec{\mathcal{T}_n})$) is equal to the smallest cardinality $|G|$ of a graph G with $\bar{\bar{\chi}}_I(G) \ge n$ (resp. $\bar{\bar{\chi}}_T(G) \ge n$). So, the characteristics $\bar{\bar{\chi}}_I$ and $\bar{\bar{\chi}}_T$ determine the isometric Ramsey numbers $\mathsf{IR}(\vec{I_n})$ and $\mathsf{IR}(\vec{\mathcal{T}_n})$.

We shall show that a graph G has $\bar{\bar{\chi}}_I(G) \le 2$ if and only if G is a comparability graph. We recall that a graph G is called a *comparability graph* if G admits a *transitive* orientation $\vec{G}$ (that is, for any directed edges (x, y) and (y, z) of $\vec{G}$ the pair (x, z) is a directed edge of $\vec{G}$); equivalently, the set V_G of vertices of G admits a partial order such that a pair $\{u, v\}$ of distinct vertices of G is an edge of G if and only if u and v are comparable in the partial order. By the results of Ghouila-Houri and of Gilmore and Hoffman (see [[4], Theorem 6.1.1]), comparability graphs can be characterized as graphs G whose every cycle of odd length has a triangular chord (more precisely, for every $(2n + 3)$-cycle on $(v_0, \dots, v_{2n+2})$ with $n \ge 1$, there is a residue i modulo $2n + 3$ such that $\{v_i, v_{i+2}\} \in E_G$). More information on comparability graphs can be found in Chapter 6 of the survey [4].

Proposition 5.1. *A graph G has $\bar{\bar{\chi}}_I(G) \le 2$ if and only if G is a comparability graph.*

Proof. If G is comparability graph, then G has a transitive orientation $\vec{G}$. It follows that for any directed path (v_0, v_1, v_2) in $\vec{G}$ the pair (v_0, v_2) is an edge of $\vec{G}$ and hence $d_G(v_0, v_2) \le 1$. This means that $G \not\Rightarrow \vec{I_3}$ and hence $\bar{\bar{\chi}}_I(G) \le 2$.

If G is not a comparability graph, then G contains an odd cycle C without a triangular chord. It is easy to see that any orientation $\vec{C}$ of the cycle C contains a directed path (v_0, v_1, v_2). Since C has no triangular chords, $d_G(v_0, v_2) = 2$, which means that $\{v_0, v_1, v_2\}$ is an isometric copy of $\vec{I_3}$ in $\vec{C}$ and in G. Therefore, $\bar{\bar{\chi}}_I(G) \ge 3$. □

Problem 5.2. Characterize graphs G with $\bar{\bar{\chi}}_I(G) \le 3$ ($\bar{\bar{\chi}}_I(G) \le n$ for $n \ge 4$).

Problem 5.3. Characterize graphs G with $\bar{\bar{\chi}}_T(G) \le 2$ ($\bar{\bar{\chi}}_T(G) \le n$ for $n \ge 3$).

Remark 5.4. Any cycle C of odd length $n \ge 5$ satisfies $\bar{\bar{\chi}}_I(C) = 3$ and $\bar{\bar{\chi}}_T(C) = 2$.

Now we prove a weak 3-space property for the number $\bar{\bar{\chi}}_I(G)$. By a *weak homomorphism* $f : G \to H$ of graphs G, H we understand a function $f: V_G \to V_H$ such that for every edge $\{u, v\}$ of G we have either $f(u) = f(v)$ or $\{f(u), f(v)\}$ is an edge of H. For a weak homomorphism $f: G \to H$ and vertex y of H the preimage $f^{-1}(y)$ is a graph with the set of edges $\{\{u, v\} \in E_G : f(u) = y = f(v)\}$.

Proposition 5.5. *If $f : G \to H$ is a weak homomorphism of finite graphs, then*

$$\bar{\bar{\chi}}_I(G) \le \max\left\{ \sum_{y \in F} \bar{\bar{\chi}}_I(f^{-1}(y)) : F \subset V_H,\ |F| \le \chi(H) \right\}.$$

Proof. By definition of the chromatic number $\chi(H)$, there exists a coloring $c: V_H \to \{1, ..., \chi(H)\}$ of the graph H such that for every edge $\{u, v\}$ of H the colors $c(u)$ and $c(v)$ are distinct. For every $y \in H$ choose an orientation $\vec{G}_y$ of the graph $G_y = f^{-1}(y)$ such that $\vec{G}_y$ contains no isometric copy of $\vec{I}_k$ for $k = \bar{\bar{\chi}}_I(G_y) + 1$. Let $\vec{G}$ be the orientation of the graph G such that for an edge $\{u, v\}$ of G the ordered pair (u, v) is an edge of $\vec{G}$ if and only if either $c(f(u)) < c(f(v))$ or (u, v) is an edge of $\vec{G}_y$ for some $y \in H$.

We claim that the digraph $\vec{G}$ contains no isometric copy of the graph $\vec{I}_{m+1}$, where

$$m = \max\left\{\sum_{y\in F} \bar{\bar{\chi}}_I(G_y) : F \subset V_H,\ |F| \le \chi(H)\right\}.$$

Suppose on the contrary that $\vec{G}$ contains a directed path $(v_0, ..., v_m)$ such that $d_G(v_0, v_m) = m$. It follows that $(c(f(v_0)), ..., c(f(v_m)))$ is a nondecreasing sequence of numbers in the interval $\{1, ..., \chi(H)\}$. Consequently, for every number i in the set $C = \{c(f(v_0)), ..., c(f(v_m))\}$ the set $J_i = \{j \in \{0, ..., m\} : c(f(v_j)) = i\}$ coincides with some subinterval $[a_i, b_i] := \{n \in \mathbb{Z} : a_i \le n \le b_i\}$ of $\{0, ..., m\}$ and the set $\{f(v_j) : j \in [a_i, b_i]\}$ is a singleton $\{y_i\}$ for some vertex $y_i \in H$. It follows that $(v_{a_i}, ..., v_{b_i})$ is a directed path isometric to $\vec{I}_{|[a_i,b_i]|}$ in the graph G_{y_i} and hence $|[a_i - b_i]| \le \bar{\bar{\chi}}_I(G_{y_i})$. The choice of the orientation $\vec{G}$ guarantees that the set $F = \{y_i : i \in C\}$ has cardinality $|F| = |C| \le \chi(H)$. Then

$$m + 1 = |[0, m]| = \sum_{i\in C} |[a_i, b_i]| \le \sum_{i\in C} \bar{\bar{\chi}}_I(G_{y_i}) = \sum_{y\in F} \bar{\bar{\chi}}_I(G_y) \le m,$$

which is a desired contradiction. □

6 | INFINITE DIRECTED PATHS IN ORIENTATIONS OF GRAPHS

Now we discuss the problem of existence of infinite directed paths in orientations of graphs. Consider the infinite digraphs $\vec{I}_\omega$ and $\vec{I}_{-\omega}$ with $V_{\vec{I}_\omega} = \omega = V_{\vec{I}_{-\omega}}$, $E_{\vec{I}_\omega} = \{(i, i+1) : i \in \omega\}$, and $E_{\vec{I}_{-\omega}} = \{(i+1, i) : i \in \omega\}$. Here $\omega = \{0\} \cup \mathbb{N}$.

First, observe that Theorem 3.3 implies the following:

Corollary 6.1. *There exists a countable graph G such that $G \Rightarrow \vec{I}_n$ for every $n \in \mathbb{N}$.*

In contrast, we shall prove that each graph G admits an orientation containing no isometric copy of the digraphs $\vec{I}_\omega$ or $\vec{I}_{-\omega}$ and, more generally, no directed paths of infinite diameter in G. (For a subset $A \subset V_G$ of a graph G its *diameter* is defined as $\mathrm{diam}(A) = \sup\{d_G(u, v): u, v \in A\} \in \omega \cup \{\infty\}$.)

A sequence $(v_n)_{n\in\omega} \in V_G^\omega$ of distinct vertices of a graph G is called an *ω-path* in G if for every $n \in \omega$ the pair $\{v_n, v_{n+1}\}$ is an edge of G. An ω-path $(v_n)_{n\in\omega}$ in a graph G is called *$\vec{\omega}$-directed* (resp. *$\overleftarrow{\omega}$-directed*) in an orientation $\vec{G}$ of G if for every $n \in \omega$ the pair (v_n, v_{n+1}) (resp. (v_{n+1}, v_n)) is a directed edge of $\vec{G}$. An ω-path in G is called *directed* in an orientation $\vec{G}$ of G if it is either $\vec{\omega}$-directed or $\overleftarrow{\omega}$-directed.

The Ramsey theorem implies that every orientation of the complete countable graph K_ω contains $\vec{I}_\omega$ or $\vec{I}_{-\omega}$. In contrast, we have the following result.

Theorem 6.2. *Every graph G has an orientation $\vec{G}$ containing no directed ω-paths of infinite diameter in G. This implies that $G \not\Rightarrow \vec{I}_\omega$ and $G \not\Rightarrow \vec{I}_{-\omega}$.*

Proof. Without loss of generality, the graph G is connected. Fix any vertex o in G and for every vertex v of G let $\|v\|$ be the smallest length of a path linking the vertices v and o. Choose an orientation $\vec{G}$ of G such that for any edge $\{u, v\}$ in G with $\|v\| = \|u\| + 1$ the pair (u, v) is an edge of $\vec{G}$ if $\|u\|$ is even and (v, u) is an edge of $\vec{G}$ if $\|u\|$ is odd.

We claim that the orientation $\vec{G}$ contains no directed ω-paths of infinite diameter. To derive a contradiction, assume that $(v_n)_{n\in\omega}$ is a directed ω-path of infinite diameter. Fix any even number $n \in \omega$ such that $\|v_0\| < n$. Since the ω-path $(v_n)_{n\in\omega}$ has infinite diameter, there exists a number $k \in \omega$ such that $\|v_k\| \geq n$. We can assume that k is the smallest number with this property. Taking into account that $|\|v_n\| - \|v_{n+1}\|| \leq 1$ for all $n \in \omega$, we conclude that $\|v_k\| = n > \|v_0\|$ and $\|v_{k-1}\| = n - 1$, and hence (v_k, v_{k-1}) is an edge of $\vec{G}$. Let also m be the smallest number such that $\|v_m\| \geq n + 1$. For this number we get $\|v_m\| = n + 1$, $\|v_{m-1}\| = n$ and hence (v_{m-1}, v_m) is a directed edge $\vec{G}$. Since both pairs (v_k, v_{k-1}) and (v_{m-1}, v_m) are directed edges of the oriented graph $\vec{G}$, the ω-path $(v_n)_{n\in\omega}$ is not directed in $\vec{G}$. Since the graphs $\vec{I}_\omega$ and $\vec{I}_{-\omega}$ have infinite diameters, the digraph $\vec{G}$ does not contain isometric copies of $\vec{I}_\omega$ or $\vec{I}_{-\omega}$. □

Remark 6.3. Theorem 6.2 implies that every locally finite graph G admits an orientation containing no directed ω-paths.

ACKNOWLEDGMENTS

The authors express their sincere thanks to Yoshiharu Kohayakawa for suggesting the proof of Theorem 2.1 (which resolves a problem posed in an earlier version of this paper) and to Tomasz Łuczak for valuable discussions related to random graphs. The authors are also grateful to anonymous referees whose valuable remarks and suggestions helped much to correct some mistakes and improve the presentation of the results of this paper. OP was supported by ERC grant 306493 and EPSRC grant EP/K012045/1.

ORCID

Taras Banakh http://orcid.org/0000-0001-6710-4611
Oleg Pikhurko http://orcid.org/0000-0002-9657-4011
Igor Protasov http://orcid.org/0000-0003-1518-6234

REFERENCES

1. N. Alon and J. Spencer, *The probabilistic method*, Wiley Series in Discrete Mathematics and Optimization, 4th ed., John Wiley & Sons, Inc., Hoboken, NJ, 2016.

2. L. Addario-Berry, F. Havet, C. L. Sales, B. Reed, and S. Thomassé, *Oriented trees in digraphs*, Discrete Math. **313** (2013), no. 8, 967–974.
3. B. Bollobás, *Chromatic number, girth and maximal degree*, Discrete Math. **24** (1978), no. 3, 311–314.
4. A. Brandstädt, B. V. Le, and J. Spinrad, *Graph classes: A survey*, SIAM Monographs on Discrete Mathematics and Applications (1999).
5. S. Burr, *Subtrees of directed graphs and hypergraphs*, Proceedings of the Eleventh Southeastern Conference on Combinatorics, Graph Theory and Computing, vol. I, Florida Atlantic University, Boca Raton, FL, Congr. Numer. 28, 1980, pp. 227–239.
6. M. Cochand and P. Duchet, *A few remarks on orientation of graphs and Ramsey theory*, Irregularities of partitions (Fertod, 1986), Algorithms Combin. Study Res. Texts, vol. **8**, Springer, Berlin, 1989, pp. 39–46.
7. D. Dellamonica Jr. and V. Rödl, *Distance preserving Ramsey graphs*, Combin. Probab. Comput. **21** (2012), no. 4, 554–581.
8. P. Erdös, *Graph theory and probability*, Canad. J. Math. **11** (1959), 34–38.
9. Y. Kohayakawa, T. Łuczak, and V. Rödl, *Ramsey-type results for oriented trees*, J. Graph Theory **22** (1996), no. 1, 1–8.
10. I. V. Protasov, *New chromatic numbers in open problems*, Mat. Stud. **42** (2016), no. 2, 115–117.
11. V. Rödl, *A generalization of Ramsey theorem*, (Zdzisław Skupień, ed.), Graphs, Hypergraphs and Block Systems (Zielona Góra), 1976, pp. 211–220.
12. J. Spencer, *Asymptotic lower bounds for Ramsey functions*, Discrete Math. **20** (1977), no. 1, 69–76.
13. J. Nešetril and P. Ossona de Mendez, *Sparsity. Graphs, structures, and algorithms*, Algorithms and Combinatorics, vol. **28**, Springer, Heidelberg, 2012.

How to cite this article: Banakh T, Idzik A, Pikhurko O, Protasov I, Pszczoła K. Isometric copies of directed trees in orientations of graphs. *J Graph Theory*. 2020;94: 175–191. https://doi.org/10.1002/jgt.22513